MATH 720, Algebra I
Exercises 12 and 13
Due Wed 21 Nov

Exercise 1. Let R be a nonzero commutative ring with identity, and let $a, b \in R$. Show that the following conditions are equivalent:

(i) $a \mid b$;
(ii) $b \in (a)$;
(iii) $(b) \subseteq (a)$.

Exercise 2. Let R be a nonzero commutative ring with identity, and let $a, b \in R$. Consider the following conditions:

(i) $a \mid b$ and $b \mid a$;
(ii) $b \in (a)$ and $a \in (b)$;
(iii) $(b) = (a)$;
(iv) There is a unit $u \in R$ such that $b = ua$.

Show the following:

(a) (i) $\iff$ (ii) $\iff$ (iii) $\Longleftarrow$ (iv).
(b) If R is an integral domain, then (iii) $\Longrightarrow$ (iv).
(c) Let $R = \mathbb{Z} \times \mathbb{Z}$. Show that $((1,1)) = R = ((1,-1))$, but $(1,1)$ is not a unit multiple of $(1,-1)$. Hence, the implication (iii) $\Longrightarrow$ (iv) can fail when R is not an integral domain.

Exercise 3. Let R be a nonzero commutative ring with identity, and let $a, b \in R$.

(a) If R is an integral domain and $(a) = (ab)$, then b is a unit.
(b) Show that the conclusion of part (b) can fail if R is not an integral domain.

Exercise 4. Let R be a nonzero commutative ring with identity. Let $a, u \in R$ and assume that u is a unit in R.

(a) Show that a is prime if and only if ua is prime.
(b) Show that a is irreducible if and only if ua is irreducible.

Exercise 5. Consider the set $\mathbb{Z}[\sqrt{10}] = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{R}$.

(a) Show that $\mathbb{Z}[\sqrt{10}]$ is an integral domain.
(b) Show that every element of $\mathbb{Z}[\sqrt{10}]$ can be factored into a product of irreducible elements, but that this fatorization need not be unique up to order and multiplication by units. (See Hungerford p. 140.)

Exercise 6. Show that the set $\mathbb{Z}[i]$ of Gaussian integers is a Euclidean domain. (See Hungerford pages 139 and 141. Do not forget to verify that $\mathbb{Z}[i]$ is an integral domain.)