MATH 721, Algebra II
Exercise 13
Due Fri 18 April

Exercise 1. [Nine Lemma] Let R be a ring and consider a commutative diagram of R-module homomorphisms:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{h_1} & C_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle k_1} & & \downarrow{\scriptstyle l_1} & & \\
0 & \longrightarrow & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{h_2} & C_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle k_2} & & \downarrow{\scriptstyle l_2} & & \\
0 & \longrightarrow & A_3 & \xrightarrow{f_3} & B_3 & \xrightarrow{h_3} & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Assume that all three columns of this diagram are exact, and that the two bottom rows are exact. Show that the top row is exact.

Exercise 2. Let R be a ring. Given two R-module homomorphisms $f\colon M \to M'$ and $g\colon N \to N'$, define $f \oplus g\colon M \oplus N \to M' \oplus N'$ by the formula

$$(f \oplus g)(m, n) = (f(m), g(n)).$$

Consider two sequences (not necessarily exact) of R-module homomorphisms

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \qquad\qquad N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3.$$

Show that the given sequences are exact if and only if the sequence

$$M_1 \oplus N_1 \xrightarrow{f_1 \oplus g_1} M_2 \oplus N_2 \xrightarrow{f_2 \oplus g_2} M_3 \oplus N_3$$

is exact.

Exercise 3. Let R be a ring and M an R-module. Let $N \subseteq M$ be an R-submodule.

(a) Show that, if M is noetherian as an R-module, then so are N and M/N.
(b) Assume that R has identity, and let $\tau\colon R \to S$ be an epimorphism of rings with identity. Show that, if R is noetherian, then so is S.
(c) Assume that R is commutative and has identity. Show that R is noetherian if and only if the polynomial ring $R[x]$ is noetherian.
(d) Must the converse of part (b) hold?
(e) Must every subring of a noetherian ring be noetherian?