MATH 725, FALL 2012, HOMEWORK 2–3
DUE FRIDAY 05 OCTOBER

Let R be a commutative ring with identity. Let M be an R-module, and let $\mathbf{x} = x_1, \ldots, x_n \in R$.

Exercise 1. (a) Prove that the following conditions are equivalent:

(i) $\mathbf{x}$ is weakly M-regular.
(ii) $\mathbf{x}$ is weakly $U^{-1}M$-regular for each multiplicatively closed subset $U \subseteq R$.
(iii) $\mathbf{x}$ is weakly $M_{\mathfrak{p}}$-regular for each $\mathfrak{p} \in \operatorname{Spec}(R)$.
(iv) $\mathbf{x}$ is weakly $M_{\mathfrak{m}}$-regular for each maximal ideal $\mathfrak{m} \in \operatorname{Supp}_R(M)$.

(b) Assume that $\mathbf{x}$ is weakly M-regular. Prove that if M is finitely generated and $\mathbf{x} \in \mathfrak{p} \in \operatorname{Supp}_R(M)$, then $\mathbf{x}$ is $M_{\mathfrak{p}}$-regular and M-regular.

Exercise 2. Consider an exact sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \to 0$ of R-module homomorphisms, and assume that $\mathbf{x}$ is weakly D-regular. Prove that the following induced sequence is exact:

$$A/(\mathbf{x})A \xrightarrow{\overline{\alpha}} B/(\mathbf{x})B \xrightarrow{\overline{\beta}} C/(\mathbf{x})C \xrightarrow{\overline{\gamma}} D/(\mathbf{x})D \to 0.$$

Exercise 3. Assume that $n = 2$ and that $\mathbf{x} = x_1, x_2$ is weakly M-regular. Observe that the proof of Exercise 4 from Homework 1 shows that for all $e_1 \geqslant 1$, the element $x_1^{e_1}$ is weakly M-regular, the sequence

$$0 \to M/x_1^{e_1}M \xrightarrow{x_1} M/x_1^{e_1+1}M \to M/x_1M \to 0$$

is exact, and $\operatorname{Ass}_R(M/x_1^{e_1}M) = \operatorname{Ass}_R(M/x_1M)$.

(a) Use the above observation to prove that the sequence $x_1^{e_1}, x_2^{e_2}$ is weakly M-regular for all $e_1, e_2 \geqslant 1$.
(b) Use Exercise 2 to prove that the following sequence is exact for all $e_1, e_2 \geqslant 1$:

$$0 \to M/(x_1^{e_1}, x_2^{e_2})M \xrightarrow{x_1} M/(x_1^{e_1+1}, x_2^{e_2})M \to M/(x_1, x_2^{e_2})M \to 0.$$

(c) Prove that $\operatorname{Ass}_R(M/(x_1^{e_1}, x_2^{e_2})M) = \operatorname{Ass}_R(M/(x_1, x_2)M)$.
(d) If $\mathbf{x}$ is M-regular, must $x_1^{e_1}, x_2^{e_2}$ also be M-regular?
(e) Bonus: For $n \geqslant 2$, prove that $x_1^{e_1}, \ldots, x_n^{e_n}$ is weakly M-regular for $e_1, \ldots, e_n \geqslant 1$.

Exercise 4. Assume that x_1, x_2 is a weakly M-regular sequence.

(a) Prove that x_1 is weakly M/x_2M-regular.
(b) Prove that if M is noetherian and $x_1, x_2 \in J(R)$, then x_2, x_1 is M-regular. (Hint: Let $K = \operatorname{Ker}(M \xrightarrow{x_2} M)$ and prove that $K = x_1K$.)

Exercise 5. Let k be a field, and set $R = k[X, Y, Z]$.

(a) Prove that the sequence $X, Y(1-X), Z(1-X)$ is R-regular.
(b) Prove that the sequence $Y(1-X), Z(1-X), X$ is not R-regular.

Exercise 6. Let k be a field, and set $k[\![X]\!][Y]$.

(a) Prove that X, Y is a maximal R-regular sequence.
(b) Prove that $1 - XY$ is a maximal R-regular sequence.

Exercise 7. [Depth Lemma] Let R be a commutative noetherian ring, and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated R-modules. Let $I \subseteq R$ be an ideal, and prove the following inequalities:

$$\operatorname{depth}_R(I;M) \geqslant \inf\{\operatorname{depth}_R(I;M'), \operatorname{depth}_R(I;M'')\}$$
$$\operatorname{depth}_R(I;M') \geqslant \inf\{\operatorname{depth}_R(I;M), \operatorname{depth}_R(I;M'')+1\}$$
$$\operatorname{depth}_R(I;M'') \geqslant \inf\{\operatorname{depth}_R(I;M')-1, \operatorname{depth}_R(I;M)\}.$$