Semidualizing Modules for Rings of Codimension Two

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Dualizing and Semidualizing Modules

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A finitely generated R-module C is semidualizing if

(1) the natural map $\chi_C^R \colon R \to \operatorname{Hom}_R(C, C)$ is bijective; and
(2) $\operatorname{Ext}_R^i(C, C) = 0$ for all $i \geq 1$.

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(i) R is Gorenstein;
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(iv) R has a unique semidualizing module.

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Focus. Cohen-Macaulay local rings of codimension 2.

Notes and Open Questions

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4. If $P \in \operatorname{Ass}(\mathrm{R})$, then $\operatorname{length}_{R_P}(C_P) = \operatorname{length}_{R_P}(R_P)$?

A Criterion for Rings of Codimension 2

Lemma. (SMC-SSW, 2009) *Let R be a Cohen-Macaulay local ring of codimension* 2*. Let C be a semidualizing R-module such that* $\beta_0^R(C) \leq \beta_1^R(C)$*. Then C is dualizing for R.*

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This is an egregious contradiction. □

Length and Betti Numbers

Lemma. (SMC-SSW, 2009) *Let R be a Cohen-Macaulay local ring. Let C be an R-module with* $\mathrm{pd}_R(C) = \infty$. *Assume that for every $P \in \mathrm{Ass}(R)$ one has* $\mathrm{length}_{R_P}(C_P) = \mathrm{length}_{R_P}(R_P)$. *Then* $\beta_0^R(C) \leq \beta_1^R(C)$.

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Proof. Count lengths in an exact sequence

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Thus, K is free, contradicting the assumption $\mathrm{pd}_R(C) = \infty$. $\square$

Proposition. (SMC-SSW, 2009) *Let R be a Cohen-Macaulay local ring of codimension 2 where* $\operatorname{length}_{R_P}(C_P) = \operatorname{length}_{R_P}(R_P)$ *for each* $P \in \operatorname{Ass}(R)$ *and each semidualizing R-module C. The only two semidualizing R-modules (up to isomorphism) are R and D.*

Proof. Combine the previous two lemmas. □

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Question. If R is a Cohen-Macaulay local ring of codimension 2, are the only two semidualizing modules (up to isomorphism) R and D?

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Proof. In each case, prove that $\mathrm{length}_{R_P}(C_P) = \mathrm{length}_{R_P}(R_P)$ for every $P \in \mathrm{Ass}(R)$ and every semidualizing R-module C.

This is straightforward in cases (1) and (2).

Special Cases

Theorem. (SMC-SSW, 2009) *Let R be a Cohen-Macaulay local ring of codimension 2. Assume that one of the following holds:*

(1) *R is generically Gorenstein, e.g., R is reduced;*

(2) *$P^2 R_P = 0$ for each $P \in \operatorname{Ass}(R)$;*

(3) *$\widehat{R} \cong S/IS$ where $S = k[\![X_0, X_1, X_2]\!]$ and $I \subset k[X_0, X_1, X_2]$ is the ideal of vanishing for a fat point scheme in $\mathbb{P}^2_k$; or*

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In case (4), use polarization to deform to the reduced case. □