MATH 721, Algebra II
Exercises 3
Due Fri 01 Feb

Exercise 1. Let R be a ring with identity, and let M be a unital R-module.

(a) Show that there is a sequence of R-module homomorphisms

$$\cdots \xrightarrow{\partial_{i+1}} R^{(\Lambda_i)} \xrightarrow{\partial_i} R^{(\Lambda_{i-1})} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} R^{(\Lambda_1)} \xrightarrow{\partial_1} R^{(\Lambda_0)} \xrightarrow{\partial_0} M \xrightarrow{\partial_{-1}} 0$$

such that $\operatorname{Im}(\partial_{i+1}) = \operatorname{Ker}(\partial_i)$ for each $i \geqslant -1$.

(b) Assume that $R = \mathbb{Z}$ and that M is finitely generated. Show that there is a sequence of $\mathbb{Z}$-module homomorphisms

$$0 \xrightarrow{\partial_2} \mathbb{Z}^n \xrightarrow{\partial_1} \mathbb{Z}^m \xrightarrow{\partial_0} M \xrightarrow{\partial_{-1}} 0$$

such that $\operatorname{Im}(\partial_{i+1}) = \operatorname{Ker}(\partial_i)$ for each $i = -1, \ldots, 2$.

Exercise 2. Let R be an integral domain with quotient field $\mathrm{Q}(R)$. Show that $\mathrm{Q}(R)$ is free as an R-module if and only if R is a field.

Exercise 3. Let R be a commutative ring with identity $1 \neq 0$. Show that, if every R-module is free, then R is a field.

Exercise 4. Let k be a field and let $f\colon V \to W$ be a epimorphism of k-vector spaces. Show that $V \cong W \oplus \operatorname{Ker}(f)$.