MATH 724, FALL 2009, HOMEWORK 3
DUE FRIDAY 25 SEPTEMBER

Exercise 1. (100 pts.) (Hom and flat base change.) Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Let M and N be R-modules.

(a) Let $s \in S$, and let $\psi \in \operatorname{Hom}_R(M, N)$. Prove that there is a well-defined S-module homomorphism

$$\psi_s\colon S \otimes_R M \to S \otimes_R N$$

such that $\psi_s(t \otimes m) = (st) \otimes \psi(m)$ for all $t \in S$ and all $m \in M$.

(b) Prove that there is a well-defined S-module homomorphism

$$\Psi_{M,N}\colon S \otimes_R \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N)$$

such that $\Psi(s \otimes \psi) = \psi_s$ for all $s \in S$ and all $\psi \in \operatorname{Hom}_R(M, N)$.

(c) Let $f\colon M \to M'$ be an R-module homomorphism. Prove that the following diagram commutes:

$$\begin{array}{ccc} S \otimes_R \operatorname{Hom}_R(M', N) & \xrightarrow{\Psi_{M',N}} & \operatorname{Hom}_S(S \otimes_R M', S \otimes_R N) \\ {\scriptstyle S\otimes_R \operatorname{Hom}_R(f,N)}\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}_S(S\otimes_R f, S\otimes_R N)} \\ S \otimes_R \operatorname{Hom}_R(M, N) & \xrightarrow{\Psi_{M,N}} & \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N). \end{array}$$

(d) Let $g\colon N \to N'$ be an R-module homomorphism. Prove that the following diagram commutes:

$$\begin{array}{ccc} S \otimes_R \operatorname{Hom}_R(M, N) & \xrightarrow{\Psi_{M,N}} & \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N) \\ {\scriptstyle S\otimes_R \operatorname{Hom}_R(M,g)}\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}_S(S\otimes_R M, S\otimes_R g)} \\ S \otimes_R \operatorname{Hom}_R(M, N') & \xrightarrow{\Psi_{M,N'}} & \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N'). \end{array}$$

(e) Assume that φ is flat and that M is finitely presented as an R-module. Prove that $\Psi_{M,N}$ is an isomorphism.

Exercise 2. (Extra Credit, 100 pts.) (Ext and flat base change) State and prove the version of Exercise 1 for Ext. (Note: For part (e) you should assume that R is noetherian.)