Midterm 3 review
Math 265
Fall 2007

§14.8. Lagrange Multipliers.

Case 1: One constraint.

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$:

Step 1: Find all values of x, y, z, λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = k$. In other words, solve the following system of equations:

$$\begin{aligned}
f_x(x, y, z) &= \lambda g_x(x, y, z) \\
f_y(x, y, z) &= \lambda g_y(x, y, z) \\
f_z(x, y, z) &= \lambda g_z(x, y, z) \\
g(x, y, z) &= k
\end{aligned}$$

Step 2: Evaluate f at all the points (x, y, z) that result from Step 1. The largest of these values is the maximum value of f; the smallest of these values is the minimum value of f.

Case 2: Two constraints.

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraints $g(x, y, z) = k$ and $h(x, y, z) = c$:

Step 1: Find all values of x, y, z, λ, μ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla g(x, y, z)$ and $g(x, y, z) = k$ and and $h(x, y, z) = c$. In other words, solve the following system of equations:

$$\begin{aligned}
f_x(x, y, z) &= \lambda g_x(x, y, z) + \mu h_x(x, y, z) \\
f_y(x, y, z) &= \lambda g_y(x, y, z) + \mu h_y(x, y, z) \\
f_z(x, y, z) &= \lambda g_z(x, y, z) + \mu h_z(x, y, z) \\
g(x, y, z) &= k \\
h(x, y, z) &= c
\end{aligned}$$

Step 2: Evaluate f at all the points (x, y, z) that result from Step 1. The largest of these values is the maximum value of f; the smallest of these values is the minimum value of f.

§15.1. Double Integrals over Rectangles.

Let R be a rectangle in the xy-plane:

$$R = [a, b] \times [c, d] = \{(x, y) \mid a \le x \le b, c \le y \le d\}.$$

Let f be a function defined on R. Be able to approximate the double integral $\iint_R f(x,y)\,dA$ using the following procedure:

Step 1: Partition R into subrectangles, each of area $\Delta A = \Delta x \cdot \Delta y$ where $\Delta x = (b-a)/m$ and $\Delta y = (d-c)/n$. The values of m and n will be given. This will yield mn rectangles $R_{i,j}$.

Step 2: For each rectangle $R_{i,j}$ choose a sample point $(x^*_{i,j}, y^*_{i,j})$ in $R_{i,j}$. In practice, the method for choosing sample points will be given to you. For instance, you may be told to choose the upper-left corner of each subrectangle. "Use the midpoint rule" means choose the midpoint of each subrectangle.

Step 3: Evaluate f at each sample point, multiply by ΔA, and add up the results:

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{i,j}, y^*_{i,j})\Delta A.$$

§15.2. Iterated Integrals.

Let R be a rectangle in the xy-plane:

$$R = [a,b] \times [c,d] = \{(x,y) \mid a \le x \le b, c \le y \le d\}.$$

Let f be a function defined on R. If f is continuous on R, then

$$\iint_R f(x,y)\,dA = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx = \int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy.$$

§15.3. Double Integrals over General Regions.

Let D be a region in the xy-plane and f a function that is continuous on D.

If there are continuous functions $g_1(x), g_2(x)$ such that

$$D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then

$$\iint_D f(x,y)\,dA = \int_{x=a}^{b}\int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

If there are continuous functions $h_1(y), h_2(y)$ such that

$$D = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

then

$$\iint_D f(x,y)\,dA = \int_{y=c}^{d}\int_{x=h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Property: The area of D is $A = \iint_D dA$.

§15.4. Double Integrals in Polar Coordinates.

Polar coordinates for the point (x, y) are (r, θ) where $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$ and $\tan\theta = y/x$.

Given a region

$$D = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

if f is continuous on D, then we have

$$\iint_D f(x, y)\, dA = \int_{\theta=\alpha}^{\beta} \int_{r=a}^{b} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Given a region

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

if f is continuous on D, then we have

$$\iint_D f(x, y)\, dA = \int_{\theta=\alpha}^{\beta} \int_{r=h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

§15.5. Applications of Double Integrals.

Consider a lamina described by the region D in the xy-plane with density function $\rho(x, y)$. The mass of the lamina is

$$m = \iint_D \rho(x, y)\, dA.$$

The moment of the lamina about the x-axis is

$$M_x = \iint_D y\rho(x, y)\, dA.$$

The moment of the lamina about the y-axis is

$$M_y = \iint_D x\rho(x, y)\, dA.$$

The center of mass of the lamina is $(\overline{x}, \overline{y})$ where $\overline{x} = M_y/m$ and $\overline{y} = M_x/m$.

§15.6. Surface Area.

The area of the surface with equation $z = f(x, y)$ where (x, y) runs through all points in a region D in the xy-plane is

$$A = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA.$$

§15.7. Triple Integrals.

Let B be a rectangular box in three-space:

$$B = [a, b] \times [c, d] \times [r, s] = \{(x, y, z) \mid a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}.$$

Let f be a function defined on B. If f is continuous on B, then

$$\iiint_R f(x, y, z)\, dV = \int_{x=a}^{b} \int_{y=c}^{d} \int_{z=r}^{s} f(x, y, z)\, dz\, dy\, dx$$
$$= \int_{y=c}^{d} \int_{z=r}^{s} \int_{x=a}^{b} f(x, y, z)\, dx\, dz\, dy$$

If there are continuous functions $g_1(x), g_2(x), h_1(x, y), h_2(x, y)$ such that $E = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x), h_1(x, y) \leq z \leq h_2(x, y)\}$ then

$$\iiint_E f(x, y, z)\, dV = \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} \int_{z=h_1(x,y)}^{h_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

Property: The volume of E is $V = \iiint_E dV$.

Property: Given an object in space described by the region E with density function $\rho(x, y, z)$, the mass of the object is $m = \iiint_E \rho(x, y, z)\, dV$.

§15.8. Triple Integrals in Cylindrical and Spherical Coordinates.

Cylindrical coordinates for the point (x, y, z) are (r, θ, z) where $x = r \cos\theta$, $y = r \sin\theta$, $x^2 + y^2 = r^2$ and $\tan\theta = y/x$. Given a region

$$D = \{(r, \theta, z) \mid 0 \leq \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta), u_1(r, \theta) \leq z \leq u_2(r, \theta)\}$$

if f is continuous on D, then we have

$$\iint_D f(x, y, z)\, dV = \int_{\theta=\alpha}^{\beta} \int_{r=h_1(\theta)}^{h_2(\theta)} \int_{z=u_1(r,\theta)}^{u_2(r,\theta)} f(r \cos\theta, r \sin\theta, z)\, r\, dz\, dr\, d\theta.$$

Spherical coordinates for the point (x, y, z) are (ρ, θ, ϕ) where $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$, and $x^2 + y^2 + z^2 = \rho^2$. Given a region

$$D = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

if f is continuous on D, then we have

$$\iint_D f(x, y, z)\, dV$$
$$= \int_{\phi=c}^{d} \int_{\theta=\alpha}^{\beta} \int_{\rho=a}^{b} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi.$$

§15.9. Changes of Variables in Multiple Integrals.

A C^1 transformation of the uv-plane to the xy-plane is a rule of assignment:

Input: a point (u, v)

Output: a point (x, y) where $x = g(u, v)$ and $y = h(u, v)$, and the functions g and h have continuous first-order partial derivatives.

The Jacobian of the transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} = g_u h_v - g_v h_u$$

Let S be a region in the uv-plane and let R be the image of S under the transformation. Then

$$\iint_R f(x, y)\, dA = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv.$$

A C^1 transformation of uvw-space to xyz-space is a rule of assignment:

Input: a point (u, v, w)

Output: a point (x, y, z) where $x = g(u, v, w)$ and $y = h(u, v, w)$ and $z = k(u, v, w)$, and the functions g, h, k have continuous first-order partials.

The Jacobian of the transformation is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} g_u & g_v & g_w \\ h_u & h_v & h_w \\ k_u & k_v & k_w \end{vmatrix}$$

Let S be a region in uvw-space and let R be the image of S under the transformation. Then

$$\iiint_R f(x, y)\, dV$$
$$= \iiint_S f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw.$$

Be sure to review the sections of the text (especially the examples), your notes, your homework, and your quizzes.

Practice Exercises:

p. 977: 59–62

pp. 1050–1051: 1–36, 38, 39, 41, 42, 47–51