# MATH 725, FALL 2012, HOMEWORK 6
# DUE FRIDAY 16 NOVEMBER

Let $R$ be a commutative noetherian ring with identity, and let $I \leqslant R$.

**Exercise 1.** Prove that $R/I$ is artinian if and only if there are maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \subset R$ and an integer $m \geqslant 1$ such that

$$(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n)^m \subseteq I \subseteq \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n.$$

**Exercise 2.** Assume that $R$ is semi-local and that $I$ is an ideal of definition for $R$. Let $J \leqslant R$ such that $J \subseteq I$, and assume that there are integers $p, q \geqslant 1$ such that $I^{q+1} = JI^q$.

(a) Prove that $J$ is an ideal of definition for $R$.

(b) Let $M$ be a finitely generated $R$-module. Let $G_I(t), G_J(t) \in \mathbb{Q}[t]$ be such that

$$G_I(n) = \text{length}_R(M/I^{n+1}M)$$
$$G_J(n) = \text{length}_R(M/J^{n+1}M).$$

Prove that $G_I(t)$ and $G_J(t)$ have the same leading coefficient. (Hint: Prove that $G_I(n+q) \geqslant G_J(n) \geqslant G_I(n)$ for $n \gg 0$.)