MATH 721, Algebra II
Exercises 11 and 12
Due Fri 11 April

Exercise 1. Find the intermediate fields of the following field extensions:
(a) $\mathbb{Q} \subseteq K$ where K is a splitting field for the polynomial $f = x^2 - 2$,
(b) $\mathbb{Q} \subseteq L$ where L is a splitting field for the polynomial $g = x^3 - 2$,
(c) $\mathbb{Q} \subseteq M$ where M is a splitting field for the polynomial $h = (x^3 - 2)(x^2 - 2)$,

Exercise 2. Let $p \geqslant 1$ be a prime integer, and assume that the polynomial $x^p - a \in \mathbb{Q}[x]$ is irreducible. Let K be a splitting field for $x^p - a$ over $\mathbb{Q}$. Show that the Galois group $\mathrm{Gal}(K\colon \mathbb{Q})$ is isomorphic to the set of all functions $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ of the form $y \mapsto ky + l$ for $k, l \in \mathbb{Z}/p\mathbb{Z}$ with $k \neq 0$.

Exercise 3. Let $p_1, \ldots, p_n$ be distinct positive prime integers. Show the Galois group of the extension $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n})$ is isomorphic to $(\mathbb{Z}/(2))^n$.

Exercise 4. Let R be a ring and let M and N be R-modules. Show that there is a natural isomorphism between the following functors:

$$\mathrm{Hom}_R(M \oplus M', -) \cong \mathrm{Hom}_R(M, -) \oplus \mathrm{Hom}_R(M', -).$$

(Here each of these functors maps from the category of left R-modules to the category of abelian groups.)

Exercise 5. Let R be a ring, and consider an exact sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0.$$

Show that the following conditions are equivalent:

(i) The exact sequence is split;
(ii) For each R-module N, the map $\mathrm{Hom}_R(f, N)\colon \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N)$ is surjective;
(iii) For each R-module N, the map $\mathrm{Hom}_R(N, g)\colon \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, M'')$ is surjective.