Midterm 1 review
Math 265
Fall 2007

§12.1. Three-Dimensional Coordinate Systems.

Coordinates of a point in space: (x, y, z)

Distance formula: If $P = (x, y, z)$ and $P_0 = (x_0, y_0, z_0)$, then

$$|PP_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

Equation of the sphere of radius r centered at the point (h, k, ℓ):

$$(x-h)^2 + (y-k)^2 + (z-\ell)^2 = r^2$$

Equations of some planes:

$z = 0$ is the xy-plane. $z = d$ is parallel to the xy-plane.

$y = 0$ is the xz-plane. $y = d$ is parallel to the xz-plane.

$x = 0$ is the yz-plane. $x = d$ is parallel to the yz-plane.

$ax + by + cz = d$ is the general plane. Graph it by graphing the intersections with the coordinate planes.

Solid regions are given by inequalities. Given inequalities, first describe the surfaces given by the corresponding equalities, then figure out which solid part of space bounded by these surfaces the inequalities describe.

§12.2. Vectors.

Vectors describe quantities that have direction and magnitude, like displacement, force, or velocity. Given two points A and B, the displacement vector $\overrightarrow{AB}$ is the vector with initial point A and terminal point B.

Vectors in the plane: $\mathbf{v} = \langle x, y \rangle$

If $A = (x_0, y_0)$ and $B = (x, y)$, then $\overrightarrow{AB} = \langle x - x_0, y - y_0 \rangle$.

length: $|\langle x, y \rangle| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$.

addition: $\langle a, b \rangle + \langle x, y \rangle = \langle a + x, b + y \rangle$. Geometrically, use the parallelogram law.

subtraction: $\langle a, b \rangle - \langle x, y \rangle = \langle a - x, b - y \rangle$.

scalar multiplication: $c \langle x, y \rangle = \langle cx, cy \rangle$. Geometrically, scalar multiplication stretches the vector and possibly reflects it in the opposite direction.

Standard vectors: $\mathbf{i} = \langle 1, 0 \rangle$, $\mathbf{j} = \langle 0, 1 \rangle$.

$\langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$

Vectors in space: $\mathbf{v} = \langle x, y, z \rangle$

If $A = (x_0, y_0, z_0)$ and $B = (x, y, z)$, then $\overrightarrow{AB} = \langle x - x_0, y - y_0, z - z_0 \rangle$.

length: $|\langle x, y, z \rangle| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$.

addition: $\langle a, b, c \rangle + \langle x, y, z \rangle = \langle a + x, b + y, c + z \rangle$. Geometrically, use the parallelogram law.

subtraction: $\langle a, b, c \rangle - \langle x, y, z \rangle = \langle a - x, b - y, c - z \rangle$.

scalar multiplication: $c \langle x, y, z \rangle = \langle cx, cy, cz \rangle$. Geometrically, scalar multiplication stretches the vector and possibly reflects it in the opposite direction.

Standard vectors: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.

$\langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

$\mathbf{u}$ is a unit vector if $|\mathbf{u}| = 1$.

If $\mathbf{v}$ is a vector and $\mathbf{v} \neq \mathbf{0}$, then $\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v}$ is a unit vector pointing in the same direction as $\mathbf{v}$.

§12.3. The Dot Product.

Input two vectors, output a scalar

$\langle a, b \rangle \cdot \langle x, y \rangle = ax + by$

$\langle a, b, c \rangle \cdot \langle x, y, z \rangle = ax + by + cz$

$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$

$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$

$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

$(c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (c\mathbf{w})$

$\mathbf{0} \cdot \mathbf{v} = 0$

If θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta$

$\mathbf{v}$ and $\mathbf{w}$ are orthogonal (or perpendicular) if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.

The vector projection ("vector shadow") of $\mathbf{v}$ onto $\mathbf{w}$ is $\text{proj}_{\mathbf{w}}\mathbf{v}$.

The scalar projection (or component) of $\mathbf{v}$ onto $\mathbf{w}$ is $\text{comp}_{\mathbf{w}}\mathbf{v}$. It is the "directed length of the vector shadow" of $\mathbf{v}$ on $\mathbf{w}$.

$$\text{comp}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|} \qquad \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}$$

§12.4. The Cross Product.

Input two vectors in V_3, output a new vector in V_3

$$\langle a,b,c\rangle \times \langle x,y,z\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix} = \begin{vmatrix} b & c \\ y & z \end{vmatrix}\mathbf{i} - \begin{vmatrix} a & c \\ x & z \end{vmatrix}\mathbf{j} + \begin{vmatrix} a & b \\ x & y \end{vmatrix}\mathbf{k}$$

$$\langle a,b,c\rangle \times \langle x,y,z\rangle = \langle bz-cy, cx-az, ay-bx\rangle$$

$\mathbf{v}\times\mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$: $(\mathbf{v}\times\mathbf{w})\cdot\mathbf{v} = 0$ and $(\mathbf{v}\times\mathbf{w})\cdot\mathbf{w} = 0$.
$|\mathbf{v}\times\mathbf{w}| = |\mathbf{v}||\mathbf{w}|\sin\theta$

$$\begin{array}{ccc} \mathbf{i}\times\mathbf{j} = \mathbf{k} & \mathbf{j}\times\mathbf{i} = -\mathbf{k} & \mathbf{i}\times\mathbf{i} = \mathbf{0} \\ \mathbf{j}\times\mathbf{k} = \mathbf{i} & \mathbf{k}\times\mathbf{j} = -\mathbf{i} & \mathbf{j}\times\mathbf{j} = \mathbf{0} \\ \mathbf{k}\times\mathbf{i} = \mathbf{j} & \mathbf{i}\times\mathbf{k} = -\mathbf{j} & \mathbf{k}\times\mathbf{k} = \mathbf{0} \end{array}$$

$\mathbf{v}\times\mathbf{w} = -\mathbf{w}\times\mathbf{v}$
$(c\mathbf{v})\times\mathbf{w} = c(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\times(c\mathbf{w})$
$\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$
$(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}$
$(\mathbf{t}+\mathbf{u})\times(\mathbf{v}+\mathbf{w}) = \mathbf{t}\times\mathbf{v}+\mathbf{t}\times\mathbf{w}+\mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$
$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$
$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$
$|\mathbf{v}\times\mathbf{w}|$ = area of parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$
$|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$ = volume of parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$

§12.5. Equations of Lines and Planes.

Given a point $P_0 = (x_0, y_0, z_0)$ and a vector $\mathbf{v} = \langle a,b,c\rangle$, the vector equation for the line passing through P_0 in the direction of $\mathbf{v}$ is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r} = \langle x,y,z\rangle$ and $\mathbf{r}_0 = \langle x_0,y_0,z_0\rangle$. The parametric equations for this line are

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

Given a point $P_0 = (x_0, y_0, z_0)$ and a vector $\mathbf{n} = \langle a,b,c\rangle$, the equation for the plane passing through P_0 perpendicular to $\mathbf{n}$ is

$$\begin{aligned} \langle a, b, c\rangle \cdot (\mathbf{r} - \mathbf{r}_0) &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\ az + by + cz &= d \end{aligned}$$

where $d = az_0 + by_0 + cz_0$. The vector $\mathbf{n}$ is the normal vector for the plane.

§12.6. Cylinders and Quadric Surfaces.

A cylinder is a surface consisting of parallel lines. If you have an equation that is missing one of the variables, then the graph is a cylinder. For instance, if the equation has no y's, first graph the curve in the xz-plane, and then stretch the curve in the y-direction.

A quadric surface is a graph of an equation of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

they come in six flavors: see p. 836 of the text for sketches. To graph them in general, look at the traces gotten by setting $x = a$ or $y = b$ or $z = c$. Then reassemble the surface from the traces.

§12.7. Cylindrical and Spherical Coordinates.

The cylindrical coordinates of a point (x, y, z) are (r, θ, z) where (r, θ) are the polar coordinates of the point (x, y).

$$\begin{aligned} x &= r\cos\theta \qquad & r^2 &= x^2 + y^2 \\ y &= r\sin\theta \qquad & \tan\theta &= y/x \\ z &= z \qquad & z &= z \end{aligned}$$

The spherical coordinates of a point $P = (x, y, z)$ are (ρ, θ, ϕ) where:
$\rho = |\overrightarrow{0, P}| = \sqrt{x^2 + y^2 + z^2}$
θ is the same angle as in polar coordinates: $\tan\theta = y/x$.
ϕ is the angle between $\mathbf{k}$ and $\overrightarrow{0P}$, so $0 \le \phi \le \pi$.

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

§13.1. Vector Functions and Space Curves.

A vector function has the form $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ where $f(t)$, $g(t)$ and $h(t)$ are real-valued functions.

The limit of $\mathbf{r}(t)$ as t approaches a is

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided each of the component limits exists.

$\mathbf{r}(t)$ is continuous at a if $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$, that is, if $\mathbf{r}(a)$ and $\lim_{t \to a} \mathbf{r}(t)$ both exist and have equal values. This is equivalent to the component functions $f(t)$, $g(t)$ and $h(t)$ being continuous at a.

§13.2. Derivatives and Integrals of Vector Functions

Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$.

The derivative of $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \frac{d}{dt}[\mathbf{r}(t)] = \frac{d\mathbf{r}}{dt} = \lim_{h \to 0} \frac{1}{h}[\mathbf{r}(t+h) - \mathbf{r}(t)]$$

provided the limit exists. This is the same as

$$\mathbf{r}'(t) = \left\langle f'(t), g'(t), h'(t) \right\rangle.$$

Assume $\mathbf{r}'(t) \neq \mathbf{0}$. The tangent vector to the curve defined by $\mathbf{r}$ at the point $P(f(t), g(t), h(t))$ is $\mathbf{r}'(t)$. The tangent line to the curve at P is the line through P with direction vector $\mathbf{r}'(t)$. The unit tangent vector to the curve at P is $\mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t)$.

The second derivative of is $\mathbf{r}''(t) = \frac{d}{dt}[\mathbf{r}'(t)]$.

The curve defined by $\mathbf{r}(t)$ on an interval I is smooth if $\mathbf{r}'$ is continuous on I and $\mathbf{r}'(t) \neq \mathbf{0}$ except possibly at any endpoints of I.

Differentiation rules:
$\frac{d}{dt}[\mathbf{r}(t) + \mathbf{s}(t)] = \mathbf{r}'(t) + \mathbf{s}'(t)$
$\frac{d}{dt}[\mathbf{r}(t) - \mathbf{s}(t)] = \mathbf{r}'(t) - \mathbf{s}'(t)$
$\frac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$
$\frac{d}{dt}[F(t)\mathbf{r}(t)] = F'(t)\mathbf{r}(t) + F(t)\mathbf{r}'(t)$
$\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{s}(t)] = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$
$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{s}(t)] = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t)$
$\frac{d}{dt}[\mathbf{r}(F(t))] = F'(t)\mathbf{r}'(F(t))$

The antiderivative of $\mathbf{r}(t)$ is

$$\mathbf{R}(t) = \int \mathbf{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle + \mathbf{C}$$

where $\mathbf{C} = \langle c_1, c_2, c_3 \rangle$ is a vector of constants. That is, $\mathbf{R}(t)$ is a vector function such that $\mathbf{R}'(t) = \mathbf{r}(t)$.

The definite integral of $\mathbf{r}(t)$ from a to b is

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle.$$

Be sure to review the sections of the text (especially the examples), your notes, your homework, and your quizzes.

Practice Exercises:
pp. 845–846: 1–7, 9–11, 13, 15–21, 23, 24(a), 26–34, 36–46
p. 882: 1–3, 5, 6(a,b), 9, 11(a)