Intersections of Symbolic Powers of Prime Ideals

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Let k be an algebraically closed field of characteristic 0, and let Y and Z be closed subvarieties of $\mathbb{A}_k^d$. Assume that Y and Z intersect at a finite number of points, including the origin, and that $\dim(XY) + \dim(Z) = d$. Let f be a nonzero regular function on $\mathbb{A}_k^n$ that vanishes on $Y \cup Z$, and let m and n denote the orders of vanishing of f along Y and Z, respectively.

Question 1. How is the order of vanishing of f at the origin related to m and n?

Answer 1. For trivial reasons f vanishes at least to order $\max\{m, n\}$ at the origin.

Write $Y = V(\mathbf{p})$ and $Z = V(\mathbf{q})$ for prime ideals $\mathbf{p}$ and $\mathbf{q}$ of $S = k[X_1, \ldots, X_d]$. Zariski's Fundamental Lemma on the Vanishing of Holomorphic Functions implies that f is in the intersection of symbolic powers $\mathbf{p}^{(m)} \cap \mathbf{q}^{(n)}$.

Since $Y \cap Z$ contains the origin, f is in the maximal ideal $\mathbf{m} = (X_1, \ldots, X_d)S$, and we can restate our question in ideal-theoretic terms.

Question 2. What is the largest power of the maximal ideal $\mathfrak{m}$ containing (in general) the ideal $\mathbf{p}^{(m)} \cap \mathbf{q}^{(n)}$ or (in particular) the function f?

For our second answer, we appeal to a theorem of Kurano and Roberts.

Theorem. Let $(R, \mathfrak{m})$ be a regular local ring containing a field with prime ideal $\mathfrak{p}$ and $\mathfrak{q}$ such that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$. Then $\mathfrak{p}^{(m)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{m+1}$ for all $m \geq 1$.

This is a simplified version of Kurano and Roberts' theorem. The complete statement also relates the positivity of intersection multiplicities over ramified regular local rings to the containment $\mathfrak{p}^{(m)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{m+1}$. They conjectured that this containment holds over an arbitrary regular local ring under the assumptions that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$.

By applying this theorem to the ring $R = k[X_1, \ldots, X_d]_{\mathbf{m}}$ and contracting back to $k[X_1, \ldots, X_d]$, we have the following.

Answer 2. The function f is in the ideal $\mathbf{m}^{(\max\{m,n\}+1)} = \mathbf{m}^{\max\{m,n\}+1}$. In other words, f vanishes at least to order $\max\{m, n\} + 1$ at the origin.

To motivate our final answer, we consider the following example.

Example. In the ring $S = k[X_1, \ldots, X_d]$, let $\mathbf{p} = (X_1, \ldots, X_i)S$ and $\mathbf{q} = (X_{i+1}, \ldots, X_d)S$. Then

$$\mathbf{p}^{(m)} \cap \mathbf{q}^{(n)} = \mathbf{p}^m \cap \mathbf{q}^n = \mathbf{p}^m\mathbf{q}^n \subseteq \mathbf{m}^{m+n}.$$

In other words, if the subvarieties $Y = V(\mathbf{p})$ and $Z = V(\mathbf{q})$ are regular and intersect transversely, then f vanishes at least to order $m+n$ at the origin. It is easy to see that this bound is sharp in this example.

Question 3. In general, does the function f vanish at least to order $m+n$ at the origin? In other words, is the intersection $\mathbf{p}^{(m)} \cap \mathbf{q}^{(n)}$ contained in the ideal $\mathbf{m}^{m+n}$?

Answer 3. Yes.

Main Theorem. Assume that $(R, \mathfrak{m})$ is a regular local ring containing a field and that $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of R such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$. Then $\mathfrak{p}^{(m)} \cap \mathfrak{q}^{(n)} \subseteq \mathfrak{m}^{m+n}$ for all $m, n \geq 1$.

Sketch of Sketch of Proof of Main Theorem: Choose an element $f \in \mathfrak{p}^{(m)} \cap \mathfrak{q}^{(n)}$ and suppose that $f \notin \mathfrak{m}^{m+n}$. Assume without loss of generality that R is complete with infinite residue field and that $f \notin \mathfrak{p}^{(m+1)} \cup \mathfrak{q}^{(n+1)}$.

Let $\overline{R} = R/fR$, $\overline{\mathfrak{p}} = \mathfrak{p}/fR$ and $\overline{\mathfrak{q}} = \mathfrak{q}/fR$. Our assumptions imply that

$$\dim(\overline{R}/\overline{\mathfrak{p}}) + \dim(\overline{R}/\overline{\mathfrak{q}}) = \dim(R) = \dim(\overline{R}) + 1.$$

Furthermore, there is a complete regular local ring $(A, \mathfrak{m}_A)$ contained in $\overline{R}$ such that:

1. The ring $\overline{R}$ is a finitely-generated, torsion-free A-module.

2. The rank of $\overline{R}$ as an A-module equals the multiplicity $e(\overline{R})$. By assumption, $e(\overline{R}) < m + n$.

It follows (here's where the work takes place) that the contractions $P = \overline{\mathfrak{p}} \cap A$ and $Q = \overline{\mathfrak{q}} \cap A$

satisfy the property $\sqrt{P+Q} = \mathfrak{m}_A$. A theorem of Serre now implies that

$$\begin{aligned}
\dim(\overline{R}) + 1 &= \dim(\overline{R}/\overline{\mathfrak{p}}) + \dim(\overline{R}/\overline{\mathfrak{q}}) \\
&= \dim(A/P) + \dim(A/Q) \\
&\leq \dim(A) \\
&= \dim(\overline{R}).
\end{aligned}$$

This is a contradiction.