MATH 720, Algebra I
Exercises 15
Due Fri 07 Dec

Exercise 1. Let R be a commutative ring with identity.

(a) Show that the set of nilpotent elements of R is an ideal.
(b) Show that, if $r \in R$ is nilpotent and $u \in R$ is a unit, then $u + r$ is a unit.

Exercise 2. Let R be a commutative ring with identity, and let $f = a_0 + a_1x + \cdots + a_dx^nd \in R[x]$.

(a) Show that f is nilpotent if and only if each a_i is nilpotent.
(b) Show that f is a unit in $R[x]$ if and only if a_0 is a unit and $a_1, \ldots, a_d$ are nilpotent. In particular, if k is a field, then the units of $k[x]$ are precisely the nonzero constant polynomials.
(c) Show that f is a zerodivisor in $R[x]$ if and only if there exists an element $0 \neq r \in R$ such that ra_i=0 for each i.
(d) State and prove the versions of parts (a)–(c) for polynomials in n variables.

Exercise 3. Let R be a UFD and $0 \neq f = a_0 + a_1x + \cdots + a_dx^d \in R[x]$.

(a) Show that $C(tf) = [t]C(f)$ for each $t \in R$.
(b) Show that, if $C(f) = [r]$, then there is a primitive polynomial g such that $f = rg$.