MATH 725, FALL 2012, HOMEWORK 1
DUE FRIDAY SEPTEMBER 7

Let R be a commutative ring with identity.

Exercise 1. (V.2.6) Let M be an R-module, and let $U \subseteq R$ be a multiplicatively closed subset.

(a) Let $m \in M$. Prove that $m/1 = 0$ in $U^{-1}M$ if and only if there exists an element $u \in U$ such that $um = 0$ in M, i.e., if and only if $U \cap \operatorname{Ann}_R(m) \neq \emptyset$.
(b) Prove that $\operatorname{Supp}_R(M) \subseteq V(\operatorname{Ann}_R(M))$.
(c) Assume that M is finitely generated. Prove that $U^{-1}M = 0$ if and only if there exists an element $u \in U$ such that $uM = 0$, i.e., if and only if $U \cap \operatorname{Ann}_R(M) \neq \emptyset$.
(d) Prove that if M is finitely generated, then $\operatorname{Supp}_R(M) = V(\operatorname{Ann}_R(M))$. (In particular, if $I \subseteq R$ is an ideal, then $\operatorname{Supp}_R(R/I) = V(I) = \operatorname{Supp}_R(R/\operatorname{rad}(I))$.)
(e) Provide an example showing that the finitely-generated assumptions in parts (c) and (d) are necessary.

Exercise 2. (V.2.9) Prove that a prime ideal $P \in \operatorname{Spec}(R)$ is an associated prime of M if and only if there is an injective R-module homomorphism $R/P \hookrightarrow M$, that is, if and only if M has a submodule $N \cong R/P$.

Exercise 3. (V.2.10) Prove that if $P \in \operatorname{Spec}(R)$, then $\operatorname{Ass}_R(R/P) = \{P\}$.

Exercise 4. (V.2.25) Assume that R is noetherian, and let $r \in R$ be an R-regular element, that is, a non-unit that is not a zero-divisor on R. Prove that $\operatorname{Ass}_R(R/r^nR) = \operatorname{Ass}_R(R/rR)$ for all $n \geqslant 1$. [Hint: Verify that the following sequence is exact:

$$0 \to R/rR \xrightarrow{r^{n-1}} R/r^nR \to R/r^{n-1}R \to 0$$

and use induction on n.]