MATH 720, Algebra I
Exercises 9
Due Fri 26 Oct

Exercise 1. Let $d_1, \ldots, d_k \in \mathbb{Z}$, and let $\mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathbb{Z}^n$ be a basis. Show that, if $n \geqslant k$, then

$$\mathbb{Z}^n / \langle d_1 \mathbf{f}_1, \ldots, d_k \mathbf{f}_k \rangle \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k\mathbb{Z} \oplus \mathbb{Z}^{n-k}.$$

Exercise 2. Let G be an abelian group, and prove the following:

(a) If G is free, then G is torsion-free.
(b) If G is finitely generated and torsion-free, then G is free.
(c) The additive abelian group $\mathbb{Q}$ is torsion-free and not free.[1]

Exercise 3. Let G be an abelian group, and prove the following:

(a) If G is finite, then G is torsion.
(b) If G is finitely generated and torsion, then G is finite.
(c) The additive abelian group $\mathbb{Q}/\mathbb{Z}$ is torsion and not finitely generated. In particular, $\mathbb{Q}/\mathbb{Z}$ is not finite.[1]

Exercise 4. Let G be an abelian group, and set

$$G_{\text{tor}} = \{g \in G \mid ng = 0 \text{ for some } n \in \mathbb{N}\}.$$

(a) Show that, if G is finitely generated, then G_{tor} is finitely generated.
(b) Give an example of an abelian group G such that G_{tor} is not finitely generated.

[1] This shows that the condition "finitely generated" in part (b) is necessary.