MATH 720, Algebra I
Exercises 14
Due Fri 30 Nov

Exercise 1. Find an example of a commutative ring R with identity and two polynomials $f, g \in R[x]$ such that $fg \neq 0$ and $\deg(fg) \neq \deg(f) + \deg(g)$.

Exercise 2. Let R be a commutative ring with identity.

(a) Fix an element $r \in R$ and let $\phi\colon R[x] \to R$ be the evaluation homomorphism $\phi(f) = f(r)$. Show that $\operatorname{Ker}(\phi) = (x - r)$.

(b) Let $n \geqslant 1$ and let $r_1, \ldots, r_n \in R$. Let $\phi\colon R[x_1, \ldots, x_n] \to R$ be the evaluation homomorphism $\phi(f) = f(r_1, \ldots, r_n)$. Show that $\operatorname{Ker}(\phi) = (x_1 - r_1, \ldots, x_n - r_n)$.

Exercise 3. Let n be a positive integer.

(a) Show that, if k is a field, then the polynomial ring $k[x]$ is an ED (and therefore is a PID and a UFD).

(b) Show that $\mathbb{Z}[x_1, \ldots, x_n]$ is not a PID (and therefore is not an ED).

(c) Show that, if k is a field, then $k[x_1, \ldots, x_n]$ is not a PID (and therefore is not an ED).