Semidualizing Modules

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ABSTRACT. Here is a survey of some aspects of semidualizing modules, theory and applications.

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CHAPTER 1

Prelude

We begin by surveying some of the "classical" aspects of homological commutative algebra, which will motivate the definition of semidualizing modules. We will focus in this section on finitely generated modules, although there are versions of these theories for non-finitely generated modules (and for chain complexes), in an attempt to keep things accessible. Note also that this section does not adhere to the original chronology of the research.

1.1. Projective Dimension

Let M be a finitely generated R-module. In a sense, the nicest R-modules are the free modules and, more generally, the projective modules. Most modules are not projective. (For instance, when R is a local ring, every R-module is projective if and only if R is a field.) However, there is a process by which one can "approximate" M by projective R-modules.

Specifically, there is a finitely generated projective R-module P_0 equipped with a surjection $\tau\colon P_0 \to M$. If M is not projective, then $M_1 = \operatorname{Ker}(\tau) \neq 0$; this "syzygy module" can be thought of as the error from the approximation of M by P_0. The module M_1 may or may not be projective, but we can approximate it by a projective R-module as we did with M.

Indeed, since R is noetherian and P_0 is finitely generated, the submodule $M_1 \subseteq P_0$ is also finitely generated. Repeat the above procedure inductively to find surjections $\tau_{i+1}\colon P_{i+1} \to M_{i+1}$ for each $i \geqslant 0$ where P_{i+1} is projective and $M_{i+1} = \operatorname{Ker}(\tau_i) \subseteq P_i$. Composing the surjections τ_{i+1} with the inclusions $M_{i+1} \subseteq P_i$, we obtain the following exact sequence

$$P^+ = \cdots \xrightarrow{\partial^P_{i+1}} P_i \xrightarrow{\partial^P_i} P_{i-1} \xrightarrow{\partial^P_{i-1}} \cdots P_1 \xrightarrow{\partial^P_1} P_0 \xrightarrow{\tau} M \to 0$$

which we call an *augmented projective resolution* of M. The *projective resolution* of M associated to P^+ is the sequence obtained by truncating:

$$P = \cdots \xrightarrow{\partial^P_{i+1}} P_i \xrightarrow{\partial^P_i} P_{i-1} \xrightarrow{\partial^P_{i-1}} \cdots P_1 \xrightarrow{\partial^P_1} P_0 \to 0.$$

Note that P is not in general exact. Indeed, one has $\operatorname{Ker}(\partial^P_i) = \operatorname{Im}(\partial^P_{i+1})$ for each $i \geqslant 1$, but $\operatorname{Coker}(\partial^P_1) \cong M$, and so P is exact if and only if $M = 0$. (One might say that P is "acyclic", but we will not use this term because it means different things to different people.) We say that P is a *free resolution* of M when each P_i is free. Note that, when R is local, an R-module is free if and only if it is projective, and so the notions of projective resolution and free resolution are the same in this setting.

If M admits a projective resolution P such that $P_i = 0$ for $i \gg 0$, then we say that M has finite projective dimension. More specifically, the *projective dimension* of M is the shortest such resolution:

$$\mathrm{pd}_R(M) = \inf\{\sup\{n \geqslant 0 \mid P_n \neq 0\} \mid P \text{ is a projective resolution of } M\}.$$

Modules with finite projective dimension are quite special, as we will see below. One need not look far to find modules of finite projective dimension: Hilbert's famous Syzygy Theorem [**14**] says that, when k is a field, every finitely generated module over the polynomial ring $k[X_1, \dots, X_d]$ has projective dimension at most d. In the local setting, this is a sort of precursor to the famous theorem of Auslander, Buchsbaum [**3**] and Serre [**20**]:

Theorem 1.1.1. *Let $(R, \mathfrak{m}, k)$ be a local ring of Krull dimension d. The following conditions are equivalent:*

(i) *R is regular, that is, the maximal ideal $\mathfrak{m}$ can be generated by d elements;*
(ii) *$\mathrm{pd}_R(M) < \infty$ for each finitely generated R-module; and*
(iii) *$\mathrm{pd}_R(k) < \infty$.*

One important application of this result is the solution of the localization problem for regular local rings: If R is a regular local ring and $\mathfrak{p} \subset R$ is a prime ideal, then the localization $R_{\mathfrak{p}}$ is also regular.

Theorem 1.1.1 substantiates the following maxim: to understand a ring is to understand its modules. If you like, the nicer the ring, the nicer its modules, and conversely. We shall see this maxim in action in numerous places below. One could say, as I often do, that module theory is representation theory for rings, with the modules taking the place of representations. This is backwards, though, since representation theory is, in fact, nothing other than the module theory of group rings.

Another feature of the projective dimension is the "Auslander-Buchsbaum formula" [**3**]:

Theorem 1.1.2. *Let $(R, \mathfrak{m}, k)$ be a local ring. If M is an R-module of finite projective dimension, then $\mathrm{pd}_R(M) = \mathrm{depth}(R) - \mathrm{depth}_R(M)$; in particular, if $M \neq 0$, then $\mathrm{depth}_R(M) \leqslant \mathrm{depth}(R)$.*

Here, the "depth" of M is the length of the longest M-regular sequence in $\mathfrak{m}$; this can be expressed homologically as

$$\mathrm{depth}_R(M) = \inf\{i \geqslant 0 \mid \mathrm{Ext}^i_R(k, M) \neq 0\}.$$

And $\mathrm{depth}(R) = \mathrm{depth}_R(R)$.

Note that this result shows how to find modules of infinite projective dimension; just find a module $M \neq 0$ with $\mathrm{depth}_R(M) > \mathrm{depth}(R)$. For instance, when k is a field, the ring $R = k[\![X, Y]\!]/(X^2, XY)$ has depth 0 and the module $M = R/XR \cong k[\![X]\!]$ has depth 1.

1.2. Complete Intersection Dimension

The class of regular local rings is not stable under specialization: if $(R, \mathfrak{m})$ is a regular local ring and $x \in \mathfrak{m}$ is an R-regular element, then R/xR may not be a regular local ring. This corresponds to the geometric fact that a hypersurface in a smooth variety need not be smooth. In a sense, this is unfortunate. However, it leads to our next class of rings.

Definition 1.2.1. A local ring $(R, \mathfrak{m})$ is a *complete intersection* if its $\mathfrak{m}$-adic completion $\widehat{R}$ has the form $\widehat{R} \cong Q/(\mathbf{x})Q$ where Q is a regular local ring and $\mathbf{x}$ is a Q-regular sequence.

Recall that Cohen's Structure Theorem [**10**] guarantees that the completion of any local ring is a homomorphic image of a regular local ring. Since the completion of a regular local ring is regular, it follows that every regular local ring is a complete intersection. It is straightforward to show that the class of complete intersection rings is closed under specialization. Furthermore, this definition of complete intersection is independent of the choice of regular local ring surjecting onto $\widehat{R}$: a theorem of Grothendieck [**13**, (19.3.2)] says that, if R is a complete intersection and $\pi\colon A \to \widehat{R}$ is a ring epimorphism where A is a regular local ring, then $\operatorname{Ker}(\pi)$ is generated by an A-regular sequence.

Avramov, Gasharov and Peeva [**6**] introduced the *complete intersection dimension* of a finitely generated R-module M, in part, to find and study modules whose free resolutions do not grow too quickly. For the sake of simplicity, we only discuss this invariant when R is local. Recall that a ring homomorphism of local rings $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is *local* when $\mathfrak{m}S \subseteq \mathfrak{n}$.

Definition 1.2.2. Let $(R, \mathfrak{m})$ be a local ring. A *quasi-deformation* of R is a diagram of local ring homomorphisms

$$R \xrightarrow{\rho} R' \xleftarrow{\tau} Q$$

where ρ is flat and τ is surjective with kernel generated by a Q-regular sequence.

A finitely generated R-module M has *finite complete intersection dimension* when there exists a quasi-deformation $R \to R' \leftarrow Q$ such that $\operatorname{pd}_Q(R' \otimes_R M)$ is finite; specifically, we have

$$\text{CI-dim}_R(M) = \inf\{\operatorname{pd}_Q(R' \otimes_R M) - \operatorname{pd}_Q(R') \mid R \to R' \leftarrow Q \text{ quasi-deformation}\}.$$

When R is a local complete intersection, it follows readily from Theorem 1.1.1 that every R-module has finite complete intersection dimension: write $\widehat{R} \cong Q/(\mathbf{x})Q$ where Q is a regular local ring and $\mathbf{x}$ is a Q-regular sequence and use the quasi-deformation $R \to \widehat{R} \leftarrow Q$. Moreover, Avramov, Gasharov and Peeva [**6**] show that the complete intersection dimension satisfies properties like those in Theorems 1.1.1 and 1.1.2:

Theorem 1.2.3. *Let $(R, \mathfrak{m}, k)$ be a local ring. The following conditions are equivalent:*

(i) *R is a complete intersection;*
(ii) *$\text{CI-dim}_R(M) < \infty$ for each finitely generated R-module; and*
(iii) *$\text{CI-dim}_R(k) < \infty$.*

Theorem 1.2.4 (AB-formula)**.** *Let R be a local ring and M a finitely generated R-module. If $R \to R' \leftarrow Q$ is a quasi-deformation such that $\operatorname{pd}_Q(R' \otimes_R M) < \infty$, then $\operatorname{pd}_Q(R' \otimes_R M) - \operatorname{pd}_Q(R') = \operatorname{depth}(R) - \operatorname{depth}_R(M)$. If $\text{CI-dim}_R(M) < \infty$, then $\text{CI-dim}_R(M) = \operatorname{depth}(R) - \operatorname{depth}_R(M)$; in particular, if $M \neq 0$, then $\operatorname{depth}_R(M) \leqslant \operatorname{depth}(R)$.*

The "AB" in the AB-formula stands for Auslander-Buchsbaum, naturally, and also Auslander-Bridger, as we shall see below. As a consequence of the AB-formula, we see that the complete intersection dimension is a refinement of the projective dimension.

Corollary 1.2.5. *Let R be a local ring and M a finitely generated R-module. There is an inequality* $\text{CI-dim}_R(M) \leqslant \text{pd}_R(M)$*, with equality when* $\text{pd}_R(M) < \infty$*.*

PROOF. Assume without loss of generality that $\text{pd}_R(M) < \infty$. Using the trivial quasi-deformation $R \to R \leftarrow R$, it is straightforward to see that M has finite complete intersection dimension. Theorems 1.1.2 and 1.2.4 show that $\text{CI-dim}_R(M) = \text{pd}_R(M)$, as desired. □

Using the work of Cohen [**10**] and Grothendieck [**12**], Avramov, Gasharov and Peeva [**6**] show one can exert a certain amount of control on the structure of quasi-deformations:

Proposition 1.2.6. *Let $(R, \mathfrak{m})$ be a local ring and M an R-module of finite complete intersection dimension. Then there exists a quasi-deformation $R \xrightarrow{\rho} R' \leftarrow Q$ such that $\text{pd}_Q(R' \otimes_R M) < \infty$ and such that Q is complete with algebraically closed residue field and such that the closed fibre $R'/\mathfrak{m}R'$ is artinian (hence, Cohen-Macaulay).*

We shall see in a theorem below how semidualizing modules allow us to improve Proposition 1.2.6.

Here is an open question that I would very much like to answer. Note that the corresponding result for modules of finite projective dimension is well-known.

Question 1.2.7. Let R be a local ring and consider an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of finitely generated R-modules. If two of the modules M_i have finite complete intersection dimension, must the third one also?

If one of the M_i has finite projective dimension, then Question 1.2.7 is readily answered in the affirmative. In particular, every module of finite complete intersection dimension has a bounded resolution by modules of complete intersection dimension zero, namely, an appropriate truncation of a projective resolution. On the other hand, it is not known whether M must have finite complete intersection dimension if it has a bounded resolution by modules of complete intersection dimension zero. Indeed, this is equivalent to one of the implications in Question 1.2.7.

1.3. G-Dimension

It is well known that R is always projective as an R-module. It is natural to ask whether it is always self-injective, i.e., injective as an R-module. The answer is "no" in general because, for instance, if a local ring R has a finitely generated injective module, then R must be artinian. One can hope to remedy this by asking whether R has finite injective dimension as an R-module, that is, when does there exist an exact sequence

$$0 \to R \to I_0 \to I_1 \to \cdots \to I_d \to 0$$

where each I_j is an injective R-module? Once again, the answer is "no" because, for instance, if a local ring R has a finitely generated module of finite injective dimension, then R must be Cohen-Macaulay.

Definition 1.3.1. A local ring R is *Gorenstein* if it has finite injective dimension as an R-module.

These rings are named after the famous group theorist Daniel Gorenstein.

It can be shown, using techniques of Auslander, Buchsbaum [**3**] and Serre [**20**], that every finitely generated module over a regular local ring has finite injective dimension; hence, every regular local ring is Gorenstein. Furthermore, the class of Gorenstein rings is stable under specialization, so every complete intersection is also Gorenstein. Thus we have the implications

$$\text{regular} \implies \text{complete intersection} \implies \text{Gorenstein} \implies \text{Cohen-Macaulay}.$$

Auslander and Bridger [**1, 2**] introduced the *G-dimension* of a finitely generated module, in part, to give a module-theoretic characterization of Gorenstein rings like that in Theorem 1.1.1. Like the projective dimension, it is defined in terms of resolutions by certain modules with good homological properties, the “totally reflexive” modules. To define these modules, we need the natural biduality map.

Definition 1.3.2. Let N be an R-module. The natural *biduality map*

$$\delta_N^R \colon N \to \operatorname{Hom}_R(\operatorname{Hom}_R(N,R),R)$$

is the R-module homomorphism given by $\delta_N^R(n)(\psi) = \psi(n)$, in other words, for each $n \in N$ the map $\delta_N^R(n) \colon \operatorname{Hom}_R(N,R) \to R$ is given by $\psi \mapsto \psi(n)$.

As the name suggests, a totally reflexive module is a reflexive module with some additional properties. The addition properties have to do with the vanishing of the Ext-modules that arise from the biduality map.

Definition 1.3.3. An R-module G is *totally reflexive* if it satisfies the following conditions:

(1) G is finitely generated over R;
(2) the biduality map $\delta_G^R \colon G \to \operatorname{Hom}_R(\operatorname{Hom}_R(G,R),R)$ is an isomorphism; and
(3) $\operatorname{Ext}_R^i(G,R) = 0 = \operatorname{Ext}_R^i(\operatorname{Hom}_R(G,R),R)$ for all $i \geqslant 1$.

We let $\mathcal{G}(R)$ denote the class of all totally reflexive R-modules.

Example 1.3.4. Every finitely generated projective R-module is totally reflexive; see Proposition 2.1.13.

Definition 1.3.5. Let M be a finitely generated R-module. An *augmented G-resolution* of M is an exact sequence

$$G^+ = \cdots \xrightarrow{\partial_{i+1}^G} G_i \xrightarrow{\partial_i^G} G_{i-1} \xrightarrow{\partial_{i-1}^G} \cdots G_1 \xrightarrow{\partial_1^G} G_0 \xrightarrow{\tau} M \to 0$$

wherein each G_i is totally reflexive. The *G-resolution* of M associated to G^+ is the sequence obtained by truncating:

$$G = \cdots \xrightarrow{\partial_{i+1}^G} G_i \xrightarrow{\partial_i^G} G_{i-1} \xrightarrow{\partial_{i-1}^G} \cdots G_1 \xrightarrow{\partial_1^G} G_0 \to 0$$

Since every finitely generated R-module has a resolution by finitely generated projective R-modules and every finitely generated projective R-module is totally reflexive, it follows that every finitely generated R-module has a G-resolution.

Definition 1.3.6. Let M be a finitely generated R-module. If M admits a G-resolution G such that $G_i = 0$ for $i \gg 0$, then we say that M has *finite G-dimension.* More specifically, the *G-dimension* of M is the shortest such resolution:

$$\text{G-dim}_R(M) = \inf\{\sup\{n \geqslant 0 \mid G_n \neq 0\} \mid G \text{ is a G-resolution of } M\}.$$

Auslander and Bridger [**1, 2**] show that the G-dimension satisfies properties like those in Theorems 1.1.1 and 1.1.2:

Theorem 1.3.7. *Let $(R, \mathfrak{m}, k)$ be a local ring. The following conditions are equivalent:*

(i) *R is Gorenstein;*
(ii) $\text{G-dim}_R(M) < \infty$ *for each finitely generated R-module; and*
(iii) $\text{G-dim}_R(k) < \infty$.

Theorem 1.3.8 (AB-formula)**.** *Let R be a local ring and M a finitely generated R-module. If* $\text{G-dim}_R(M) < \infty$*, then* $\text{G-dim}_R(M) = \text{depth}(R) - \text{depth}_R(M)$*; in particular, if $M \neq 0$, then* $\text{depth}_R(M) \leqslant \text{depth}(R)$.

Corollary 1.3.9. *Let R be a local ring and M a finitely generated R-module. There are inequalities* $\text{G-dim}_R(M) \leqslant \text{CI-dim}_R(M) \leqslant \text{pd}_R(M)$*, with equality to the left of any finite quantity.*

Here is another open question that I would very much like to answer. It is a special case (though, equivalent to the general case) of Avramov and Foxby's Composition Question for ring homomorphisms of finite G-dimension [**5**, (4.8)]. Note that it is straightforward to answer the corresponding result for homomorphisms of finite projective dimension in the affirmative. The analogue for complete intersection dimension is also open.

Question 1.3.10. Let $R \to S \to T$ be surjective local ring homormophisms. If $\text{G-dim}_R(S)$ and $\text{G-dim}_S(T)$ are finite, must $\text{G-dim}_R(T)$ also be finite?

We shall see in a theorem below how semidualizing modules allow us to give a partial answer to Question 1.3.10.

CHAPTER 2

Semidualizing Basics

In this section, we survey the basic properties of semidualizing modules.

2.1. Definitions and Basic Properties

One can modify Definition 1.3.3 to consider dualities with respect to modules other than R. However, not every class of modules which arises in this way is well-suited for building a homological dimension. We shall see next that, in a sense, the best class of modules arise from considering dualities with respect to semidualizing modules.

Definition 2.1.1. Let M and N be R-modules. The natural *biduality map*

$$\delta_N^M : N \to \mathrm{Hom}_R(\mathrm{Hom}_R(N, M), M)$$

is the R-module homomorphism given by $\delta_N^M(n)(\psi) = \psi(n)$, in other words, for each $n \in N$ the map $\delta_N^M(n)\colon \mathrm{Hom}_R(N, M) \to M$ is given by $\psi \mapsto \psi(n)$.

Remark 2.1.2. Let $f\colon M \to M'$ and $g\colon N \to N'$ be R-module homomorphisms. It is straightforward to show that the map δ_N^M from Definition 2.1.1 is a well-defined R-module homomorphism and that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}_R(\mathrm{Hom}_R(N,M'),M') & \xrightarrow{\mathrm{Hom}_R(\mathrm{Hom}_R(N,f),M')} & \mathrm{Hom}_R(\mathrm{Hom}_R(N,M),M') \\
\uparrow{\scriptstyle \delta_N^{M'}} & & \uparrow{\scriptstyle \mathrm{Hom}_R(\mathrm{Hom}_R(N,M),f)} \\
N & \xrightarrow{\delta_N^M} & \mathrm{Hom}_R(\mathrm{Hom}_R(N,M),M) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mathrm{Hom}_R(\mathrm{Hom}_R(g,M),M)} \\
N' & \xrightarrow{\delta_{N'}^M} & \mathrm{Hom}_R(\mathrm{Hom}_R(N',M),M).
\end{array}$$

Golod introduced the following notion, though elements of it can be traced to Foxby and Vasconcelos.

Definition 2.1.3. Let C be a finitely generated R-module. An R-module G is *totally C-reflexive* if it satisfies the following conditions:

(1) G is finitely generated over R;
(2) the biduality map $\delta_G^C\colon G \to \mathrm{Hom}_R(\mathrm{Hom}_R(G, C), C)$ is an isomorphism; and
(3) $\mathrm{Ext}_R^i(G, C) = 0 = \mathrm{Ext}_R^i(\mathrm{Hom}_R(G, C), C)$ for all $i \geqslant 1$.

We let $\mathcal{G}_C(R)$ denote the class of all totally C-reflexive R-modules.

Proposition 2.1.4. *Let C, M and N be R-modules. Then $M \oplus N$ is totally C-reflexive if and only if M and N are both totally C-reflexive.*

Proof. It is straightforward to show that $M \oplus N$ is finitely generated if and only if M and N are both finitely generated. The isomorphism

$$\operatorname{Ext}_R^i(M \oplus N, C) \cong \operatorname{Ext}_R^i(M, C) \oplus \operatorname{Ext}_R^i(N, C)$$

shows that $\operatorname{Ext}_R^i(M \oplus N, C) = 0$ for all $i \geqslant 1$ if and only if $\operatorname{Ext}_R^i(M, C) = 0 = \operatorname{Ext}_R^i(N, C)$ for all $i \geqslant 1$. The isomorphisms

$$\begin{aligned} \operatorname{Ext}_R^i(\operatorname{Hom}_R(M \oplus N, C), C) &\cong \operatorname{Ext}_R^i(\operatorname{Hom}_R(M, C) \oplus \operatorname{Hom}_R(N, C), C) \\ &\cong \operatorname{Ext}_R^i(\operatorname{Hom}_R(M, C), C) \oplus \operatorname{Ext}_R^i(\operatorname{Hom}_R(N, C), C) \end{aligned}$$

show that $\operatorname{Ext}_R^i(\operatorname{Hom}_R(M, C), C) = 0 = \operatorname{Ext}_R^i(\operatorname{Hom}_R(N, C), C)$ for all $i \geqslant 1$ if and only if $\operatorname{Ext}_R^i(\operatorname{Hom}_R(M \oplus N, C), C) = 0$ for all $i \geqslant 1$. Finally, there is a commutative diagram

$$\begin{array}{ccc} M \oplus N & \xrightarrow{\delta^C_{M\oplus N}} & \operatorname{Hom}_R(\operatorname{Hom}_R(M \oplus N, C), C) \\ & \searrow^{\delta^C_M \oplus \delta^C_N} & \downarrow \cong \\ & & \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C) \oplus \operatorname{Hom}_R(\operatorname{Hom}_R(N, C), C) \end{array}$$

so $\delta^C_{M\oplus N}$ is an isomorphism if and only if $\delta^C_M \oplus \delta^C_N$ is an isomorphism, that is, if and only if δ^C_M and δ^C_N are both isomorphisms. The desired result now follows. $\square$

Now we are finally ready to define the main players of this article.

Definition 2.1.5. Let C be an R-module. The natural *homothety map*

$$\chi^R_C \colon R \to \operatorname{Hom}_R(C, C)$$

is the R-module homomorphism given by $\chi^R_C(r)(c) = rc$, that is, for each $r \in R$ the map $\chi^R_C(r) \colon C \to C$ is given by $c \mapsto rc$.

Remark 2.1.6. Let $f \colon C \to C'$ be an R-module homomorphism. It is straightforward to show that the map χ^R_C from Definition 2.1.5 is a well-defined R-module homomorphism and that the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{\chi^R_C} & \operatorname{Hom}_R(C, C) \\ {\scriptstyle \chi^R_{C'}}\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}_R(C,f)} \\ \operatorname{Hom}_R(C', C') & \xrightarrow{\operatorname{Hom}_R(f,C')} & \operatorname{Hom}_R(C, C'). \end{array}$$

Fact 2.1.7. If C is an R-module, then $\operatorname{Ann}_R(C) = \operatorname{Ker}(\chi^R_C)$.

The following notion was introduced, seemingly independently and using different terminology, by Foxby, Golod, Wakamatsu, and Vasconcelos.

Definition 2.1.8. The R-module C is *semidualizing* if it satisfies the following:

(1) C is finitely generated;
(2) the homothety map $\chi^R_C \colon R \to \operatorname{Hom}_R(C, C)$ is an isomorphism; and
(3) $\operatorname{Ext}_R^i(C, C) = 0$ for all $i > 0$.

The set of isomorphism classes of semidualizing R-modules is denoted $\mathfrak{S}_0(R)$, and the isomorphism class of a module C is denoted $[C]$.

An R-module D is *point-wise dualizing* if it is semidualizing and $\mathrm{id}_{R_\mathfrak{m}}(D_\mathfrak{m}) < \infty$ for each maximal ideal $\mathfrak{m}$. The R-module D is *dualizing* if it is semidualizing and has finite injective dimension.

Example 2.1.9. It is straightforward to show that the free R-module R^1 is semidualizing.

Definition 2.1.10. The ring R is *point-wise Gorenstein* if it is locally Gorenstein, that is, if $\mathrm{id}_{R_\mathfrak{m}}(R_\mathfrak{m}) < \infty$ for each maximal ideal $\mathfrak{m}$. The ring R is *Gorenstein* if $\mathrm{id}_R(R) < \infty$.

Example 2.1.11. It is straightforward to show that the free R-module R^1 is (point-wise) dualizing if and only if R is (point-wise) Gorenstein.

If R is Gorenstein, then it is point-wise Gorenstein. If D is dualizing for R, then it is point-wise dualizing. The converses of these statements hold when $\dim(R) < \infty$, and they fail when $\dim(R) = \infty$. Nagata's famous example of a noetherian ring of infinite Krull dimension is point-wise Gorenstein and not Gorenstein.

Here is what we mean when we say that duality with respect to a semidualizing module is, in a sense, best. We shall see in Proposition 2.1.13 below that the conditions in this result are equivalent to *every* finitely generated projective R-module being totally C-reflexive.

Proposition 2.1.12. *Let C be a finitely generated R-module. The following conditions are equivalent:*

(i) *C is a semidualizing R-module;*
(ii) *R is a totally C-reflexive R-module; and*
(iii) *C is a totally C-reflexive R-module and* $\mathrm{Ann}_R(C) = 0$.

Proof. Let $f\colon \mathrm{Hom}_R(R, C) \to C$ be the Hom cancellation isomorphism given by $f(\psi) = \psi(1)$. It is readily shown that the following diagrams commute:

$$\begin{array}{ccc} R & \xrightarrow{\chi_C^R} & \mathrm{Hom}_R(C,C) \\ & \searrow^{\delta_R^C} & \downarrow{\scriptstyle \cong\ \mathrm{Hom}_R(f,C)} \\ & & \mathrm{Hom}_R(\mathrm{Hom}_R(R,C),C) \end{array} \tag{2.1.12.1}$$

$$\begin{array}{ccc} C & \xrightarrow{\delta_C^C} & \mathrm{Hom}_R(\mathrm{Hom}_R(C,C),C) \\ {\scriptstyle \mathrm{id}_C}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_R(\chi_C^R,C)} \\ C & \xleftarrow[\cong]{f} & \mathrm{Hom}_R(R,C). \end{array} \tag{2.1.12.2}$$

(i) $\iff$ (ii). For $i \geqslant 1$, we have $\mathrm{Ext}^i_R(R, C) = 0$ because R is projective, and

$$\mathrm{Ext}^i_R(\mathrm{Hom}_R(R, C), C) \cong \mathrm{Ext}^i_R(C, C).$$

In particular, we have $\mathrm{Ext}^i_R(\mathrm{Hom}_R(R, C), C) = 0$ for all $i \geqslant 1$ if and only if $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geqslant 1$. Furthermore, diagram (2.1.12.1) shows that δ^C_R is an isomorphism if and only if χ^R_C is an isomorphism. Thus C is semidualizing if and only if R is totally C-reflexive.

(i) $\Longrightarrow$ (iii). Assume that C is semidualizing. The isomorphism $\mathrm{Hom}_R(C, C) \cong R$ implies that $\mathrm{Ann}_R(C) \subseteq \mathrm{Ann}_R(R) = 0$.

We next show that C is totally C-reflexive. For $i \geqslant 1$, we have

$$\operatorname{Ext}^i_R(C,C)=0 \qquad \text{and} \qquad \operatorname{Ext}^i_R(\operatorname{Hom}_C(C,C),C)\cong \operatorname{Ext}^i_R(R,C)=0$$

because C is semidualizing. Since χ^R_C is an isomorphism, diagram (2.1.12.2) shows that δ^C_C is an isomorphism, so C is totally C-reflexive.

(iii) $\Longrightarrow$ (i). Assume that C is a totally C-reflexive R-module and $\operatorname{Ann}_R(C)=0$. Note that it follows that $\operatorname{Supp}_R(C)=V(\operatorname{Ann}_R(C))=V(0)=\operatorname{Spec}(R)$. It also follows that $\operatorname{Ext}^i_R(C,C)=0$, so it remains to show that χ^R_C is an isomorphism. We have $0=\operatorname{Ann}_R(C)=\operatorname{Ker}(\chi^R_C)$, so χ^R_C is injective. Set $N=\operatorname{Coker}(\chi^R_C)$ and consider the exact sequence

$$0\to R\xrightarrow{\chi^R_C}\operatorname{Hom}_R(C,C)\to N\to 0.$$

The associated long exact sequence in $\operatorname{Ext}_R(-,C)$ begins as follows

$$0\to\operatorname{Hom}_R(N,C)\to\operatorname{Hom}_R(\operatorname{Hom}_R(C,C),C)\xrightarrow[\cong]{\operatorname{Hom}_R(\chi^R_C,C)}\operatorname{Hom}_R(R,C).$$

The fact that $\operatorname{Hom}_R(\chi^R_C,C)$ is an isomorphism follows from diagram (2.1.12.2) because C is totally C-reflexive. We conclude that $\operatorname{Hom}_R(N,C)=0$. The next piece of the long exact sequence has the following form

$$\operatorname{Hom}_R(\operatorname{Hom}_R(C,C),C)\xrightarrow{\cong}\operatorname{Hom}_R(R,C)\to\operatorname{Ext}^1_R(N,C)\to\underbrace{\operatorname{Ext}^1_R(\operatorname{Hom}_R(C,C),C)}_{=0}$$

so $\operatorname{Ext}^1_R(N,C)=0$. Other pieces of the long exact sequence have the form

$$\underbrace{\operatorname{Ext}^{i-1}_R(R,C)}_{=0}\to\operatorname{Ext}^i_R(N,C)\to\underbrace{\operatorname{Ext}^i_R(\operatorname{Hom}_R(C,C),C)}_{=0}$$

and it follows that $\operatorname{Ext}^i_R(N,C)=0$ for all $i\geqslant 0$.

We will be done once we show that $N=0$, so suppose that $N\neq 0$ and let $\mathfrak{p}\in\operatorname{Supp}_R(N)$. It follows that $\mathfrak{p}\in\operatorname{Spec}(R)=\operatorname{Supp}_R(C)$, so $C_\mathfrak{p}$ and $N_\mathfrak{p}$ are nonzero finitely generated $R_\mathfrak{p}$-modules. From [**16**, (16.6)] it follows that there exists some $i\geqslant 0$ such that

$$0\neq\operatorname{Ext}^i_{R_\mathfrak{p}}(N_\mathfrak{p},C_\mathfrak{p})=\operatorname{Ext}^i_R(N,C)_\mathfrak{p}=0$$

which is clearly a contradiction. □

Proposition 2.1.13. *Let C be a semidualizing R-module, and let P be a finitely generated projective R-module.*

(a) *If G is totally C-reflexive, then so is $G\otimes_R P$.*
(b) *The modules P and $C\otimes_R P$ are totally C-reflexive.*
(c) *For each integer $n\geqslant 0$, the modules R^n and C^n are totally C-reflexive.*

Proof. (a) Since P is a finitely generated projective R-module, there is a second finitely generated projective R-module Q such that $P\oplus Q\cong R^n$ for some integer $n\geqslant 0$. Since G is totally C-reflexive we conclude from Proposition 2.1.4 that $G^n\cong(G\otimes_R P)\oplus(G\otimes_R Q)$ is totally C-reflexive and then that $G\otimes_R P$ and $G\otimes_R Q$ are totally C-reflexive.

(b) Proposition 2.1.12 shows that R and C are totally C-reflexive, so the desired conclusions follow from part (a) using $G=R$ and $G=C$.

(c) This is the special case of part (b) with $P=R^n$. □

The following corollary is a complement to the first statement in Example 2.1.9.

Corollary 2.1.14. *If C is a cyclic semidualizing R-module, then $C \cong R$.*

Proof. Assuming that C is a cyclic semidualizing R-module, the equality in the next sequence is from Proposition 2.1.12

$$C \cong R/\operatorname{Ann}_R(C) = R/(0) \cong R$$

and the isomorphisms are standard. □

The next example shows that when C is totally C-reflexive, one may have $\operatorname{Ann}_R(C) \neq 0$. In other words, the conditions in Proposition 2.1.12 are not equivalent to the condition "C is totally C-reflexive".

Example 2.1.15. Let k be a field, and set $R = k \times k$. The subset $I = 0 \times k \subseteq R$ is an ideal, and we set $C = R/I$. It is straightforward to show that the natural map $\psi\colon \operatorname{Hom}_R(R/I, R/I) \to R/I$ given by $\psi(\alpha) = \alpha(\overline{1})$ is an R-module isomorphism. It is routine to show that R/I is projective as an R-module. This explains the vanishing in the next sequence

$$0 = \operatorname{Ext}^i_R(R/I, R/I) \cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(R/I, R/I), R/I)$$

while the isomorphism is induced by ψ. There is a commutative diagram of R-module homomorphisms

$$\begin{array}{ccc} \operatorname{Hom}_R(R/I,R/I) & \xrightarrow[\cong]{\psi} & R/I \\ {\scriptstyle \operatorname{Hom}_R(\psi,R/I)}\Big\downarrow\cong & \swarrow{\scriptstyle \delta^{R/I}_{R/I}} & \\ \operatorname{Hom}_R(\operatorname{Hom}_R(R/I,R/I),R/I) & & \end{array}$$

and it follows that $\delta^{R/I}_{R/I}$ is an isomorphism. By definition, this implies that $C = R/I$ is C-reflexive. However, we have $\operatorname{Ann}_R(C) = I \neq 0$, so C is not semidualizing.

Proposition 2.1.16. *Let C be a semidualizing R-module.*

(a) *One has* $\operatorname{Supp}_R(C) = \operatorname{Spec}(R)$ *and* $\operatorname{Ass}_R(C) = \operatorname{Ass}_R(R)$.
(b) *One has* $\dim_R(C) = \dim(R)$ *and* $\operatorname{Ann}_R(C) = 0$.
(c) *Given an ideal $I \subseteq R$, one has $IC = C$ if and only if $I = R$.*
(d) *An element $x \in R$ is R-regular if and only if it is C-regular.*

Proof. (a) and (b) The equality $\operatorname{Ann}_R(C) = 0$ is shown in Proposition 2.1.12. This implies $\operatorname{Supp}_R(C) = V(\operatorname{Ann}_R(C)) = V(0) = \operatorname{Spec}(R)$, and the equality $\dim_R(C) = \dim(R)$ follows directly. The isomorphism $\operatorname{Hom}_R(C, C) \cong R$ implies

$$\begin{aligned} \operatorname{Ass}_R(R) &= \operatorname{Ass}_R(\operatorname{Hom}_R(C, C)) = \operatorname{Supp}_R(C) \cap \operatorname{Ass}_R(C) \\ &= \operatorname{Spec}(R) \cap \operatorname{Ass}_R(C) = \operatorname{Ass}_R(C). \end{aligned}$$

(c) One implication is immediate. For the nontrivial implication, assume that $IC = C$. It follows that, for each maximal ideal $\mathfrak{m} \subset R$, we have $I_\mathfrak{m} C_\mathfrak{m} = C_\mathfrak{m}$; since $C_\mathfrak{m} \neq 0$, Nakayama's lemma implies that $I_\mathfrak{m} = R_\mathfrak{m}$ and thus $I \not\subseteq \mathfrak{m}$. Since this is so for each maximal ideal, we conclude that $I = R$.

(d) Assume without loss of generality that x is not a unit in R, i.e., that $xC \neq C$. Then x is a non-zerodivisor on R if and only if $x \notin \cup_{\mathfrak{p} \in \operatorname{Ass}_R(R)} \mathfrak{p} = \cup_{\mathfrak{p} \in \operatorname{Ass}_R(C)} \mathfrak{p}$, that is, if and only if x is a non-zerodivisor on C. □

Corollary 2.1.17. *Let M be a non-zero R-module. If C is a semidualizing R-module, then* $\operatorname{Hom}_R(C, M) \neq 0$ *and* $C \otimes_R M \neq 0$.

PROOF. Proposition 2.1.16(a) and Lemma A.2.1. □

2.2. Base Change

Proposition 2.2.1. *Let $\varphi\colon R \to S$ be a flat ring homomorphism, and let C be a finitely generated R-module. If C is a semidualizing R-module, then $C \otimes_R S$ is a semidualizing S-module. The converse holds when φ is faithfully flat.*

PROOF. The S-module $C \otimes_R S$ is finitely generated because C is a finitely generated R-module. For $i \geqslant 1$, we have

$$\operatorname{Ext}^i_S(C \otimes_R S, C \otimes_R S) \cong \operatorname{Ext}^i_R(C, C) \otimes_R S.$$

If $\operatorname{Ext}^i_R(C, C) = 0$, then this shows that $\operatorname{Ext}^i_S(C \otimes_R S, C \otimes_R S) = 0$. The converse holds when φ is faithfully flat.

Finally, there is a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\cong} & R \otimes_R S \\ {\scriptstyle \chi^S_{C\otimes_R S}} \downarrow & & \downarrow {\scriptstyle \chi^R_C \otimes_R S} \\ \operatorname{Hom}_S(C \otimes_R S, C \otimes_R S) & \xleftarrow[\cong]{} & \operatorname{Hom}_R(C, C) \otimes_R S \end{array}$$

where the unlabeled isomorphisms are the natural ones. If the homothety map χ^R_C is an isomorphism, then so is $\chi^R_C \otimes_R S$, and so the diagram shows that $\chi^S_{C\otimes_R S}$ is an isomorphism. Conversely, if $\chi^S_{C\otimes_R S}$ is an isomorphism, then the diagram shows that $\chi^R_C \otimes_R S$ is an isomorphism; if we also assume that φ is faithfully flat, then χ^R_C is an isomorphism. This yields desired result. □

The next result is from unpublished notes by Foxby. See also Avramov, Iyengar, and Lipman [**7**]. It will be quite handy, saving us from worrying about certain commutative diagrams. Note that it can also be derived as a corollary of Proposition 5.4.1 in the special case $M = R$.

Proposition 2.2.2. *Let C be an R-module.*

(a) *If there is an R-module isomorphism $\alpha\colon R \xrightarrow{\cong} \operatorname{Hom}_R(C, C)$, then the natural homothety map $\chi^R_C\colon R \to \operatorname{Hom}_R(C, C)$ is an isomorphism.*
(b) *Assume that C is finitely generated. If for every maximal ideal $\mathfrak{m} \subset R$ there is an $R_\mathfrak{m}$-module isomorphism $R_\mathfrak{m} \cong \operatorname{Hom}_R(C, C)_\mathfrak{m}$, then the natural homothety map $\chi^R_C\colon R \to \operatorname{Hom}_R(C, C)$ is an isomorphism.*

PROOF. (a) Let $\mathrm{id}_C\colon C \to C$ be the identity map, and set $u = \alpha^{-1}(\mathrm{id}_C)$. Using the condition $\mathrm{id}_C = \alpha(u) = u\alpha(1)$, it is straightforward to show that u is a unit. Furthermore, we have $\chi^R_C = u\alpha$. Since u is a unit and α is an isomorphism, it follows that χ^R_C is an isomorphism.

(b) The assumption $R_\mathfrak{m} \cong \operatorname{Hom}_R(C, C)_\mathfrak{m} \cong \operatorname{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m}, C_\mathfrak{m})$ conspires with part (a) to imply that the natural homothety map $\chi^{R_\mathfrak{m}}_{C_\mathfrak{m}}\colon R_\mathfrak{m} \to \operatorname{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m}, C_\mathfrak{m})$ is an isomorphism for each maximal ideal $\mathfrak{m} \subset R$. Furthermore, for each $\mathfrak{m}$ there is a commutative diagram

$$\begin{array}{ccc} R_{\mathfrak{m}} & \xrightarrow[\cong]{\chi^{R_{\mathfrak{m}}}_{C_{\mathfrak{m}}}} & \operatorname{Hom}_{R_{\mathfrak{m}}}(C_{\mathfrak{m}}, C_{\mathfrak{m}}) \\ {\scriptstyle (\chi^R_C)_{\mathfrak{m}}} \big\downarrow & \nearrow_{\cong} & \\ \operatorname{Hom}_R(C, C)_{\mathfrak{m}} & & \end{array}$$

where the unspecified map is the natural isomorphism. It follows that $(\chi^R_C)_{\mathfrak{m}}$ is an isomorphism for each $\mathfrak{m}$, and so χ^R_C is an isomorphism. □

Here is a compliment to part of Proposition 2.2.1. It is a local-global principle.

Proposition 2.2.3. *Let C be a finitely generated R-module. The following conditions are equivalent:*

(i) *C is a semidualizing R-module;*
(ii) *$U^{-1}C$ is a semidualizing $U^{-1}R$-module for each multiplicatively closed subset $U \subset R$;*
(iii) *$C_{\mathfrak{p}}$ is a semidualizing $R_{\mathfrak{p}}$-module for each prime ideal $\mathfrak{p} \subset R$; and*
(iv) *$C_{\mathfrak{m}}$ is a semidualizing $R_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m} \subset R$.*

Proof. The implication (i) $\Longrightarrow$ (ii) is in Proposition 2.2.1; use the flat homomorphism $R \to U^{-1}R$. The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are straightforward.

(iv) $\Longrightarrow$ (i). Assume that $C_{\mathfrak{m}}$ is a semidualizing $R_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m} \subset R$. For each $i \geqslant 1$ and each $\mathfrak{m}$ this provides the vanishing

$$\operatorname{Ext}^i_R(C, C)_{\mathfrak{m}} \cong \operatorname{Ext}^i_{R_{\mathfrak{m}}}(C_{\mathfrak{m}}, C_{\mathfrak{m}}) = 0$$

where the isomorphism is standard because C is finitely generated and R is noetherian. Since this is so for each maximal ideal $\mathfrak{m}$, we conclude that $\operatorname{Ext}^i_R(C, C) = 0$ for each $i \geqslant 1$. Furthermore, for each $\mathfrak{m}$, we have $R_{\mathfrak{m}} \cong \operatorname{Hom}_{R_{\mathfrak{m}}}(C_{\mathfrak{m}}, C_{\mathfrak{m}}) \cong \operatorname{Hom}_R(C, C)_{\mathfrak{m}}$, so Proposition 2.2.2(b) implies that χ^R_C is an isomorphism. Hence, C is a semidualizing R-module, as desired. □

Remark 2.2.4. With the notation of Proposition 2.2.1, assume that C is dualizing for R. While $C \otimes_R S$ will be semidualizing for S, it may not be dualizing for S. For example, let R be a field and let S be a non-Gorenstein local R-algebra; then R^1 is dualizing for R, but $S^1 \cong R^1 \otimes_R S$ is not dualizing for S.

On the other hand, the $U^{-1}R$-module $U^{-1}C$ will be dualizing because of the inequality $\operatorname{id}_{U^{-1}R}(U^{-1}C) \leqslant \operatorname{id}_R(C) < \infty$.

Corollary 2.2.5. *Let C be a finitely generated R-module, and let P be a finitely generated projective R-module of rank 1.*

(a) *The R-module P is semidualizing.*
(b) *The R-module $C \otimes_R P$ is semidualizing if and only if C is semidualizing.*
(c) *The R-module C is (point-wise) dualizing if and only if $C \otimes_R P$ is (point-wise) dualizing.*

Proof. By assumption, we have $P_{\mathfrak{m}} \cong R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m} \subset R$. Since $R_{\mathfrak{m}}$ is a semidualizing $R_{\mathfrak{m}}$-module, Proposition 2.2.3 implies that P is a semidualizing R-module. This establishes part (a). Part (b) follows similarly, using the sequence of isomorphisms

$$(C \otimes_R P)_{\mathfrak{m}} \cong C_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} P_{\mathfrak{m}} \cong C_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} R_{\mathfrak{m}} \cong C_{\mathfrak{m}}.$$

Since P is projective and non-zero, we have $\mathrm{id}_R(C \otimes_R P) = \mathrm{id}_R(C)$, and this explains part (c). $\square$

Part (a) of the next result generalizes Proposition 2.1.16(d). See also Corollary 3.4.3.

Theorem 2.2.6. *Let C be a semidualizing R-module, and let $\mathbf{x} = x_1, \ldots, x_n \in R$.*

(a) *The sequence $\mathbf{x}$ is C-regular if and only if $\mathbf{x}$ is R-regular.*
(b) *If $\mathbf{x}$ is R-regular, then $C/\mathbf{x}C$ is a semidualizing $R/\mathbf{x}R$-module.*
(c) *Given a proper ideal $I \subsetneq R$, one has $\mathrm{depth}_R(I; C) = \mathrm{depth}(I; R)$. In particular, if R is local, then $\mathrm{depth}_R(C) = \mathrm{depth}(R)$.*

Proof. Part (c) follows from part (a). We prove parts (a) and (b) by induction on n. For the base case $n = 1$, part (a) is contained in Proposition 2.1.16(d). Thus, for the base case, we assume that x_1 is R-regular (and hence C-regular) and prove that $\overline{C} = C/x_1C$ is a semidualizing $\overline{R}$-module where $\overline{R} = R/x_1R$.

We claim that $\mathrm{Ext}^i_R(C, \overline{C}) = 0$ for all $i \geqslant 1$. To see this, consider the following sequence, which is exact since x_1 is C-regular:

$$0 \to C \xrightarrow{x_1} C \to \overline{C} \to 0. \tag{2.2.6.1}$$

Since $\mathrm{Ext}^i_R(C, C) = 0$, the associated long exact sequence in $\mathrm{Ext}_R(C, -)$ yields the desired vanishing.

The fact that x_1 is R-regular and C-regular yields an isomorphism

$$\mathrm{Ext}^i_{\overline{R}}(\overline{C}, \overline{C}) \cong \mathrm{Ext}^i_R(C, \overline{C}) \tag{2.2.6.2}$$

for $i \geqslant 0$; see, e.g., [**16**, p. 140, Lem. 2]. Because of the previous paragraph, we conclude that $\mathrm{Ext}^i_{\overline{R}}(\overline{C}, \overline{C}) = 0$ for all $i \geqslant 1$.

There is a commutative diagram of R-module homomorphisms

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{x_1} & R & \longrightarrow & \overline{R} & \longrightarrow & 0 \\
 & & {\scriptstyle \chi^R_C}\downarrow{\scriptstyle \cong} & & {\scriptstyle \chi^R_C}\downarrow{\scriptstyle \cong} & & {\scriptstyle \gamma}\downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(C, C) & \xrightarrow{x_1} & \mathrm{Hom}_R(C, C) & \longrightarrow & \mathrm{Hom}_R(C, \overline{C}) & \longrightarrow & 0
\end{array}$$

where $\gamma(\overline{r})(c) = \overline{rc}$. The top row of this diagram is exact because x_1 is R-regular. The bottom row is the sequence obtained by applying the functor $\mathrm{Hom}_R(C, -)$ to the exact sequence (2.2.6.1), and it is exact because $\mathrm{Ext}^1_R(C, C) = 0$. Hence, the snake lemma implies that γ is an isomorphism. This is the first step in the next sequence, and the second step is from (2.2.6.2):

$$\overline{R} \cong \mathrm{Hom}_R(C, \overline{C}) \cong \mathrm{Hom}_{\overline{R}}(\overline{C}, \overline{C}).$$

These are R-module isomorphisms of $\overline{R}$-modules, hence $\overline{R}$-module isomorphisms. Proposition 2.2.2(a) implies that $\chi^{\overline{R}}_{\overline{C}}$ is an isomorphism, so $\overline{C}$ is a semidualizing $\overline{R}$-module. This completes the base case. The induction step is routine. $\square$

Corollary 2.2.7. *Let $\mathbf{x} = x_1, \ldots, x_n \in R$ be an R-regular sequence. If C is a (point-wise) dualizing R-module, then $C/\mathbf{x}C$ is a (point-wise) dualizing $R/\mathbf{x}R$-module.*

Proof. Set $\overline{R} = R/\mathbf{x}R$ and $\overline{C} = C/\mathbf{x}C$.

Assume first that C is point-wise dualizing. Theorem 2.2.6 implies that $\mathbf{x}$ is C-regular and that $\overline{C}$ is a semidualizing $\overline{R}$-module. For each maximal ideal $\mathfrak{m} \subset R$ such that $\mathbf{x} \in \mathfrak{m}$, set $\overline{\mathfrak{m}} = \mathfrak{m}/\mathbf{x}R$. From [**8**, (3.1.15)], we have

$$\mathrm{id}_{\overline{R}_{\overline{\mathfrak{m}}}}(\overline{C}_{\overline{\mathfrak{m}}}) = \mathrm{id}_{R_\mathfrak{m}}(C_\mathfrak{m}) - n < \infty.$$

Since every maximal ideal of $\overline{R}$ is of the form $\overline{\mathfrak{m}}$, it follows that $\overline{C}$ is point-wise dualizing for $\overline{R}$.

Assume next that C is dualizing, and set $d = \mathrm{id}_R(C) < \infty$. For each maximal ideal $\overline{\mathfrak{m}} \subset \overline{R}$, we have

$$\mathrm{id}_{\overline{R}_{\overline{\mathfrak{m}}}}(\overline{C}_{\overline{\mathfrak{m}}}) = \mathrm{id}_{R_\mathfrak{m}}(C_\mathfrak{m}) - n \leqslant d - n.$$

As every maximal ideal of $\overline{R}$ is of the form $\overline{\mathfrak{m}}$, we conclude that $\mathrm{id}_{\overline{R}}(\overline{C}) \leqslant d-n < \infty$. It follows that $\overline{C}$ is dualizing for $\overline{R}$. □

Corollary 2.2.8. *Let C be a semidualizing R-module of finite projective dimension. Then C is a rank 1 projective R-module. If R is local, then $C \cong R$.*

Proof. Assume first that R is local. The Auslander-Buchsbaum formula explains the first equality in the next sequence

$$\mathrm{pd}_R(C) = \mathrm{depth}(R) - \mathrm{depth}_R(C) = 0$$

and the second equality is from Corollary 2.2.6(c). This shows that C is projective, and, since R is local, that C is free. Hence $C \cong R^n$ for some integer $n \geqslant 1$. From the isomorphisms

$$R \cong \mathrm{Hom}_R(C, C) \cong \mathrm{Hom}_R(R^n, R^n) \cong R^{n^2}$$

it follows that $n = 1$ and so $C \cong R$.

Assume now that R is not necessarily local. For each maximal ideal $\mathfrak{m} \subset R$, the $R_\mathfrak{m}$-module $C_\mathfrak{m}$ is semidualizing and has finite projective dimension, and so we have $C_\mathfrak{m} \cong R_\mathfrak{m}$ for each $\mathfrak{m}$. It follows that C is a rank 1 projective R-module. □

Corollary 2.2.9. *The ring R is (point-wise) Gorenstein if and only if R has a (point-wise) dualizing module of finite projective dimension.*

Proof. If R is Gorenstein, then R is a dualizing module for R that has finite projective dimension.

Conversely, assume that P is a dualizing module for R that has finite projective dimension. Corollary 2.2.8 implies that P is a rank 1 projective module. Since P is projective and has finite injective dimension, it follows that the module $R \cong \mathrm{Hom}_R(P, P)$ has finite injective dimension, that is, that R is Gorenstein.

Since the conditions "point-wise Gorenstein" and "point-wise dualizing" are local conditions, the equivalence of point-wise conditions follows from the non-point-wise statements proved above. □

The next result is like [**8**] and has a similar proof.

Proposition 2.2.10. *Let C be a finitely generated R-module. Then C is semidualizing for R if and only if the following conditions are satisfied:*

(1) *For each $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathrm{depth}(R_\mathfrak{p}) \geqslant 2$, one has $\mathrm{depth}_{R_\mathfrak{p}}(C_\mathfrak{p}) \geqslant 2$;*
(2) *For each $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathrm{depth}(R_\mathfrak{p}) \leqslant 1$, there is an $R_\mathfrak{p}$-isomorphism $R_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p}, C_\mathfrak{p})$ is an isomorphism; and*

(3) *One has* $\operatorname{Ext}^i_R(C,C)=0$ *for all* $i\geqslant 1$.

Proof. Assume first that C is semidualizing. Then $\operatorname{Ext}^i_R(C,C)=0$ for all $i\geqslant 1$, by definition. Proposition 2.2.1 shows that $C_\mathfrak{p}$ is semidualizing for $R_\mathfrak{p}$ for each $\mathfrak{p}\in\operatorname{Spec}(R)$. So the map $\chi^{R_\mathfrak{p}}_{C_\mathfrak{p}}\colon R_\mathfrak{p}\to\operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p},C_\mathfrak{p})$ is an isomorphism, and $\operatorname{depth}_{R_\mathfrak{p}}(C_\mathfrak{p})=\operatorname{depth}(R_\mathfrak{p})$ by Theorem 2.2.6(c).

Assume next that conditions (1)–(3) are satisfied. To show that C is semidualizing, it suffices to show that the homothety map $\chi^R_C\colon R\to\operatorname{Hom}_R(C,C)$ is an isomorphism. Combining condition (2) and Proposition 2.2.2(a), we conclude that for each $\mathfrak{p}\in\operatorname{Spec}(R)$ such that $\operatorname{depth}(R_\mathfrak{p})\leqslant 1$, the homothety homomorphism $\chi^{R_\mathfrak{p}}_{C_\mathfrak{p}}\colon R_\mathfrak{p}\to\operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p},C_\mathfrak{p})$ is an isomorphism.

The condition $\operatorname{Ker}(\chi^R_C)\subseteq R$ implies that $\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Ker}(\chi^R_C)_\mathfrak{p})\geqslant 1$ whenever $\operatorname{depth}(R_\mathfrak{p})\geqslant 1$. On the other hand, when $\operatorname{depth}(R_\mathfrak{p})=0$, the map $\chi^{R_\mathfrak{p}}_{C_\mathfrak{p}}$ is an isomorphism, so we have $\operatorname{Ker}(\chi^R_C)_\mathfrak{p}\cong\operatorname{Ker}(\chi^{R_\mathfrak{p}}_{C_\mathfrak{p}})=0$ for all such primes. In particular, there are no primes $\mathfrak{p}$ such that $\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Ker}(\chi^R_C)_\mathfrak{p})=0$, implying that $\operatorname{Ass}_R(\operatorname{Ker}(\chi^R_C))=\emptyset$. Hence, we have $\operatorname{Ker}(\chi^R_C)=0$, so χ^R_C is injective.

To show that χ^R_C is surjective, we similarly show that there are no primes $\mathfrak{p}$ such that $\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Coker}(\chi^R_C)_\mathfrak{p})=0$. Since χ^R_C is injective, we have an exact sequence

$$0\to R\xrightarrow{\chi^R_C}\operatorname{Hom}_R(C,C)\to\operatorname{Coker}(\chi^R_C)\to 0$$

which we localize at a prime $\mathfrak{p}$ to obtain the next exact sequence

$$0\to R_\mathfrak{p}\xrightarrow{\chi^{R_\mathfrak{p}}_{C_\mathfrak{p}}}\operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p},C_\mathfrak{p})\to\operatorname{Coker}(\chi^R_C)_\mathfrak{p}\to 0. \qquad (2.2.10.1)$$

If $\operatorname{depth}(R_\mathfrak{p})\leqslant 1$, then $\chi^{R_\mathfrak{p}}_{C_\mathfrak{p}}$ is an isomorphism, so $\operatorname{Coker}(\chi^R_C)_\mathfrak{p}=0$ and it follows that $\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Coker}(\chi^R_C)_\mathfrak{p})=\infty$ in this case. Assume that $\operatorname{depth}(R_\mathfrak{p})\geqslant 2$. Since $\operatorname{depth}_{R_\mathfrak{p}}(C_\mathfrak{p})\geqslant 2$ in this case, we have

$$\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p},C_\mathfrak{p}))\geqslant\min\{2,\operatorname{depth}_{R_\mathfrak{p}}(C_\mathfrak{p})\}=2.$$

Hence, the sequence (2.2.10.1) implies that

$$\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Coker}(\chi^R_C)_\mathfrak{p})\geqslant\min\{\operatorname{depth}(R_\mathfrak{p})-1,\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p},C_\mathfrak{p}))\}\geqslant 1$$

as desired. □

Definition 2.2.11. Assume that R is local with residue field k. The *type* of a finitely generated R-module N is $\operatorname{rank}_k(\operatorname{Ext}^{\operatorname{depth}_R(N)}_R(k,N))$, and N is *maximal Cohen-Macaulay* if $\operatorname{depth}_R(N)=\dim(R)$.

Assume that R is Cohen-Macaulay and local. A *canonical module* for R is a maximal Cohen-Macaulay R-module of finite injective dimension and type 1.

Definition 2.2.12. Assume that R is Cohen-Macaulay, though not necessarily local. A *canonical module* for R is a finitely generated R-module C that is local canonical, that is, such that, for each maximal ideal $\mathfrak{m}\subset R$, the localization $C_\mathfrak{m}$ is a canonical module for $R_\mathfrak{m}$.

Corollary 2.2.13. *An R-module C is point-wise dualizing if and only if R is Cohen-Macaulay and C is a canonical module for R.*

Proof. Since the conditions under consideration are local by definition, we may assume for the rest of the proof that R is local. Under this assumption, note that C is dualizing if and only if it is point-wise dualizing.

If R is Cohen-Macaulay and C is a canonical module for R, then C is dualizing for R by [**8**, (3.3.4(d)) and (3.3.10(d)(ii)].

For the converse, assume that C is dualizing for R. Since C is a non-zero finitely generated R-module of finite injective dimension, a corollary of the new intersection theorem implies that R is Cohen-Macaulay, and it follows from Theorem 2.2.6(c) that C is maximal Cohen-Macaulay. To show that C has type 1, let $\mathbf{x} = x_1, \ldots, x_n \in R$ be a maximal R-regular sequence. Set $\overline{R} = R/\mathbf{x}R$ and $\overline{C} = C/\mathbf{x}C$. Corollary 2.2.7 implies that $\overline{C}$ is a dualizing $\overline{R}$-module. Since $\overline{R}$ is artinian and local, this implies that $\overline{C} \cong E^c$ for some $c \geqslant 1$, where E is the injective hull of the residue field of $\overline{R}$. Hence, we have

$$\overline{R} \cong \operatorname{Hom}_{\overline{R}}(\overline{C}, \overline{C}) \cong \operatorname{Hom}_{\overline{R}}(E^c, E^c) \cong \overline{R}^{c^2}.$$

From this, we conclude that $c = 1$, so $\overline{C} \cong E$. In other words, the type of $\overline{C}$ is 1, that is, the type of C is 1, as desired. □

Proposition 2.2.14. *Let $\varphi \colon R \to S$ be a ring homomorphism such that S is a finitely generated projective R-module. (Note that this implies that S is noetherian.)*

(a) *If C is a semidualizing R-module, then $\operatorname{Hom}_R(S, C)$ is a semidualizing S-module.*
(b) *If D is a dualizing R-module, then $\operatorname{Hom}_R(S, D)$ is a dualizing S-module.*

Proof. (a) Since S is finitely generated and projective as an R-module, it is totally C-reflexive by Proposition 2.1.13. Since C is finitely generated, the module $\operatorname{Hom}_R(S, C)$ is finitely generated over R. Since the S-module structure on $\operatorname{Hom}_R(S, C)$ is compatible with the R-module structure via φ, it follows that $\operatorname{Hom}_R(S, C)$ is finitely generated over S.

In the next sequence, the first isomorphism is from the natural biduality map the second isomorphism is induced by tensor-cancellation, and the third isomorphism is Hom-tensor adjointness:

$$\begin{aligned} S &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(S, C), C) \\ &\cong \operatorname{Hom}_R(S \otimes_S \operatorname{Hom}_R(S, C), C) \\ &\cong \operatorname{Hom}_S(\operatorname{Hom}_R(S, C), \operatorname{Hom}_R(S, C)). \end{aligned}$$

It is straightforward to show that these isomorphisms are S-linear. From Proposition 2.2.2, we conclude that $\chi^S_{\operatorname{Hom}_R(S,C)}$ is an isomorphism.

Let I be an injective resolution of C over R. It is straightforward to show that $\operatorname{Hom}_R(S, I_j)$ is an injective S-module for each j. Since S is projective as an R-module and the augmented resolution ${}^+I$ is exact, the complex $\operatorname{Hom}_R(S, {}^+I)$ is exact. It follows that $\operatorname{Hom}_R(S, I)$ is an injective resolution of $\operatorname{Hom}_R(S, C)$ as an S-module. (In particular, we have $\operatorname{id}_S(\operatorname{Hom}_R(S, C)) \leqslant \operatorname{id}_R(C)$.) This yields the first isomorphism in the next sequence:

$$\begin{aligned}\operatorname{Ext}^i_S(\operatorname{Hom}_R(S,C),\operatorname{Hom}_R(S,C)) &\cong \operatorname{H}_{-i}(\operatorname{Hom}_S(\operatorname{Hom}_R(S,C),\operatorname{Hom}_R(S,I))\\ &\cong \operatorname{H}_{-i}(\operatorname{Hom}_R(S\otimes_S \operatorname{Hom}_R(S,C),I)\\ &\cong \operatorname{H}_{-i}(\operatorname{Hom}_R(\operatorname{Hom}_R(S,C),I)\\ &\cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(S,C),C).\end{aligned}$$

The second isomorphism is induced by Hom-tensor adjointness. The third isomorphism is induced by tensor-cancellation. The fourth isomorphism follows from the fact that I is an injective resolution of C over R. Since S is totally C-reflexive, we have $\operatorname{Ext}^i_S(\operatorname{Hom}_R(S,C),\operatorname{Hom}_R(S,C)) \cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(S,C),C) = 0$ for all $i \geqslant 1$, so $\operatorname{Hom}_R(S,C)$ is a semidualizing S-module.

(b) Assume that D is a dualizing R-module. Part (a) implies that $\operatorname{Hom}_R(S,D)$ is a semidualizing S-module. The proof of part (a) shows that $\operatorname{id}_S(\operatorname{Hom}_R(S,D)) \leqslant \operatorname{id}_R(D) < \infty$, so $\operatorname{Hom}_R(S,D)$ is dualizing for S. □

Proposition 2.2.15. *Let $\varphi\colon R \to S$ be a flat local ring homomorphism, and let C be a finitely generated R-module. The S-module $C \otimes_R S$ is dualizing for S if and only if C is dualizing for R and the ring $S/\mathfrak{m}S$ is Gorenstein.*

Proof. Proposition 2.2.1 implies that C is semidualizing for R if and only if $C \otimes_R S$ is semidualizing for S. From [**8**, (1.2.16(b)] we know that $\operatorname{type}_S(C \otimes_R S) = \operatorname{type}_R(C)\operatorname{type}(S/\mathfrak{m}S)$, and [**8**, (2.1.7)] says that S is Cohen-Macaulay if and only if R and $S/\mathfrak{m}S$ are Cohen-Macaulay. Thus, the desired conclusion follows from Corollary 2.2.13. □

2.3. Examples

To this point, we have not provided an example of a nontrivial semidualizing module, that is, one that is not projective and not dualizing. The goal of this subsection is to provide such examples.

Example 2.3.1. Let A be a local Gorenstein ring, and set $R = A[X,Y]/(X,Y)^2$. Then R is a local Cohen-Macaulay ring, and it is free (of rank 3) as an A-module. Also, we have $\operatorname{type}(R) = 2$, so R is not Gorenstein. The R-modules $R = R \otimes_A A$ and $D^R = \operatorname{Hom}_A(R,A)$ are semidualizing. Moreover, the R-module D^R is dualizing by Proposition 2.2.14(b), and we have $D^R \ncong R$ by Example 2.1.11.

Example 2.3.2. Let $(R,\mathfrak{m},k)$ be a local artinian ring that is not Gorenstein, with dualizing module $D^R \ncong R$; for instance, if A is a field, then the ring from Example 2.3.1 satisfies these conditions. The ring $S = R[U,V]/(U,V)^2$ is a local Cohen-Macaulay ring with residue field k, and S is free (of rank 3) as an R-module. Also, we have $\operatorname{type}(S) = 2\operatorname{type}(R)$, so S is not Gorenstein. The following S-modules are semidualizing

$$S = S \otimes_R R \qquad B = \operatorname{Hom}_R(S,R)$$
$$C = S \otimes_R D^R \qquad D^S = \operatorname{Hom}_R(S,D^R)$$

and D^S is dualizing; see Propositions 2.2.1 and 2.2.14. We claim that the modules B and C have infinite projective dimension and are not dualizing. (In fact, we also have $B \ncong C$, but we will not show this.) Note that $D^S \ncong S$ by Example 2.1.11. Moreover, since S is local, Corollary 2.2.8 implies that $\operatorname{pd}_S(D^S) = \infty$.

Since $D^R \not\cong R$, Corollary 2.1.14 implies that D^R is not cyclic. It follows that $C = S \otimes_R D^R$ is not cyclic, hence $C \not\cong S$. As in the previous paragraph, this implies that $\mathrm{pd}_S(C) = \infty$.

To show that B is not dualizing, we need to show that $\mathrm{type}_S(B) = \mathrm{type}(R) \geqslant 2$; see Corollary 2.2.13. Since S is artinian, the equality $\mathrm{type}_S(B) = \mathrm{type}(R)$ follows from the next sequence

$$\begin{aligned}\mathrm{Hom}_S(k, B) &\cong \mathrm{Hom}_S(k, \mathrm{Hom}_R(S, R)) \cong \mathrm{Hom}_R(S \otimes_S k, R) \\ &\cong \mathrm{Hom}_R(k, R) \cong k^{\mathrm{type}(R)}.\end{aligned}$$

Of course, we have $\mathrm{type}(R) \geqslant 2$ since R is not Gorenstein.

Suppose by way of contradiction that $\mathrm{pd}_S(B) < \infty$. Corollary 2.2.8 shows that $B \cong S$, hence the second isomorphism in the next sequence:

$$k^{\mathrm{type}(R)} \cong \mathrm{Hom}_S(k, C) \cong \mathrm{Hom}_S(k, S) \cong k^{\mathrm{type}(S)} \cong k^{2\,\mathrm{type}(R)}.$$

The first isomorphism is from the previous paragraph. This implies that $\mathrm{type}(R) = 0$, a contradiction, so $\mathrm{pd}_S(B) = \infty$.

The fact that C is not dualizing for S follows from Proposition 2.2.15 because the ring $S/\mathfrak{m}S \cong k[U, V]/(U^2, UV, V^2)$ is not Gorenstein.

Direct products provide another way to build nontrivial semidualizing modules. We provide some background since we will use the ideas in a couple of places.

Fact 2.3.3. Let R_1 and R_2 be noetherian rings and set $R = R_1 \times R_2$. Every R-module is (isomorphic to one) of the form $M_1 \times M_2$ with the R-module structure given coordinate-wise as $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$. Indeed, if $e_1 = (1, 0)$ and $e_2 = (0, 1)$, then $M_i = e_iM$.

Each prime ideal is of the form $P = P_1 \times R_2$ or $P = R_1 \times P_2$ for some $P_i \in \mathrm{Spec}(R_i)$, and P is a maximal ideal of R if an only if P_i is a maximal ideal of R_i. There is an isomorphism of local rings $R_P \cong (R_i)_{P_i}$. Moreover, we have $M_P \cong (M_i)_{P_i}$. This isomorphism is verified by first showing that $R_{e_i} \cong R_i$ and $M_{e_i} \cong M_i$.

Proposition 2.3.4. *Let $R_1, \ldots, R_n$ be noetherian rings, and set $R = R_1 \times \cdots \times R_n$. There is a bijection $\pi\colon \mathfrak{S}_0(R_1) \times \cdots \times \mathfrak{S}_0(R_n) \xrightarrow{\sim} \mathfrak{S}_0(R)$ given by $([C_1], \ldots, [C_n]) \mapsto [C_1 \times \cdots \times C_n]$.*

Proof. We prove the case when $n = 2$; the case when $n > 2$ follows readily by induction on n.

First, we show that the map π is well-defined. Let $[C_i] \in \mathfrak{S}_0(R_i)$ for $i = 1, 2$. We need to show that the $R_1 \times R_2$-module $C_1 \times C_2$ is semidualizing. (It is straightforward to show that the class $[C_1 \times C_2]$ is independent of the choice of representatives of the classes of the $[C_i]$.) Proposition 2.2.3 says that it suffices to show that the localization $(C_1 \times C_2)_P$ is an $(R_1 \times R_2)_P$-semidualizing module for each prime ideal $P \subset R_1 \times R_2$. Fact 2.3.3 says that P is of the form $P = P_1 \times R_2$ or $P = R_1 \times P_2$ for some $P_i \in \mathrm{Spec}(R_i)$, and there is an isomorphism $(C_1 \times C_2)_P \cong (C_i)_{P_i}$ as a module over the ring $(R_1 \times R_2)_P \cong (R_i)_{P_i}$; since C_i is a semidualizing R_i-module, this localization is semidualizing for $(R_i)_{P_i}$.

To show that π is surjective, let $[C] \in \mathfrak{S}_0(R_1 \times R_2)$. Fact 2.3.3 says that $C \cong C_1 \times C_2$ for the R_i-modules $C_i = e_iC$. Since C is finitely generated, so are the C_i. The argument of the previous paragraph shows that, since C is a semidualizing

$R_1 \times R_2$-module, the C_i are semidualizing R_i-modules for $i = 1, 2$. Thus, we have $[C] = \pi([C_1], [C_2])$.

Finally, to show that π is injective, assume that $\pi([B_1], [B_2]) = \pi([C_1], [C_2])$, that is, that there is an isomorphism of $R_1 \times R_2$-modules $B_1 \times B_2 \cong C_1 \times C_2$. For $i = 1, 2$ this implies that

$$B_i \cong e_i(B_1 \times B_2) \cong e_i(C_1 \times C_2) \cong C_i$$

so we have $([B_1], [B_2]) = ([C_1], [C_2])$. □

Remark 2.3.5. In the proof of the injectivity of π in Proposition 2.3.4, note that we cannot simply check that the kernel of π is trivial. Indeed, the sets $\mathfrak{S}_0(R_i)$ and $\mathfrak{S}_0(R_1 \times R_2)$ do not have any meaningful group structure (that we know of) so it does not make sense for π to be a group homomorphism.

Proposition 2.3.6. *Let k be a field, and let R and S be k-algebras. If B is a semidualizing R-module, and C is a semidualizing S-module, then $B \otimes_k C$ is semidualizing for $R \otimes_k S$.*

Proof. From Proposition A.1.5, there are $R \otimes_k S$-isomorphisms

$$\operatorname{Ext}^i_{R\otimes_k S}(B \otimes_k C, B \otimes_k C) \cong \oplus_{j=0}^i \operatorname{Ext}^j_R(B, B) \otimes_k \operatorname{Ext}^{i-j}_S(C, C)$$

for each $i \geqslant 0$. Hence, the condition $\operatorname{Ext}^i_R(B, B) = 0 = \operatorname{Ext}^i_S(C, C)$ for $i \geqslant 1$ implies that $\operatorname{Ext}^i_{R\otimes_k S}(B \otimes_k C, B \otimes_k C) = 0$ for $i \geqslant 1$. When $i = 0$, we have

$$\operatorname{Hom}_{R\otimes_k S}(B \otimes_k C, B \otimes_k C) \cong \operatorname{Hom}_R(B, B) \otimes_k \operatorname{Hom}_S(C, C) \cong R \otimes_R S$$

so the desired conclusion follows from Proposition 2.2.2(a). □

CHAPTER 3

Foxby Classes

3.1. Definitions and Basic Properties

We begin with a spot of motivation.

Remark 3.1.1. An analysis of the proof of Theorem 2.2.6 shows that there are three conditions that allow us to conclude that $\overline{C}$ is a semidualizing $\overline{R}$-module:

(1) $\operatorname{Tor}_i^R(C,\overline{R}) = 0$ for all $i \geqslant 1$;
(2) $\operatorname{Ext}_R^i(C, C \otimes_R \overline{R}) \cong \operatorname{Ext}_R^i(C,\overline{C}) = 0$ for $i \geqslant 1$; and
(3) there is a natural isomorphism $\gamma\colon \overline{R} \to \operatorname{Hom}_R(C,\overline{C}) = \operatorname{Hom}_R(C, C \otimes_R \overline{R})$.

These are essentially the defining conditions for membership in the Auslander class.

Definition 3.1.2. Let M and N be R-modules. The natural *evaluation map*

$$\xi_N^M : M \otimes_R \operatorname{Hom}_R(M,N) \to N$$

is the R-module homomorphism given by $\xi_N^M(m \otimes \psi) = \psi(m)$. The natural map

$$\gamma_N^M : N \to \operatorname{Hom}_R(M, M \otimes_R N)$$

is the R-module homomorphism given by $\gamma_N^M(n)(m) = m \otimes n$.

Remark 3.1.3. Let $f\colon M \to M'$ and $g\colon N \to N'$ be R-module homomorphisms. It is straightforward to show that the maps ξ_N^M and γ_N^M from Definition 3.1.2 are well-defined R-module homomorphisms and that the following diagrams commute:

$$\begin{array}{ccc}
M \otimes_R \operatorname{Hom}_R(M',N) & \xrightarrow{f \otimes_R \operatorname{Hom}_R(M',N)} & M' \otimes_R \operatorname{Hom}_R(M',N) \\
{\scriptstyle M \otimes_R \operatorname{Hom}_R(f,N)}\big\downarrow & & \big\downarrow{\scriptstyle \xi_N^{M'}} \\
M \otimes_R \operatorname{Hom}_R(M,N) & \xrightarrow{\xi_N^M} & N \\
{\scriptstyle M \otimes_R \operatorname{Hom}_R(M,g)}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
M \otimes_R \operatorname{Hom}_R(M,N') & \xrightarrow{\xi_{N'}^M} & N'.
\end{array}$$

$$\begin{array}{ccc}
\operatorname{Hom}_R(M', M' \otimes_R N) & \xrightarrow{\operatorname{Hom}_R(f, M' \otimes_R N)} & \operatorname{Hom}_R(M, M' \otimes_R N) \\
{\scriptstyle \gamma_N^{M'}}\big\uparrow & & \big\uparrow{\scriptstyle \operatorname{Hom}_R(M, f \otimes_R N)} \\
N & \xrightarrow{\gamma_N^M} & \operatorname{Hom}_R(M, M \otimes_R N) \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Hom}_R(M, M \otimes_R g)} \\
N' & \xrightarrow{\gamma_{N'}^M} & \operatorname{Hom}_R(M, M \otimes_R N').
\end{array}$$

The classes defined next are collectively known as *Foxby classes*. The definitions are due to Foxby; see Avramov and Foxby [**5**] and Christensen [**9**]. Note that we do not assume in the definition that C is semidualizing.

Definition 3.1.4. Let C be a finitely generated R-module. The *Auslander class* $\mathcal{A}_C(R)$ is the class of all R-modules M satisfying the following conditions:

(1) the natural map $\gamma^C_M\colon M \to \operatorname{Hom}_R(C, C\otimes_R M)$ is an isomorphism; and
(2) $\operatorname{Tor}^R_i(C,M)=0=\operatorname{Ext}^i_R(C,C\otimes_R M)$ for all $i\geqslant 1$.

The *Bass class* $\mathcal{B}_C(R)$ is the class of all R-modules M satisfying the following:

(1) the evaluation map $\xi^C_M\colon C\otimes_R \operatorname{Hom}_R(C,M)\to M$ is an isomorphism; and
(2) $\operatorname{Ext}^i_R(C,M)=0=\operatorname{Tor}^R_i(C,\operatorname{Hom}_R(C,M))$ for all $i\geqslant 1$.

The following example is readily verified.

Example 3.1.5. In the case $C=R$: the classes $\mathcal{A}_R(R)$ and $\mathcal{B}_R(R)$ are both equal to the class of all R-module.

Here is a useful property of Foxby classes.

Proposition 3.1.6. *Let C be a finitely generated R-module, and let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules.*

(a) *One has $\coprod_{\lambda\in\Lambda} M_\lambda\in\mathcal{A}_C(R)$ if and only if $M_\lambda\in\mathcal{A}_C(R)$ for all $\lambda\in\Lambda$.*
(b) *One has $\coprod_{\lambda\in\Lambda} M_\lambda\in\mathcal{B}_C(R)$ if and only if $M_\lambda\in\mathcal{B}_C(R)$ for all $\lambda\in\Lambda$.*

In particular, the classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ are closed under coproducts and direct summands.

Proof. (a) Recall that there is an isomorphism

$$\alpha\colon C\otimes_R (\textstyle\coprod_{\lambda\in\Lambda} M_\lambda) \xrightarrow{\cong} \coprod_{\lambda\in\Lambda}(C\otimes_R M_\lambda)$$

given by $\alpha(c\otimes(m_\lambda))=(c\otimes m_\lambda)$. Furthermore, we have

$$\operatorname{Tor}^R_i(C,\textstyle\coprod_{\lambda\in\Lambda} M_\lambda)\cong\coprod_{\lambda\in\Lambda}\operatorname{Tor}^R_i(C,M_\lambda)$$

for all i. It follows that $\operatorname{Tor}^R_i(C,\coprod_{\lambda\in\Lambda} M_\lambda)=0$ for all $i\geqslant 1$ if and only if $\operatorname{Tor}^R_i(C,M_\lambda)=0$ for all $\lambda\in\Lambda$ and all $i\geqslant 1$.

For each $\mu\in\Lambda$, let $\delta_\mu\colon \coprod_{\lambda\in\Lambda}(C\otimes_R M_\lambda)\to C\otimes_R M_\mu$ be given by $\delta_\mu(x_\lambda)=x_\mu$; in words, δ_μ is the projection onto the μth coordinate. Because C is finitely generated and R is noetherian, the map

$$\Delta\colon \operatorname{Hom}_R(C,\textstyle\coprod_{\lambda\in\Lambda}(C\otimes_R M_\lambda))\to\coprod_{\lambda\in\Lambda}\operatorname{Hom}_R(C,C\otimes_R M_\lambda)$$

given by $\phi\mapsto(\delta_\lambda\circ\phi)$ is an isomorphism. A similar construction (using a degree-wise finite free resolution of C) yields the second isomorphism in the following sequence, while the first isomorphism is induced by α:

$$\begin{aligned}\operatorname{Ext}^i_R(C,C\otimes_R(\textstyle\coprod_{\lambda\in\Lambda} M_\lambda)) &\cong \operatorname{Ext}^i_R(C,\textstyle\coprod_{\lambda\in\Lambda}(C\otimes_R M_\lambda))\\ &\cong \textstyle\coprod_{\lambda\in\Lambda}\operatorname{Ext}^i_R(C,C\otimes_R M_\lambda).\end{aligned}$$

We conclude that $\operatorname{Ext}^i_R(C,C\otimes_R(\coprod_{\lambda\in\Lambda} M_\lambda))=0$ for all $i\geqslant 1$ if and only if $\operatorname{Ext}^i_R(C,C\otimes_R M_\lambda)=0$ for all $\lambda\in\Lambda$ and all $i\geqslant 1$.

Finally, there is a commutative diagram of R-module homomorphisms:

$$\begin{array}{ccc}
\coprod_{\lambda\in\Lambda} M_\lambda & \xrightarrow{\gamma^C_{\coprod_{\lambda\in\Lambda} M_\lambda}} & \operatorname{Hom}_R(C, C\otimes_R \coprod_{\lambda\in\Lambda} M_\lambda) \\
{\scriptstyle \coprod_{\lambda\in\Lambda}\gamma^C_{M_\lambda}}\downarrow & & \cong \downarrow {\scriptstyle \operatorname{Hom}_R(C,\alpha)} \\
\coprod_{\lambda\in\Lambda} \operatorname{Hom}_R(C, C\otimes_R M_\lambda) & \underset{\cong}{\xleftarrow{\Delta}} & \operatorname{Hom}_R(C, \coprod_{\lambda\in\Lambda}(C\otimes_R M_\lambda)).
\end{array}$$

This shows that $\gamma^C_{\coprod_{\lambda\in\Lambda} M_\lambda}$ is an isomorphism if and only if $\coprod_{\lambda\in\Lambda}\gamma^C_{M_\lambda}$ is an isomorphism, that is, if and only if $\gamma^C_{M_\lambda}$ is an isomorphism for all $\lambda \in \Lambda$. This completes the proof of part (a).

The proof of part (b) is similar. □

Here is one of the most frequently used properties of Foxby classes. It says that the Foxby classes satisfy the two-of-three property.

Proposition 3.1.7. *Let C be a semidualizing R-module, and consider an exact sequence of R-module homomorphisms*

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0.$$

(a) *If two of the M_i are in $\mathcal{A}_C(R)$, then so is the third.*
(b) *If two of the M_i are in $\mathcal{B}_C(R)$, then so is the third.*

PROOF. Assume first that $M_1, M_2 \in \mathcal{A}_C(R)$. Consider the long exact sequence in $\operatorname{Tor}_i^R(C, -)$ associated to the given sequence. Since $\operatorname{Tor}_{i-1}^R(C, M_1) = 0 = \operatorname{Tor}_i^R(C, M_2)$ for each $i > 1$, we see readily that $\operatorname{Tor}_i^R(C, M_3) = 0$ for each $i > 1$. For the remaining Tor-module, consider the following piece of the long exact sequence

$$0 \to \operatorname{Tor}_1^R(C, M_3) \to C\otimes_R M_1 \xrightarrow{C\otimes_R f} C\otimes_R M_2.$$

Apply $\operatorname{Hom}_R(C, -)$ to obtain the bottom exact sequence in the following commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & M_1 & \xrightarrow{f} & M_2 \\
 & & {\scriptstyle \gamma^C_{M_1}}\downarrow\cong & & {\scriptstyle \gamma^C_{M_2}}\downarrow\cong \\
\operatorname{Hom}(C, \operatorname{Tor}_1(C, M_3)) & \hookrightarrow & \operatorname{Hom}(C, C\otimes M_1) & \xrightarrow{\operatorname{Hom}(C, C\otimes f)} & \operatorname{Hom}(C, C\otimes M_2).
\end{array}$$

The top row is from our original sequence. Since f is injective, it follows that $\operatorname{Hom}_R(C, C\otimes f)$ is also injective, so $\operatorname{Hom}_R(C, \operatorname{Tor}_1^R(C, M_3)) = 0$. Now apply Corollary 2.1.17 to conclude that $\operatorname{Tor}_1^R(C, M_3) = 0$.

It follows that we have an exact sequence

$$0 \to C\otimes_R M_1 \xrightarrow{C\otimes_R f} C\otimes_R M_2 \xrightarrow{C\otimes_R g} C\otimes_R M_3 \to 0.$$

Consider the associated long exact sequence in $\operatorname{Ext}_R^i(C, -)$. As above, it is straightforward to show that the vanishing $\operatorname{Ext}_R^i(C, C\otimes_R M_1) = 0 = \operatorname{Ext}_R^i(C, C\otimes_R M_2)$ for all $i \geqslant 1$ implies $\operatorname{Ext}_R^i(C, C\otimes_R M_3) = 0$ for all $i \geqslant 1$. Finally, the remainder of the long exact sequence fits into the bottom row of the next commutative diagram

$$\begin{array}{ccccc} M_1 & \xrightarrow{\quad f \quad} & M_2 & \xrightarrow{\quad g \quad} & M_3 \\ {\scriptstyle \gamma^C_{M_1}}\big\downarrow\cong & & {\scriptstyle \gamma^C_{M_2}}\big\downarrow\cong & & {\scriptstyle \gamma^C_{M_3}}\big\downarrow \\ \operatorname{Hom}(C, C \otimes M_1) & \xrightarrow{\operatorname{Hom}(C,C\otimes f)} & \operatorname{Hom}(C, C \otimes M_2) & \xrightarrow{\operatorname{Hom}(C,C\otimes g)} & \operatorname{Hom}(C, C \otimes M_3) \end{array}$$

and a diagram chase shows that $\gamma^C_{M_3}$ is an isomorphism.

The other cases are verified similarly. □

Corollary 3.1.8. *Let C be a semidualizing R-module, and consider an exact sequence of R-module homomorphisms*

$$0 \to M_1 \to M_2 \to \cdots \to M_n \to 0.$$

(a) *If $M_i \in \mathcal{A}_C(R)$ for all $i \neq j$, then $M_j \in \mathcal{A}_C(R)$.*
(b) *If $M_i \in \mathcal{B}_C(R)$ for all $i \neq j$, then $M_j \in \mathcal{B}_C(R)$.*

Proof. By induction on n, using Proposition 3.1.7. □

Here is another frequently cited property of Foxby classes. Recall finite flat dimension from the appendix.

Proposition 3.1.9. *Let C be a fintely generated R-module. The following conditions are equivalent:*

(i) *The R-module C is semidualizing;*
(ii) *The class $\mathcal{A}_C(R)$ contains every R-module of finite flat dimension;*
(iii) *The class $\mathcal{A}_C(R)$ contains every flat R-module; and*
(iv) *The class $\mathcal{A}_C(R)$ contains a faithfully flat R-module.*

Proof. Let F be a flat R-module, and let P be a free resolution of C such that each P_i is finitely generated. Lemma A.1.2 provides an isomorphism of complexes

$$\operatorname{Hom}_R(P, C \otimes_R F) \cong \operatorname{Hom}_R(P, C) \otimes_R F$$

and it follows readily that there is an isomorphism

$$\operatorname{Ext}^i_R(C, C \otimes_R F) \cong \operatorname{Ext}^i_R(C, C) \otimes_R F \tag{3.1.9.1}$$

for each integer i. Also, there is a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\quad \gamma^C_F \quad} & \operatorname{Hom}_R(C, C \otimes_R F) \\ \big\downarrow\cong & & \cong\big\uparrow{\scriptstyle \omega_{CCF}} \\ R \otimes_R F & \xrightarrow{\chi^R_C \otimes_R F} & \operatorname{Hom}_R(C, C) \otimes_R F \end{array} \tag{3.1.9.2}$$

where the unspecified isomorphism is the tensor-cancellation isomorphism; see Definitions 2.1.5, 3.1.2 and A.1.1 for the other maps.

(i) $\Longrightarrow$ (iii) and (iv) $\Longrightarrow$ (i): The flatness of F implies that $\operatorname{Tor}^R_i(C, F) = 0$ for all $i \geqslant 1$. From the isomorphism (3.1.9.1) we know that if $\operatorname{Ext}^i_R(C, C) = 0$ for all $i \geqslant 1$, then $\operatorname{Ext}^i_R(C, C \otimes_R F) = 0$ for all $i \geqslant 1$; and the converse holds when F is faithfully flat. From the diagram (3.1.9.2) we know that if χ^R_C is an isomorphism, then γ^C_F is an isomorphism; and the converse holds when F is faithfully flat. The desired implications now follow.

(iii) $\Longrightarrow$ (ii) Assume that C is semidualizing and that $\mathrm{fd}_R(M)$ is finite. Consider a flat resolution

$$0 \to F_n \to \cdots \to F_0 \to M \to 0.$$

Since each F_i is flat, condition (iii) implies that $F_0, \ldots, F_n \in \mathcal{A}_C(R)$. Corollary 3.1.8 implies that $M \in \mathcal{A}_C(R)$.

The implication (ii) $\Longrightarrow$ (iv) is routine. □

The proof of the next result is similar to the proof of the previous result.

Proposition 3.1.10. *Let C be a fintely generated R-module. The following conditions are equivalent:*

(i) *The R-module C is semidualizing;*
(ii) *The class $\mathcal{B}_C(R)$ contains every R-module of finite injective dimension;*
(iii) *The class $\mathcal{B}_C(R)$ contains every injective R-module; and*
(iv) *The class $\mathcal{B}_C(R)$ contains a faithfully injective R-module.*

Corollary 3.1.11. *Let C be a semidualizing R-module and let M be an R-module. Fix a flat (e.g., projective) resolution F of M and an injective resolution I of M.*

(a) *One has $M \in \mathcal{A}_C(R)$ if and only if $\mathrm{Im}(\partial_i^F) \in \mathcal{A}_C(R)$ for some (equivalently, every) $i \geqslant 0$.*
(b) *One has $M \in \mathcal{B}_C(R)$ if and only if $\mathrm{Im}(\partial_i^I) \in \mathcal{A}_C(R)$ for some (equivalently, every) $i \geqslant 0$.*

Here are two lemmas for later use.

Lemma 3.1.12. *Let C be a semidualizing R-module, and fix an exact sequence*

$$X = \cdots \xrightarrow{\partial_{i+1}} X_i \xrightarrow{\partial_i} X_{i-1} \xrightarrow{\partial_{i-1}} \cdots.$$

(a) *Assume that each $X_i \in \mathcal{A}_C(R)$ and that some $\mathrm{Im}(\partial_j) \in \mathcal{A}_C(R)$. Then every $\mathrm{Im}(\partial_i) \in \mathcal{A}_C(R)$, and the induced sequence $C \otimes_R X$ is exact.*
(b) *Assume that each $X_i \in \mathcal{B}_C(R)$ and that some $\mathrm{Im}(\partial_j) \in \mathcal{B}_C(R)$. Then every $\mathrm{Im}(\partial_i) \in \mathcal{B}_C(R)$, and the induced sequence $\mathrm{Hom}_R(C, X)$ is exact.*

Proof. (a) Set $M_i = \mathrm{Im}(\partial_i)$ for each i, and consider the following exact sequence:

$$0 \to M_{i+1} \to X_i \to M_i \to 0. \tag{3.1.12.1}$$

Since each $X_i \in \mathcal{A}_C(R)$ and at least one $M_j \in \mathcal{A}_C(R)$, a straightforward induction argument using Proposition 3.1.7(a) shows that every M_i is in $\mathcal{A}_C(R)$. Hence, applying the functor $C \otimes_R -$ to the exact sequence (3.1.12.1) yields an exact sequence

$$0 \to C \otimes_R M_{i+1} \to C \otimes_R X_i \to C \otimes_R M_i \to 0.$$

It follows readily that the sequence $C \otimes_R X$ is exact, as desired.

The proof of part (b) is similar. □

Lemma 3.1.13. *Let C be a semidualizing R-module, and let M and N be R-modules.*

(a) *If $\mathrm{Tor}_i^R(C, M) = 0$ for all $i \geqslant 1$ (e.g., if $M \in \mathcal{A}_C(R)$) and $N \in \mathcal{A}_C(R)$, then $\mathrm{Ext}_R^i(C \otimes_R M, C \otimes_R N) \cong \mathrm{Ext}_R^i(M, N)$ for all $i \geqslant 0$.*
(b) *If $\mathrm{Ext}_R^i(C, N) = 0$ for all $i \geqslant 1$ (e.g., if $N \in \mathcal{B}_C(R)$) and $M \in \mathcal{B}_C(R)$, then $\mathrm{Ext}_R^i(\mathrm{Hom}_R(C, M), \mathrm{Hom}_R(C, N)) \cong \mathrm{Ext}_R^i(M, N)$ for all $i \geqslant 0$.*

(c) *If* $\operatorname{Tor}_i^R(C,M)=0$ *for all* $i\geqslant 1$ *(e.g., if* $M\in\mathcal{A}_C(R)$*) and* $N\in\mathcal{B}_C(R)$*, then* $\operatorname{Tor}_i^R(C\otimes_R M,\operatorname{Hom}_R(C,N))\cong\operatorname{Tor}_i^R(M,N)$ *for all* $i\geqslant 0$.

Proof. We prove part (a); the proofs of (b) and (c) are similar.

First, the following sequence deals with the case $i=0$:

$$\operatorname{Hom}_R(C\otimes_R M,C\otimes_R N)\cong\operatorname{Hom}_R(M,\operatorname{Hom}_R(C,C\otimes_R N))\cong\operatorname{Hom}_R(M,N).$$

The first isomorphism is Hom-tensor adjointness, and the second one is from the assumption $N\in\mathcal{A}_C(R)$.

Second, we consider the case where M is projective and $i\geqslant 1$. For this case, it suffices to show that $\operatorname{Ext}_R^i(C\otimes_R M,C\otimes_R N)=0$. If I is an injective resolution of $C\otimes_R N$, then the first and fourth steps in the next sequence are by definition:

$$\begin{aligned}\operatorname{Ext}_R^i(C\otimes_R M,C\otimes_R N)&\cong\operatorname{H}_{-i}(\operatorname{Hom}_R(C\otimes_R M,I))\\&\cong\operatorname{H}_{-i}(\operatorname{Hom}_R(M,\operatorname{Hom}_R(C,I)))\\&\cong\operatorname{Hom}_R(M,\operatorname{H}_{-i}(\operatorname{Hom}_R(C,I)))\\&\cong\operatorname{Hom}_R(M,\operatorname{Ext}_R^i(C,C\otimes_R N))\\&=0.\end{aligned}$$

The second step is Hom-tensor adjointness. The third step follows from the assumption that M is projective, and the last step is justified by the condition $N\in\mathcal{A}_C(R)$.

For the remainder of the proof, fix an exact sequence

$$0\to M'\xrightarrow{f}P\xrightarrow{g}M\to 0 \tag{3.1.13.1}$$

such that P is projective. From the long exact sequence in $\operatorname{Tor}_R(C,-)$, the condition $\operatorname{Tor}_i^R(C,M)=0=\operatorname{Tor}_i^R(C,P)$ for all $i\geqslant 1$ implies that $\operatorname{Tor}_i^R(C,M')=0$ for all $i\geqslant 1$.

Finally, we prove the result by induction on $i\geqslant 1$.

Base case: $i=1$. Since $\operatorname{Tor}_1^R(C,M)=0$, the long exact sequence in $\operatorname{Tor}^R(C,-)$ associated to (3.1.13.1) yields an exact sequence

$$0\to C\otimes_R M'\xrightarrow{C\otimes_R f}C\otimes_R P\xrightarrow{C\otimes_R g}C\otimes_R M\to 0. \tag{3.1.13.2}$$

Using the vanishing guaranteed by the second part of this proof, the long exact sequence in $\operatorname{Ext}_R(-, C \otimes_R N)$ implies that the third column of the following commutative diagram is exact:

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Hom}(M,N) & \xrightarrow{\cong} & \operatorname{Hom}(M,\operatorname{Hom}(C,C\otimes N)) & \xrightarrow{\cong} & \operatorname{Hom}(C\otimes M, C\otimes N) \\
\downarrow{\scriptstyle \operatorname{Hom}(g,N)} & & \downarrow{\scriptstyle \operatorname{Hom}(g,\operatorname{Hom}(C,C\otimes N))} & & \downarrow{\scriptstyle \operatorname{Hom}(C\otimes g,C\otimes N)} \\
\operatorname{Hom}(P,N) & \xrightarrow{\cong} & \operatorname{Hom}(P,\operatorname{Hom}(C,C\otimes N)) & \xrightarrow{\cong} & \operatorname{Hom}(C\otimes P, C\otimes N) \\
\downarrow{\scriptstyle \operatorname{Hom}(f,N)} & & \downarrow{\scriptstyle \operatorname{Hom}(f,\operatorname{Hom}(C,C\otimes N))} & & \downarrow{\scriptstyle \operatorname{Hom}(C\otimes f,C\otimes N)} \\
\operatorname{Hom}(M',N) & \xrightarrow{\cong} & \operatorname{Hom}(M',\operatorname{Hom}(C,C\otimes N)) & \xrightarrow{\cong} & \operatorname{Hom}(C\otimes M', C\otimes N) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Ext}^1(M,N) & & \operatorname{Ext}^1(M,\operatorname{Hom}(C,C\otimes N)) & & \operatorname{Ext}^1(C\otimes M, C\otimes N) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}
$$

The first and second columns are long exact sequences associated to (3.1.13.1). Half of the unspecified horizontal isomorphisms are induced by the isomorphism $\gamma_n^C \colon N \xrightarrow{\cong} \operatorname{Hom}_R(C, C \otimes_R N)$, and the others are Hom-tensor adjointness. The commutativity of the diagram yields the second step in the next sequence:

$$
\begin{aligned}
\operatorname{Ext}^1(M,N) &\cong \operatorname{Coker}(\operatorname{Hom}(f,N)) \\
&\cong \operatorname{Coker}(\operatorname{Hom}(C\otimes f, C\otimes N)) \\
&\cong \operatorname{Ext}^1(C\otimes M, C\otimes N).
\end{aligned}
$$

The other steps follow from the exactness of the first and third columns of the diagram.

Induction step: Assume that $i \geqslant 2$ and that $\operatorname{Ext}_R^{i-1}(C \otimes_R M_1, C \otimes_R N) \cong \operatorname{Ext}_R^{i-1}(M_1, N)$ for all R-modules M_1 such that $\operatorname{Tor}_j^R(C, M_1) = 0$ for all $j \geqslant 1$. Long exact sequences associated to (3.1.13.1) and (3.1.13.2) yield the next exact sequences:

$$
0 \longrightarrow \operatorname{Ext}_R^{i-1}(M',N) \longrightarrow \operatorname{Ext}_R^{i}(M,N) \longrightarrow 0
$$

$$
0 \longrightarrow \operatorname{Ext}_R^{i-1}(C\otimes_R M', C\otimes_R N) \longrightarrow \operatorname{Ext}_R^{i}(C\otimes_R M, C\otimes_R N) \longrightarrow 0.
$$

These sequences explain the first and third isomorphisms in the next sequence:

$$
\begin{aligned}
\operatorname{Ext}_R^i(M,N) &\cong \operatorname{Ext}_R^{i-1}(M',N) \\
&\cong \operatorname{Ext}_R^{i-1}(C\otimes_R M', C\otimes_R N) \\
&\cong \operatorname{Ext}_R^{i}(C\otimes_R M, C\otimes_R N).
\end{aligned}
$$

The second isomorphism is from the inductive hypothesis. □

3.2. Foxby Equivalence

Here is Foxby equivalence.

Theorem 3.2.1. *Let C be a semidualizing R-module.*

(a) *An R-module M is in $\mathcal{A}_C(R)$ if and only if $C \otimes_R M$ is in $\mathcal{B}_C(R)$.*
(b) *An R-module M is in $\mathcal{B}_C(R)$ if and only if $\operatorname{Hom}_R(C, M)$ is in $\mathcal{A}_C(R)$.*
(c) *The functors $C \otimes_R -\colon \mathcal{A}_C(R) \to \mathcal{B}_C(R)$ and $\operatorname{Hom}_R(C, -)\colon \mathcal{B}_C(R) \to \mathcal{A}_C(R)$ are inverse equivalences.*

Proof. (a) We begin by noting the readily verified equality

$$\xi^C_{C\otimes_R M} \circ (C \otimes_R \gamma^C_M) = \mathrm{id}_{C\otimes_R M}\colon C \otimes_R M \to C \otimes_R M. \tag{3.2.1.1}$$

For the first implication, assume that $M \in \mathcal{A}_C(R)$. By definition, this means that $\operatorname{Tor}^R_i(C, M) = 0 = \operatorname{Ext}^i_R(C, C \otimes_R M)$ for all $i \geqslant 1$, and the natural map $\gamma^C_M\colon M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism. To show that $C \otimes_R M$ is in $\mathcal{B}_C(R)$, we need to verify the following three conditions.

(1) $\operatorname{Ext}^i_R(C, C \otimes_R M) = 0$ for all $i \geqslant 1$: this is true by assumption.

(2) $\operatorname{Tor}^R_i(C, \operatorname{Hom}_R(C, C \otimes_R M)) = 0$ for all $i \geqslant 1$: this follows from our assumptions because $\operatorname{Tor}^R_i(C, \operatorname{Hom}_R(C, C \otimes_R M)) \cong \operatorname{Tor}^R_i(C, M) = 0$ for each $i \geqslant 1$.

(3) The natural evaluation map $\xi^C_{C\otimes_R M}\colon C \otimes_R \operatorname{Hom}_R(C, C \otimes_R M) \to C \otimes_R M$ is an isomorphism: Since γ^C_M is an isomorphism by assumption, it follows that $C \otimes_R \gamma^C_M$ is an isomorphism. Equation (3.2.1.1) implies that $\xi^C_{C\otimes_R M} = (C \otimes_R \gamma^C_M)^{-1}$ is also an isomorphism.

For the converse, assume that $C \otimes_R M$ is in $\mathcal{B}_C(R)$. Since this implies, in particular, that $\operatorname{Ext}^i_R(C, C \otimes_R M) = 0$ for all $i \geqslant 1$, we need only check the next two conditions to show that $M \in \mathcal{A}_C(R)$:

(4) The natural map $\gamma^C_M\colon M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism: By assumption, the map $\xi^C_{C\otimes_R M}$ is an isomorphism. Using equation (3.2.1.1) as in (3) above, we conclude that $C \otimes_R \gamma^C_M$ is an isomorphism. Set $N = \operatorname{Coker}(\gamma^C_M)$ and consider the exact sequence

$$M \xrightarrow{\gamma^C_M} \operatorname{Hom}_R(C, C \otimes_R M) \to N \to 0.$$

The right-exactness of $C \otimes_R -$ yields the next exact sequence

$$C \otimes_R M \xrightarrow[\cong]{C\otimes_R \gamma^C_M} C \otimes_R \operatorname{Hom}_R(C, C \otimes_R M) \to C \otimes_R N \to 0.$$

It follows that $C \otimes_R N = 0$, so $N = 0$ by Corollary 2.1.17. Hence γ^C_M is surjective. Set $K = \operatorname{Ker}(\gamma^C_M)$ and consider the exact sequence

$$0 \to K \to M \xrightarrow{\gamma^C_M} \operatorname{Hom}_R(C, C \otimes_R M) \to 0.$$

The long exact sequence in $\operatorname{Tor}^R(C, -)$ yields the next exact sequence

$$\underbrace{\operatorname{Tor}^R_1(C, \operatorname{Hom}_R(C, C \otimes_R M))}_{=0} \to C \otimes K \to C \otimes M \xrightarrow[\cong]{C\otimes_R \gamma^C_M} C \otimes \operatorname{Hom}(C, C \otimes M) \to 0.$$

It follows that $C \otimes_R K = 0$, and so $K = 0$ by Corollary 2.1.17. Hence γ^C_M is also injective.

(5) $\operatorname{Tor}^R_i(C, M) = 0$ for all $i \geqslant 1$: this follows from our assumptions along with item (4) because $\operatorname{Tor}^R_i(C, M) \cong \operatorname{Tor}^R_i(C, \operatorname{Hom}_R(C, C \otimes_R M)) = 0$ for all $i \geqslant 1$.

This completes the proof of part (a). The proof of part (b) is similar, and part (c) follows from parts (a) and (b). □

Corollary 3.2.2. *Let C be a semidualizing R-module.*

(a) *If M is an R-module of finite flat dimension (e.g., if M is flat or projective), then $C \otimes_R M \in \mathcal{B}_C(R)$. In particular $C \in \mathcal{B}_C(R)$.*
(b) *If M is an R-module of finite injective dimension (e.g., if M is injective), then $\operatorname{Hom}_R(C, M) \in \mathcal{A}_C(R)$.*

Proof. Part (a) follows from Proposition 3.1.9 and Theorem 3.2.1(a), and similarly for part (b). □

Proposition 3.2.3. *Let C be a semidualizing R-module. Let M be an R-module, and fix a positive integer n.*

(a) *One has $\operatorname{fd}_R(M) \leqslant n$ if and only if $C \otimes_R M$ admits a bounded resolution $0 \to C \otimes_R F_n \to \cdots \to C \otimes_R F_0 \to C \otimes_R M \to 0$ with each F_i flat.*
(b) *One has $\operatorname{fd}_R(\operatorname{Hom}_R(C, M)) \leqslant n$ if and only if M admits a bounded resolution $0 \to C \otimes_R F_n \to \cdots \to C \otimes_R F_0 \to M \to 0$ with each F_i flat.*
(c) *If M admits a bounded resolution $0 \to C \otimes_R F_n \to \cdots \to C \otimes_R F_0 \to M \to 0$ with each F_i flat, then $M \in \mathcal{B}_C(R)$.*

Proof. Step 1. Assume first that $\operatorname{fd}_R(M) \leqslant n$ and fix a bounded resolution

$$0 \to F_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$$

with each F_i flat. Proposition 3.1.9 implies that each module in this sequence is in $\mathcal{A}_C(R)$. Lemma 3.1.12(a) implies that the induced sequence

$$0 \to C \otimes_R F_n \xrightarrow{C \otimes_R \partial_n} \cdots \xrightarrow{C \otimes_R \partial_1} C \otimes_R F_0 \xrightarrow{C \otimes_R \partial_0} C \otimes_R M \to 0$$

is exact, as desired.

Step 2. Assume next that M admits a bounded resolution

$$0 \to C \otimes_R F_n \to \cdots \to C \otimes_R F_0 \to M \to 0 \tag{3.2.3.1}$$

with each F_i flat. Since each $C \otimes_R F_i$ is in $\mathcal{B}_C(R)$ by Corollary 3.2.2(a), we conclude from Corollary 3.1.8(b) that $M \in \mathcal{B}_C(R)$. Hence, Theorem 3.2.1(b) implies that $\operatorname{Hom}_R(C, M) \in \mathcal{A}_C(R)$. Since each module in the sequence (3.2.3.1) is in $\mathcal{B}_C(R)$, Lemma 3.1.12(b) implies that the next induced sequence is exact:

$$0 \to \underbrace{\operatorname{Hom}_R(C, C \otimes F_n)}_{\cong F_n} \to \cdots \to \underbrace{\operatorname{Hom}_R(C, C \otimes F_0)}_{\cong F_0} \to \operatorname{Hom}_R(C, M) \to 0.$$

The isomorphisms are from the condition $F_i \in \mathcal{A}_C(R)$. This sequence shows that $\operatorname{fd}_R(\operatorname{Hom}_R(C, M)) \leqslant n$, as desired.

Step 3. Assume next that $\operatorname{fd}_R(\operatorname{Hom}_R(C, M)) \leqslant n$. Proposition 3.1.9 implies that $\operatorname{Hom}_R(C, M) \in \mathcal{A}_C(R)$, so we have $M \in \mathcal{B}_C(R)$ by Theorem 3.2.1(b). Step 1 implies that $C \otimes_R \operatorname{Hom}_R(C, M)$ admits a bounded resolution

$$0 \to C \otimes_R F_n \to \cdots \to C \otimes_R F_0 \to \underbrace{C \otimes_R \operatorname{Hom}_R(C, M)}_{\cong M} \to 0$$

where each F_i is flat. The isomorphism $M \cong C \otimes_R \operatorname{Hom}_R(C, M)$ is from the condition $M \in \mathcal{B}_C(R)$.

Step 4. Assume that $C \otimes_R M$ admits a bounded resolution

$$0 \to C \otimes_R F_n \to \cdots \to C \otimes_R F_0 \to C \otimes_R M \to 0$$

with each F_i flat. An argument as in Step 3 shows that $\operatorname{fd}_R(M) \leqslant n$. $\square$

Remark 3.2.4. Proposition 3.2.3 augments Theorem 3.2.1 as follows. Let $\mathcal{F}(R)_{\leqslant n}$ denote the class of all R-modules M such that $\operatorname{fd}_R(M) \leqslant n$, and let $\mathcal{F}_C(R)_{\leqslant n}$ denote the class of all R-modules N that admit a bounded resolution

$$0 \to C \otimes_R F_n \to \cdots \to C \otimes_R F_0 \to N \to 0$$

with each F_i flat.

Theorem 3.2.1 shows that the functors $C \otimes_R -$ and $\operatorname{Hom}_R(C, -)$ provide inverse equivalences between the Auslander and Bass classes, as we indicate in the third row of the following diagram:

$$\begin{array}{ccc}
\mathcal{P}(R)_{\leqslant n} & \underset{\operatorname{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}}\ \sim & \mathcal{P}_C(R)_{\leqslant n} \\
\downarrow & & \downarrow \\
\mathcal{F}(R)_{\leqslant n} & \underset{\operatorname{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}}\ \sim & \mathcal{F}_C(R)_{\leqslant n} \\
\downarrow & & \downarrow \\
\mathcal{A}_C(R) & \underset{\operatorname{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}}\ \sim & \mathcal{B}_C(R) \\
\uparrow & & \uparrow \\
\mathcal{I}_C(R)_{\leqslant n} & \underset{\operatorname{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}}\ \sim & \mathcal{I}(R)_{\leqslant n}.
\end{array}$$

The class $\mathcal{F}(R)_{\leqslant n}$ is contained in $\mathcal{A}_C(R)$ by Proposition 3.1.9. From Proposition 3.2.3, we know that the image of $\mathcal{F}(R)_{\leqslant n}$ in $\mathcal{B}_C(R)$ under the equivalence is exactly $\mathcal{F}_C(R)_{\leqslant n}$ and that $C \otimes_R -$ and $\operatorname{Hom}_R(C, -)$ provide inverse equivalences between these classes; this is indicated in the second row of the preceding diagram. The remaining aspects of the diagram are explained by the next two results; they are proved like Proposition 3.2.3.

Proposition 3.2.5. *Let C be a semidualizing R-module.*

(a) *One has $\operatorname{pd}_R(M) \leqslant n$ if and only if $C \otimes_R M$ admits a bounded resolution $0 \to C \otimes_R P_n \to \cdots \to C \otimes_R P_0 \to C \otimes_R M \to 0$ with each P_i projective.*
(b) *One has $\operatorname{pd}_R(\operatorname{Hom}_R(C, M)) \leqslant n$ if and only if M admits a bounded resolution $0 \to C \otimes_R P_n \to \cdots \to C \otimes_R P_0 \to M \to 0$ with each P_i projective.*
(c) *If M admits a bounded resolution $0 \to C \otimes_R P_n \to \cdots \to C \otimes_R P_0 \to M \to 0$ with each P_i projective, then $M \in \mathcal{B}_C(R)$.*

Proposition 3.2.6. *Let C be a semidualizing R-module.*

(a) *One has $\operatorname{id}_R(M) \leqslant n$ if and only if $\operatorname{Hom}_R(C, M)$ admits a bounded resolution $0 \to \operatorname{Hom}_R(C, M) \to \operatorname{Hom}_R(C, J^0) \to \cdots \to \operatorname{Hom}_R(C, J^n) \to 0$ with each J^i injective.*
(b) *One has $\operatorname{id}_R(C \otimes_R M) \leqslant n$ if and only if M admits a bounded resolution $0 \to M \to \operatorname{Hom}_R(C, J^0) \to \cdots \to \operatorname{Hom}_R(C, J^n) \to 0$ with each J^i injective.*
(c) *If M has a resolution $0 \to M \to \operatorname{Hom}_R(C, J^0) \to \cdots \to \operatorname{Hom}_R(C, J^n) \to 0$ with each J^i injective, then $M \in \mathcal{A}_C(R)$.*

3.3. Other Operators on Foxby Classes

Here is a dualizing equivalence for Foxby classes. See the appendix for some information about faithfully injective modules.

Proposition 3.3.1. *Let C be a semidualizing R-module, let E be an injective R-module, and let M be an R-module.*

(a) *If M is in $\mathcal{B}_C(R)$, then $\operatorname{Hom}_R(M,E)$ is in $\mathcal{A}_C(R)$.*
(b) *If M is in $\mathcal{A}_C(R)$, then $\operatorname{Hom}_R(M,E)$ is in $\mathcal{B}_C(R)$.*
(c) *The converses of* (a) *and* (b) *hold when E is faithfully injective.*

PROOF. We prove part (a), and we prove its converse when E is faithfully injective.

We begin by recalling the following isomomorphism for each $i \geqslant 0$:

$$\operatorname{Tor}_i^R(C, \operatorname{Hom}_R(M,E)) \cong \operatorname{Hom}_R(\operatorname{Ext}_R^i(C,M),E).$$

To explain the isomorphism, let P be a projective resolution of C such that each P_i is finitely generated. The Hom-evaluation isomorphism from Lemma A.1.3 yields an isomorphism of complexes

$$P \otimes_R \operatorname{Hom}_R(M,E) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(P,M),E)$$

and this explains the second isomorphism in the next sequence

$$\begin{aligned}\operatorname{Tor}_i^R(C, \operatorname{Hom}_R(M,E)) &\cong \operatorname{H}_i(P \otimes_R \operatorname{Hom}_R(M,E))\\ &\cong \operatorname{H}_i(\operatorname{Hom}_R(\operatorname{Hom}_R(P,M),E))\\ &\cong \operatorname{Hom}_R(\operatorname{H}_{-i}(\operatorname{Hom}_R(P,M)),E)\\ &\cong \operatorname{Hom}_R(\operatorname{Ext}_R^i(C,M),E).\end{aligned}$$

The first and fourth isomorphisms are by definition, and the third isomorphism follows from the injectivity of E. One concludes that, if $\operatorname{Ext}_R^i(C,M) = 0$, then $\operatorname{Tor}_i^R(C, \operatorname{Hom}_R(M,E)) = 0$; and the converse holds when E is faithfully injective.

The case $i = 0$ in the previous display reads as

$$C \otimes_R \operatorname{Hom}_R(M,E) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(C,M),E)$$

and this explains the first isomorphism in the next sequence

$$\begin{aligned}\operatorname{Ext}_R^i(C, C \otimes_R \operatorname{Hom}_R(M,E)) &\cong \operatorname{Ext}_R^i(C, \operatorname{Hom}_R(\operatorname{Hom}_R(C,M),E))\\ &\cong \operatorname{Hom}_R(\operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C,M)),E).\end{aligned}$$

The second isomorphism follows as in the previous paragraph, using Hom-tensor adjointness in place of Hom-evaluation. From these isomorphisms, one concludes that, if $\operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C,M)) = 0$, then $\operatorname{Ext}_R^i(C, C \otimes_R \operatorname{Hom}_R(M,E)) = 0$; and the converse holds when E is faithfully injective.

It is routine to show that the following diagram commutes

$$\begin{array}{ccc} \operatorname{Hom}_R(M,E) & \xrightarrow{\gamma^C_{\operatorname{Hom}(M,E)}} & \operatorname{Hom}_R(C, C \otimes_R \operatorname{Hom}_R(M,E)) \\ {\scriptstyle \operatorname{Hom}(\xi^C_M,E)}\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}(C,\theta_{CME})}\ \cong \\ \operatorname{Hom}_R(C \otimes_R \operatorname{Hom}_R(C,M),E) & \xleftarrow{\cong} & \operatorname{Hom}_R(C, \operatorname{Hom}_R(\operatorname{Hom}_R(C,M),E)). \end{array}$$

where the unspecified isomorphism is Hom-tensor adjointness. From this, one sees that, if ξ_M^C is an isomorphism, then so is $\gamma^C_{\operatorname{Hom}(M,E)}$; and the converse holds when E is faithfully injective. This establishes (a) and the first half of (c).

The proofs of (b) and the rest of (c) are similar. □

The next results are proved similarly to the previous one. Note that Proposition 3.3.3 uses Lemma A.1.4.

Proposition 3.3.2. *Let C be a semidualizing R-module, let F be a flat R-module, and let M be an R-module.*

(a) *If M is in $\mathcal{A}_C(R)$, then $M \otimes_R F$ is in $\mathcal{A}_C(R)$.*
(b) *If M is in $\mathcal{B}_C(R)$, then $M \otimes_R F$ is in $\mathcal{B}_C(R)$.*
(c) *The converses of* (a) *and* (b) *hold when F is faithfully flat.*

Proposition 3.3.3. *Let C be a semidualizing R-module, let P be a projective R-module, and let M be an R-module.*

(a) *If M is in $\mathcal{A}_C(R)$, then* $\operatorname{Hom}_R(P,M)$ *is in $\mathcal{A}_C(R)$.*
(b) *If M is in $\mathcal{B}_C(R)$, then* $\operatorname{Hom}_R(P,M)$ *is in $\mathcal{B}_C(R)$.*
(c) *The converses of* (a) *and* (b) *hold when P is faithfully projective.*

The following examples explain the need for the faithful hypotheses in the converses of the previous results.

Example 3.3.4. Let $(R,\mathfrak{m},k)$ be a local Cohen-Macaulay ring that admits a semidualizing module $C \ncong R$. Assume that $\dim(R) \geqslant 1$ and let $z \in \mathfrak{m}$ be an R-regular element. (For instance, let k be a field and set $R_0 = k[X,Y]/(X,Y)^2$ and $R = R_0[\![Z]\!]$. Then $\dim(R) = 1$. The module $C_0 = \operatorname{Hom}_k(R_0,k)$ is dualizing for R_0 and such that $C_0 \ncong R_0$; see Example 2.3.1. Proposition 2.2.1 implies that the R-module $C = R \otimes_{R_0} C_0$ is semidualizing for R. (Actually, it is dualizing, but we do not need that fact.) Since $C_0 \ncong R_0$, Corollary 2.1.14 implies that C_0 is not cyclic. Hence, the R-module C is not cyclic, and it follows that $C \ncong R$.)

Corollary 2.2.8 implies that $\operatorname{pd}_R(C) = \infty$, so $\operatorname{Tor}_i^R(C,k) \neq 0$ and $\operatorname{Ext}^i_R(C,k) \neq 0$ for all $i \geqslant 0$. In particular, we have $k \notin \mathcal{A}_C(R)$ and $k \notin \mathcal{B}_C(R)$. The localization R_z is a flat R-module. Since $zk = 0$, we have $R_z \otimes_R k = 0 \in \mathcal{A}_C(R) \cap \mathcal{B}_C(R)$. Thus, the condition "faithfully flat" is necessary in Proposition 3.3.2(c).

Let $\mathfrak{p} \subset R$ be a nonmaximal prime, and set $J = E_R(R/\mathfrak{p})$. it follows that $\operatorname{Hom}_R(k,J) = 0 \in \mathcal{A}_C(R) \cap \mathcal{B}_C(R)$. Thus, the condition "faithfully injective" is necessary in Proposition 3.3.1(c).

The previous example will not work to explain the need for "faithfully projective" in Proposition 3.3.3(c). Indeed, when R is local, an R-module is projective if and only if it is free, and it follows that a module is projective if and only if it is faithfully projective. Hence, we need a nonlocal example.

Example 3.3.5. For $i = 1,2$ let $(R_i,\mathfrak{m}_i,k_i)$ be a local ring with a semidualizing module C_i. Assume that $C_1 \ncong R_1$. Set $R = R_1 \times R_2$ and consider the semidualizing R-module $C = C_1 \times C_2 \ncong R$; see Proposition 2.3.4.

The R-module $P = 0 \times R_2$ is projective. As in Example 3.3.4, the R-module $M = k_1 \times 0$ is not in $\mathcal{A}_C(R)$ because

$$\begin{aligned}\operatorname{Tor}_i^R(C,M) &= \operatorname{Tor}^{R_1\times R_2}(C_1\times C_2, k_1\times 0) \cong \operatorname{Tor}_i^{R_1}(C_1,k_1)\times\operatorname{Tor}_i^{R_2}(C_2,0)\\ &\cong \operatorname{Tor}_i^{R_1}(C_1,k_1) \neq 0.\end{aligned}$$

A similar computation of $\mathrm{Ext}^i_R(C,M)$ shows that $M \notin \mathcal{B}_C(R)$. However, we have

$$\mathrm{Hom}_R(P,M) = \mathrm{Hom}_{R_1\times R_2}(0\times R_2, k_1\times 0) \cong \mathrm{Hom}_{R_1}(0,k_1)\times \mathrm{Hom}_2(R_2,0) = 0$$

which is in $\mathcal{A}_C(R)\cap\mathcal{B}_C(R)$.

Next, we document some interactions between the operations thus far considered. The first one is a consequence of Hom-tensor adjointness.

Proposition 3.3.6. *Let C be a semidualizing R-module, and let I be an injective R-module. Then the following diagram commutes:*

$$\begin{CD}
\mathcal{A}_C(R) @>{\mathrm{Hom}_R(-,I)}>> \mathcal{B}_C(R) \\
@V{C\otimes_R -}VV @VV{\mathrm{Hom}_R(C,-)}V \\
\mathcal{B}_C(R) @>{\mathrm{Hom}_R(-,I)}>> \mathcal{A}_C(R)
\end{CD}
\qquad
\begin{CD}
\mathcal{B}_C(R) @>{\mathrm{Hom}_R(-,I)}>> \mathcal{A}_C(R) \\
@V{\mathrm{Hom}_R(C,-)}VV @VV{C\otimes_R -}V \\
\mathcal{A}_C(R) @>{\mathrm{Hom}_R(-,I)}>> \mathcal{B}_C(R).
\end{CD}$$

Proposition 3.3.7. *Let C be a semidualizing R-module, and let F be a flat R-module. Then the following diagram commutes:*

$$\begin{CD}
\mathcal{A}_C(R) @>{-\otimes_R F}>> \mathcal{A}_C(R) \\
@V{C\otimes_R -}VV @VV{C\otimes_R -}V \\
\mathcal{B}_C(R) @>{-\otimes_R F}>> \mathcal{B}_C(R)
\end{CD}
\qquad
\begin{CD}
\mathcal{B}_C(R) @>{-\otimes_R F}>> \mathcal{B}_C(R) \\
@V{\mathrm{Hom}_R(C,-)}VV @VV{\mathrm{Hom}_R(C,-)}V \\
\mathcal{A}_C(R) @>{-\otimes_R F}>> \mathcal{A}_C(R).
\end{CD}$$

Proposition 3.3.8. *Let C be a semidualizing R-module, and let P be a projective R-module. Then the following diagram commutes:*

$$\begin{CD}
\mathcal{A}_C(R) @>{\mathrm{Hom}_R(P,-)}>> \mathcal{A}_C(R) \\
@V{C\otimes_R -}VV @VV{C\otimes_R -}V \\
\mathcal{B}_C(R) @>{\mathrm{Hom}_R(P,-)}>> \mathcal{B}_C(R)
\end{CD}
\qquad
\begin{CD}
\mathcal{B}_C(R) @>{\mathrm{Hom}_R(P,-)}>> \mathcal{B}_C(R) \\
@V{\mathrm{Hom}_R(C,-)}VV @VV{\mathrm{Hom}_R(C,-)}V \\
\mathcal{A}_C(R) @>{\mathrm{Hom}_R(P,-)}>> \mathcal{A}_C(R).
\end{CD}$$

Remark 3.3.9. Let C be a semidualizing R-module, and let I be an injective R-module. Using Theorem 3.2.1 and Proposition 3.3.6 one sees that the following diagrams commute:

$$\begin{CD}
\mathcal{A}_C(R) @>{\mathrm{Hom}_R(-,I)}>> \mathcal{B}_C(R) \\
@V{C\otimes_R -}VV @AA{C\otimes_R -}A \\
\mathcal{B}_C(R) @>>{\mathrm{Hom}_R(-,I)}> \mathcal{A}_C(R)
\end{CD}
\qquad
\begin{CD}
\mathcal{B}_C(R) @>{\mathrm{Hom}_R(-,I)}>> \mathcal{A}_C(R) \\
@V{\mathrm{Hom}_R(C,-)}VV @AA{\mathrm{Hom}_R(C,-)}A \\
\mathcal{A}_C(R) @>>{\mathrm{Hom}_R(-,I)}> \mathcal{B}_C(R).
\end{CD}$$

Similar diagrams follow from Propositions 3.3.7 and 3.3.8.

Remark 3.3.10. Let C be a semidualizing R-module, and let F,G be flat R-modules. It is straightforward to show that the following diagrams commute:

$$\begin{array}{ccc}
\mathcal{A}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{A}_C(R) \\
{\scriptstyle -\otimes_R G}\downarrow & \searrow^{-\otimes_R(F\otimes_R G)} & \downarrow{\scriptstyle -\otimes_R G} \\
\mathcal{A}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{A}_C(R)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{B}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{B}_C(R) \\
{\scriptstyle -\otimes_R G}\downarrow & \searrow^{-\otimes_R(F\otimes_R G)} & \downarrow{\scriptstyle -\otimes_R G} \\
\mathcal{B}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{B}_C(R).
\end{array}$$

Similarly, if P and Q are projective R-modules, then the next diagrams commute:

$$\begin{tikzcd} \mathcal{A}_C(R) \arrow[r, "\mathrm{Hom}(P,-)"] \arrow[d, "\mathrm{Hom}(Q,-)"'] \arrow[dr, "\mathrm{Hom}(P\otimes Q,-)"'] & \mathcal{A}_C(R) \arrow[d, "\mathrm{Hom}(Q,-)"] \\ \mathcal{A}_C(R) \arrow[r, "\mathrm{Hom}(P,-)"'] & \mathcal{A}_C(R) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}_C(R) \arrow[r, "\mathrm{Hom}(P,-)"] \arrow[d, "\mathrm{Hom}(Q,-)"'] \arrow[dr, "\mathrm{Hom}(P\otimes Q,-)"'] & \mathcal{B}_C(R) \arrow[d, "\mathrm{Hom}(Q,-)"] \\ \mathcal{B}_C(R) \arrow[r, "\mathrm{Hom}(P,-)"'] & \mathcal{B}_C(R). \end{tikzcd}$$

However, the next examples show that the analogous diagrams do not commute for the operators $\mathrm{Hom}_R(-,I)$ and $\mathrm{Hom}_R(-,J)$ when I and J are injective.

Example 3.3.11. Let k be the field with two elements. Then the k-module k is injective. We claim that the following diagrams do not commute:

$$\begin{tikzcd} \mathcal{A}_k(k) \arrow[r, "\mathrm{Hom}_k(-,k)"] \arrow[dr, "-\otimes_R \mathrm{Hom}_k(k,k)"'] & \mathcal{B}_k(k) \arrow[d, "\mathrm{Hom}(-,k)"] \\ & \mathcal{A}_k(k) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}_k(k) \arrow[r, "\mathrm{Hom}_k(-,k)"] \arrow[dr, "-\otimes_R \mathrm{Hom}_k(k,k)"'] & \mathcal{A}_k(k) \arrow[d, "\mathrm{Hom}(-,k)"] \\ & \mathcal{A}_k(k). \end{tikzcd}$$

Indeed, the classes $\mathcal{A}_k(k)$ and $\mathcal{B}_k(k)$ contain every k-module, so it suffices to show that the first diagram does not commute.

We consider the module $k^{(\mathbb{N})}$, and claim that

$$\mathrm{Hom}_k(k^{\mathbb{N}},k) \ncong k^{(\mathbb{N})}.$$

By way of contradiction, suppose that $\mathrm{Hom}_k(k^{\mathbb{N}},k) \cong k^{(\mathbb{N})}$, and consider the exact sequence

$$0 \to k^{(\mathbb{N})} \to k^{\mathbb{N}} \to k^{\mathbb{N}}/k^{(\mathbb{N})} \to 0.$$

Apply the exact functor $\mathrm{Hom}_k(-,k)$ to obtain the next exact sequence

$$0 \to \mathrm{Hom}_k(k^{\mathbb{N}}/k^{(\mathbb{N})},k) \to \underbrace{\mathrm{Hom}_k(k^{\mathbb{N}},k)}_{\cong k^{(\mathbb{N})}} \to \underbrace{\mathrm{Hom}_k(k^{(\mathbb{N})},k)}_{\cong k^{\mathbb{N}}} \to 0.$$

However, this sequence can not be exact because $k^{(\mathbb{N})}$ is countable and $k^{\mathbb{N}}$ is uncountable. Thus, we have our contradiction.

Now, we compute

$$\mathrm{Hom}_k(\mathrm{Hom}_k(k^{(\mathbb{N})},k),k) \cong \mathrm{Hom}_k(k^{\mathbb{N}},k) \ncong k^{(\mathbb{N})} \cong k^{(\mathbb{N})} \otimes_k \mathrm{Hom}_k(k,k)$$

to see that the displayed diagrams do not commute.

Example 3.3.12. Let $(R,\mathfrak{m},k)$ be a complete local ring that has a prime ideal $\mathfrak{p} \neq \mathfrak{m}$. We claim that neither square in the following diagram commutes:

$$\begin{tikzcd} \mathcal{A}_C(R) \arrow[r, "\mathrm{Hom}(-,E_R(k))"] \arrow[d, "\mathrm{Hom}(-,E_R(R/\mathfrak{p}))"'] & \mathcal{B}_C(R) \arrow[r, "\mathrm{Hom}(-,E_R(k))"] \arrow[d, "\mathrm{Hom}(-,E_R(R/\mathfrak{p}))"'] & \mathcal{A}_C(R) \arrow[d, "\mathrm{Hom}(-,E_R(R/\mathfrak{p}))"] \\ \mathcal{B}_C(R) \arrow[r, "\mathrm{Hom}(-,E_R(k))"'] & \mathcal{A}_C(R) \arrow[r, "\mathrm{Hom}(-,E_R(k))"'] & \mathcal{B}_C(R). \end{tikzcd}$$

For the first square, this follows from the next sequence:

$$\mathrm{Hom}_R(\mathrm{Hom}_R(R,E_R(R/\mathfrak{p})),E_R(k)) \cong \mathrm{Hom}_R(E_R(R/\mathfrak{p}),E_R(k)) = 0$$
$$\mathrm{Hom}_R(\mathrm{Hom}_R(R,E_R(k)),E_R(R/\mathfrak{p})) \cong \mathrm{Hom}_R(E_R(k),E_R(R/\mathfrak{p})) \neq 0.$$

The second square fails to commute because of the next sequence:

$$\operatorname{Hom}_R(\operatorname{Hom}_R(E_R(k), E_R(R/\mathfrak{p})), E_R(k)) \cong \operatorname{Hom}_R(0, E_R(k)) = 0$$
$$\operatorname{Hom}_R(\operatorname{Hom}_R(E_R(k), E_R(k)), E_R(R/\mathfrak{p})) \cong \operatorname{Hom}_R(R, E_R(R/\mathfrak{p})) \neq 0.$$

The next result provides the diagrams that we do know to commute. That they commute can be verified directly or by combining Theorem 3.2.1 with Propositions 3.3.6–3.3.8.

Proposition 3.3.13. *Let C be a semidualizing R-module, and let I, J be injective R-modules. Let F be a flat R-module, and let P be a projective R-module. Then the following diagrams commute:*

$$\begin{array}{ccccc}
\mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,I)} & \mathcal{B}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,J)} & \mathcal{A}_C(R) \\
{\scriptstyle C\otimes_R -}\Big\downarrow & & & & \Big\uparrow{\scriptstyle \operatorname{Hom}_R(C,-)} \\
\mathcal{B}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,I)} & \mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,J)} & \mathcal{B}_C(R) \\
{\scriptstyle \operatorname{Hom}_R(C,-)}\Big\downarrow & & & & \Big\uparrow{\scriptstyle C\otimes_R -} \\
\mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,I)} & \mathcal{B}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,J)} & \mathcal{A}_C(R)
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(P,-)} & \mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,J)} & \mathcal{B}_C(R) \\
{\scriptstyle C\otimes_R -}\Big\downarrow & & & & \Big\uparrow{\scriptstyle C\otimes_R -} \\
\mathcal{B}_C(R) & \xrightarrow{\operatorname{Hom}_R(P,-)} & \mathcal{B}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,J)} & \mathcal{A}_C(R) \\
{\scriptstyle \operatorname{Hom}_R(C,-)}\Big\downarrow & & & & \Big\uparrow{\scriptstyle \operatorname{Hom}_R(C,-)} \\
\mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(P,-)} & \mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,J)} & \mathcal{B}_C(R)
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,I)} & \mathcal{B}_C(R) & \xrightarrow{\operatorname{Hom}_R(P,-)} & \mathcal{B}_C(R) \\
{\scriptstyle C\otimes_R -}\Big\downarrow & & & & \Big\uparrow{\scriptstyle C\otimes_R -} \\
\mathcal{B}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,I)} & \mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(P,-)} & \mathcal{A}_C(R) \\
{\scriptstyle \operatorname{Hom}_R(C,-)}\Big\downarrow & & & & \Big\uparrow{\scriptstyle \operatorname{Hom}_R(C,-)} \\
\mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,I)} & \mathcal{B}_C(R) & \xrightarrow{\operatorname{Hom}_R(P,-)} & \mathcal{B}_C(R)
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{A}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,J)} & \mathcal{B}_C(R) \\
{\scriptstyle C\otimes_R -}\Big\downarrow & & & & \Big\uparrow{\scriptstyle C\otimes_R -} \\
\mathcal{B}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{B}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,J)} & \mathcal{A}_C(R) \\
{\scriptstyle \operatorname{Hom}_R(C,-)}\Big\downarrow & & & & \Big\uparrow{\scriptstyle \operatorname{Hom}_R(C,-)} \\
\mathcal{A}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,J)} & \mathcal{B}_C(R)
\end{array}$$

$$
\begin{array}{ccccc}
\mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,I)} & \mathcal{B}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{B}_C(R) \\
{\scriptstyle C\otimes_R -}\Big\downarrow & & & & \Big\uparrow{\scriptstyle C\otimes_R -} \\
\mathcal{B}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,I)} & \mathcal{A}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{A}_C(R) \\
{\scriptstyle \operatorname{Hom}_R(C,-)}\Big\downarrow & & & & \Big\uparrow{\scriptstyle \operatorname{Hom}_R(C,-)} \\
\mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(-,I)} & \mathcal{B}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{B}_C(R)
\end{array}
$$

$$
\begin{array}{ccccc}
\mathcal{A}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(P,-)} & \mathcal{A}_C(R) \\
{\scriptstyle C\otimes_R -}\Big\downarrow & & & & \Big\uparrow{\scriptstyle \operatorname{Hom}_R(C,-)} \\
\mathcal{B}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{B}_C(R) & \xrightarrow{\operatorname{Hom}_R(P,-)} & \mathcal{B}_C(R) \\
{\scriptstyle \operatorname{Hom}_R(C,-)}\Big\downarrow & & & & \Big\uparrow{\scriptstyle C\otimes_R -} \\
\mathcal{A}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(P,-)} & \mathcal{A}_C(R)
\end{array}
$$

$$
\begin{array}{ccccc}
\mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(P,-)} & \mathcal{A}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{A}_C(R) \\
{\scriptstyle C\otimes_R -}\Big\downarrow & & & & \Big\uparrow{\scriptstyle \operatorname{Hom}_R(C,-)} \\
\mathcal{B}_C(R) & \xrightarrow{\operatorname{Hom}_R(P,-)} & \mathcal{B}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{B}_C(R) \\
{\scriptstyle \operatorname{Hom}_R(C,-)}\Big\downarrow & & & & \Big\uparrow{\scriptstyle C\otimes_R -} \\
\mathcal{A}_C(R) & \xrightarrow{\operatorname{Hom}_R(P,-)} & \mathcal{A}_C(R) & \xrightarrow{-\otimes_R F} & \mathcal{A}_C(R).
\end{array}
$$

Here are some companions for the previous results. We let $\mathrm{J}(R)$ denote the Jacobson radical of R.

Proposition 3.3.14. *Let C be a semidualizing R-module, let N be an R-module of finite flat dimension, and let M be an R-module such that $\operatorname{Tor}_i^R(M,N)=0$ for all $i \geqslant 1$.*

(a) *If M is in $\mathcal{A}_C(R)$, then $M\otimes_R N$ is in $\mathcal{A}_C(R)$.*
(b) *If M is in $\mathcal{B}_C(R)$, then $M\otimes_R N$ is in $\mathcal{B}_C(R)$.*
(c) *The converses of* (a) *and* (b) *hold when M is finitely generated and $N \cong R/(\mathbf{x})$ for some sequence $\mathbf{x}=x_1,\ldots,x_n \in \mathrm{J}(R)$ that is R-regular.*

Proof. (a) Assume that M is in $\mathcal{A}_C(R)$. The assumption $\operatorname{fd}_R(N)<\infty$ implies that there is an exact sequence

$$F^+ \qquad = 0 \to F_s \to \cdots F_0 \to N \to 0$$

such that each F_i is flat. The assumption $\operatorname{Tor}_i^R(M,N)=0$ for all $i \geqslant 1$ implies that the induced sequence

$$M\otimes_R F^+ \qquad = 0 \to M\otimes_R F_s \to \cdots M\otimes_R F_0 \to M\otimes_R N \to 0$$

is exact. Proposition 3.3.2(a) implies that $M \otimes_R F_i \in \mathcal{A}_C(R)$ for $i = 0, \ldots, s$. Hence, we have $M \otimes_R N \in \mathcal{A}_C(R)$ by Corollary 3.1.8(a).

The proof of part (b) is similar.

(c) Assume that M is finitely generated and $N \cong R/(\mathbf{x})$ for some sequence $\mathbf{x} = x_1, \ldots, x_n \in \mathrm{J}(R)$ that is R-regular. Set $\overline{M} = M \otimes_R N$.

We now prove the converse of part (a); the other converse is proved similarly.

Assume that $\overline{M} \in \mathcal{A}_C(R)$. We prove that $M \in \mathcal{A}_C(R)$ by induction on n.

Base case: $n = 1$. In this case, there is an exact sequence

$$0 \to R \xrightarrow{x_1} R \to N \to 0.$$

Since $\mathrm{Tor}_1^R(M, N) = 0$, an application of the functor $M \otimes_R -$ yields the next exact sequence:

$$0 \to M \xrightarrow{x_1} M \to \overline{M} \to 0.$$

Using the long exact sequence in $\mathrm{Tor}_i^R(C, -)$, the fact that $\mathrm{Tor}_i^R(C, \overline{M}) = 0$ for $i \geqslant 1$ implies that the map

$$\mathrm{Tor}_i^R(C, M) \xrightarrow{x_1} \mathrm{Tor}_i^R(C, M)$$

is surjective, that is, we have $\mathrm{Tor}_i^R(C, M) = x_1 \mathrm{Tor}_i^R(C, M)$. The R-modules M and C are finitely generated, so $\mathrm{Tor}_i^R(C, M)$ is finitely generated. Since $x_1 \in \mathrm{J}(R)$, Nakayama's Lemma implies that $\mathrm{Tor}_i^R(C, M) = 0$ for all $i \geqslant 1$.

The vanishing $\mathrm{Tor}_1^R(C, \overline{M}) = 0$ implies that the next sequence is exact

$$0 \to C \otimes_R M \xrightarrow{x_1} C \otimes_R M \to C \otimes_R \overline{M} \to 0.$$

As in the previous paragraph, the condition $\mathrm{Ext}_R^i(C, C \otimes_R \overline{M}) = 0$ for $i \geqslant 1$ implies that $\mathrm{Ext}_R^i(C, C \otimes_R M) = 0$ for $i \geqslant 1$.

Finally, there is a commutative diagram of R-module homomorphisms where the bottom row is gotten by applying $\mathrm{Hom}_R(C, -)$ to the previous exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{x_1} & M & \longrightarrow & \overline{M} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma_M^C} & & \downarrow{\scriptstyle \gamma_M^C} & & \downarrow{\scriptstyle \gamma_{M \otimes_R N}^C}\,\cong & & \\
0 & \longrightarrow & \mathrm{Hom}(C, C \otimes M) & \xrightarrow{x_1} & \mathrm{Hom}(C, C \otimes M) & \longrightarrow & \mathrm{Hom}(C, C \otimes \overline{M}) & \longrightarrow & 0.
\end{array}$$

The top row is exact by assumption; the bottom row is exact as $\mathrm{Ext}_R^1(C, C \otimes_R M) = 0$. The snake lemma implies that the following maps are isomorphisms:

$$\mathrm{Ker}(\gamma_M^C) \underset{\cong}{\xrightarrow{x_1}} \mathrm{Ker}(\gamma_M^C) \qquad \mathrm{Coker}(\gamma_M^C) \underset{\cong}{\xrightarrow{x_1}} \mathrm{Coker}(\gamma_M^C).$$

From Nakayama's lemma, we conclude that $\mathrm{Ker}(\gamma_M^C) = 0 = \mathrm{Coker}(\gamma_M^C)$, so γ_M^C is an isomorphism. This implies that $M \in \mathcal{A}_C(R)$, as desired.

Induction step: Assume that the result holds for sequences of length $n-1$. Let $\mathbf{x}' = x_1, \ldots, x_{n-1}$ and set $N' = R/(\mathbf{x}')$.

We first show that $M \otimes_R N' \in \mathcal{A}_C(R)$. Indeed, since $\mathbf{x}$ is R-regular and contained in $\mathrm{J}(R)$, we know that x_n is R-regular. Furthermore, the element x_n is N'-regular, so we have $\mathrm{Tor}_i^R(N', R/(x_n)) = 0$ for all $i \geqslant 1$. Since $N' \otimes_R R/(x_n) \cong N \in \mathcal{A}_C(R)$ the base case implies that $M \otimes_R N' \in \mathcal{A}_C(R)$.

In order to use our induction hypothesis to conclude that $M \in \mathcal{A}_C(R)$, we need to show that $\mathrm{Tor}_i^R(M, N') = 0$ for all $i \geqslant 1$. By assumption, the element x_n

is N'-regular, so there is an exact sequence

$$0 \to N' \xrightarrow{x_n} N' \to N \to 0.$$

Because of the vanishing $\operatorname{Tor}_i^R(M,N)=0$, the associated long exact sequence in $\operatorname{Tor}_i^R(M,-)$ shows that the map

$$\operatorname{Tor}_i^R(M,N') \xrightarrow{x_n} \operatorname{Tor}_i^R(M,N')$$

is bijective for all $i \geqslant 1$. Hence, Nakayama's Lemma implies that $\operatorname{Tor}_i^R(M,N')=0$ for all $i \geqslant 1$. □

Remark 3.3.15. Under the hypotheses of Proposition 3.3.14(c), the sequence $\mathbf{x}$ is M-regular. Indeed, since $\mathbf{x}$ is R-regular, the Koszul complex $K = K^R(\mathbf{x})$ is a free resolution of $R/(\mathbf{x}) \cong N$. This explains the second step in the next sequence where $\mathrm{H}_i(\mathbf{x};M)$ is the ith Koszul homology of M with respect to $\mathbf{x}$:

$$0 = \operatorname{Tor}_i^R(M,N) \cong \mathrm{H}_i(M \otimes_R K) = \mathrm{H}_i(\mathbf{x};M).$$

The Tor-vanishing is by assumption for $i \geqslant 1$, and the third step is by definition. Since $\mathbf{x} \in \mathrm{J}(R)$, it follows that $\mathbf{x}$ is M-regular.

Proposition 3.3.16. *Let C be a semidualizing R-module, let N be an R-module of finite injective dimension, and let M be an R-module such that $\operatorname{Ext}_R^i(M,N)=0$ for all $i \geqslant 1$.*

(a) *If M is in $\mathcal{B}_C(R)$, then $\operatorname{Hom}_R(M,N)$ is in $\mathcal{A}_C(R)$.*
(b) *If M is in $\mathcal{A}_C(R)$, then $\operatorname{Hom}_R(M,N)$ is in $\mathcal{B}_C(R)$.*
(c) *The converses of* (a) *and* (b) *hold when M is finitely generated and $N \cong \operatorname{Hom}_R(R/(\mathbf{x}),E)$ for some sequence $\mathbf{x} \in \mathrm{J}(R)$ that is R-regular and some faithfully injective R-module E.*

Proof. The proofs of (a) and (b) are similar to the corresponding proofs in Proposition 3.3.14.

(c) Assume that $\operatorname{Hom}_R(M,N) \in \mathcal{A}_C(R)$ where $N \cong \operatorname{Hom}_R(R/(\mathbf{x}),E)$ for some sequence $\mathbf{x} \in \mathrm{J}(R)$ that is R-regular and some faithfully injective R-module E. By assumption, the module

$$\operatorname{Hom}_R(M,N) \cong \operatorname{Hom}_R(M,\operatorname{Hom}_R(R/(\mathbf{x}),E)) \cong \operatorname{Hom}_R(M \otimes_R R/(\mathbf{x}),E)$$

is in $\mathcal{A}_C(R)$. Since E is faithfully injective, Proposition 3.3.1(c) implies that $M \otimes_R R/(\mathbf{x}) \in \mathcal{B}_C(R)$.

As in the proof of Proposition 3.3.1, we have for all $i \geqslant 1$

$$0 = \operatorname{Ext}_R^i(M,N) \cong \operatorname{Hom}_R(\operatorname{Tor}_i^R(M,R'),E).$$

The fact that E is faithfully injective implies that $\operatorname{Tor}_i^R(M,R')=0$ for all $i \geqslant 1$. Hence, we have $M \in \mathcal{B}_C(R)$ by Proposition 3.3.14(c).

Similar reasoning shows that if $\operatorname{Hom}_R(M,N) \in \mathcal{B}_C(R)$, then $M \in \mathcal{A}_C(R)$ as desired. □

The next result is proved like Proposition 3.3.14.

Proposition 3.3.17. *Let C be a semidualizing R-module. Let M and N be R-modules such that $p = \operatorname{pd}_R(M) < \infty$ and $\operatorname{Ext}_R^i(M,N)=0$ for all $i \neq p$.*

(a) *If $N \in \mathcal{A}_C(R)$, then $\operatorname{Ext}_R^p(M,N) \in \mathcal{A}_C(R)$.*
(b) *If $N \in \mathcal{B}_C(R)$, then $\operatorname{Ext}_R^p(M,N) \in \mathcal{B}_C(R)$.*

(c) *The converses of parts* (a) *and* (b) *hold when N is finitely generated and $M \cong R/(\mathbf{x})R$ for some R-regular sequence $\mathbf{x}$ in the Jacobson radical of R.*

Remark 3.3.18. Under the hypotheses of Proposition 3.3.16(c), the sequence $\mathbf{x}$ is M-regular. Indeed, the proof shows that $\operatorname{Tor}_i^R(M, R') = 0$ for all $i \geqslant 1$, and we can apply the reasoning of Remark 3.3.15. The same conclusion holds under the hypotheses of Proposition 3.3.17(c).

3.4. Base Change

Theorem 3.4.1. *Let C be a semidualizing R-module and let $\varphi\colon R \to S$ be a ring homomorphism. Then $S \in \mathcal{A}_C(R)$ if and only if $\operatorname{Tor}_i^R(C, S) = 0$ for all $i \geqslant 1$ and $C \otimes_R S$ is a semidualizing S-module.*

Proof. Since the condition $S \in \mathcal{A}_C(R)$ includes the vanishing $\operatorname{Tor}_i^R(C, S) = 0$ for all $i \geqslant 1$, we assume without loss of generality that $\operatorname{Tor}_i^R(C, S) = 0$ for all $i \geqslant 1$. Let P be an R-free resolution of C such that each P_i is finitely generated.

In the following sequence, the first isomorphism is Hom-tensor adjointness

$$\begin{aligned} \operatorname{Hom}_S(P \otimes_R S, C \otimes_R S) &\cong \operatorname{Hom}_R(P, \operatorname{Hom}_S(S, C \otimes_R S)) \\ &\cong \operatorname{Hom}_R(P, C \otimes_R S) \end{aligned}$$

The second isomorphism is induced by Hom-cancellation. The vanishing assumption $\operatorname{Tor}_i^R(C, S) = 0$ for all $i \geqslant 1$ implies that the complex $P \otimes_R S$ is an S-free resolution of $C \otimes_R S$, and this explains the first isomorphism in the next sequence

$$\begin{aligned} \operatorname{Ext}_S^i(C \otimes_R S, C \otimes_R S) &\cong \mathrm{H}_{-i}(\operatorname{Hom}_S(P \otimes_R S, C \otimes_R S)) \\ &\cong \mathrm{H}_{-i}(\operatorname{Hom}_R(P, C \otimes_R S)) \\ &\cong \operatorname{Ext}_R^i(C, C \otimes_R S). \end{aligned}$$

The second isomorphism is from the previous displayed sequence, and the third isomorphism comes from the fact that P is an R-free resolution of C. From this, we see that $\operatorname{Ext}_S^i(C \otimes_R S, C \otimes_R S) = 0$ for all $i \geqslant 1$ if and only if $\operatorname{Ext}_R^i(C, C \otimes_R S) = 0$ for all $i \geqslant 1$.

Let $f\colon \operatorname{Hom}_S(S, C \otimes_R S) \to C \otimes_R S$ be the Hom-cancellation isomorphism given by $f(\psi) = \psi(1)$. This fits into a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\chi^S_{C \otimes_R S}} & \operatorname{Hom}_S(C \otimes_R S, C \otimes_R S) \\ {\scriptstyle \gamma^C_S}\downarrow & & \downarrow{\scriptstyle \cong} \\ \operatorname{Hom}_R(C, C \otimes_R S) & \xleftarrow[\cong]{\operatorname{Hom}_R(C,f)} & \operatorname{Hom}_R(C, \operatorname{Hom}_S(S, C \otimes_R S)) \end{array}$$

where the unspecified isomorphism is Hom-tensor adjointness. This diagram shows that $\chi^S_{C \otimes_R S}$ is an isomorphism if and only if γ^C_S is an isomorphism. This completes the proof. □

The next result generalizes Proposition 2.2.1.

Corollary 3.4.2. *Let C be a semidualizing R-module and let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension. Then $\operatorname{Tor}_i^R(C, S) = 0$ for all $i \geqslant 1$, and $C \otimes_R S$ is a semidualizing S-module.*

Proof. Since S has finite flat dimension as an R-module, Proposition 3.1.9 implies that S is in $\mathcal{A}_C(R)$. Hence, the result follows from Theorem 3.4.1. $\square$

The next result recovers Theorem 2.2.6(b).

Corollary 3.4.3. *Let C be a semidualizing R-module. If $\mathbf{x} \in R$ is an R-regular sequence, then $C/\mathbf{x}C$ is a semidualizing $R/\mathbf{x}R$-module.*

Proof. If $\mathbf{x}$ is R-regular, then $\mathrm{pd}_R(R/\mathbf{x}R) < \infty$, so Corollary 3.4.2 implies that and $C \otimes_R R/\mathbf{x}R \cong C/\mathbf{x}C$ is a semidualizing $R/\mathbf{x}R$-module. $\square$

Here is a partial converse to Theorem 2.2.6(b), and a compliment to Proposition 2.2.1.

Proposition 3.4.4. *Let $\mathbf{x} = x_1, \ldots, x_n \in \mathrm{J}(R)$ be an R-regular sequence. Let C be a finitely generated R-module such that $\mathbf{x}$ is C-regular. If $C/(\mathbf{x})C$ is a semidualizing $R/(\mathbf{x})R$-module, then C is a semidualizing R-module.*

Proof. Arguing by induction on n we may assume that $n = 1$. Set $x = x_1$ and $\overline{R} = R/(x)R$. Note that x is $\mathrm{Hom}_R(C, C)$-regular. Indeed, use the exact sequence

$$0 \to C \xrightarrow{x} C \to C/xC \to 0 \tag{3.4.4.1}$$

and the left-exactness of $\mathrm{Hom}_R(C, -)$ to conclude that the sequence

$$0 \to \mathrm{Hom}_R(C, C) \xrightarrow{x} \mathrm{Hom}_R(C, C)$$

is exact.

The element x is R-regular, so there is an exact sequence

$$0 \to R \xrightarrow{x} R \to \overline{R} \to 0.$$

Part of the associated long exact sequence in $\mathrm{Tor}_1^R(-, \mathrm{Hom}_R(C, C))$ has the form

$$0 \to \mathrm{Tor}_1^R(\overline{R}, \mathrm{Hom}_R(C, C)) \to \mathrm{Hom}_R(C, C) \xrightarrow{x} \mathrm{Hom}_R(C, C).$$

As the labeled map in this sequence is injective, we have $\mathrm{Tor}_1^R(\overline{R}, \mathrm{Hom}_R(C, C)) = 0$.

Consider the following commutative diagram

$$\begin{array}{ccccc}
\overline{R} & \xrightarrow[\cong]{\chi^{\overline{R}}_{C\otimes_R\overline{R}}} & \mathrm{Hom}_{\overline{R}}(C \otimes_R \overline{R}, C \otimes_R \overline{R}) & \xrightarrow{\cong} & \mathrm{Hom}_R(C, \mathrm{Hom}_{\overline{R}}(\overline{R}, C \otimes_R \overline{R})) \\
{\scriptstyle\cong}\downarrow & & & & \downarrow{\scriptstyle\cong} \\
\overline{R} \otimes_R R & \xrightarrow{\overline{R}\otimes_R \chi^R_C} & \overline{R} \otimes_R \mathrm{Hom}_R(C, C) & \xrightarrow[\cong]{\omega_{CC\overline{R}}} & \mathrm{Hom}_R(C, C \otimes_R \overline{R})
\end{array}$$

wherein the unspecified vertical isomorphisms are induced by Hom- and tensor-cancellation, and the unspecified horizontal isomorphism is Hom-tensor adjointness. The diagram shows that $\overline{R} \otimes_R \chi^R_C$ is an isomorphism.

We claim that χ^R_C is surjective. To see this, consider the exact sequence

$$R \xrightarrow{\chi^R_C} \mathrm{Hom}_R(C, C) \to \mathrm{Coker}(\chi^R_C) \to 0.$$

Use the right-exactness of $\overline{R} \otimes_R -$ to see that the next sequence is exact

$$\overline{R} \otimes_R R \xrightarrow[\cong]{\overline{R}\otimes_R\chi^R_C} \overline{R} \otimes_R \mathrm{Hom}_R(C, C) \to \overline{R} \otimes_R \mathrm{Coker}(\chi^R_C) \to 0.$$

It follows that $\overline{R} \otimes_R \mathrm{Coker}(\chi^R_C) = 0$, and the fact that x is in the Jacobson radical of R implies that $\mathrm{Coker}(\chi^R_C) = 0$ by Nakayama's Lemma.

We claim that χ^R_C is injective. To see this, consider the exact sequence

$$0 \to \operatorname{Ker}(\chi^R_C) \to R \xrightarrow{\chi^R_C} \operatorname{Hom}_R(C,C) \to 0$$

and take the long exact sequence in $\operatorname{Tor}^R(\overline{R},-)$:

$$\underbrace{\operatorname{Tor}^R_1(\overline{R},\operatorname{Hom}_R(C,C))}_{=0} \to \overline{R}\otimes_R \operatorname{Ker}(\chi^R_C) \to \overline{R}\otimes_R R \xrightarrow[\cong]{\overline{R}\otimes_R \chi^R_C} \overline{R}\otimes_R \operatorname{Hom}(C,C).$$

It follows that $\overline{R}\otimes \operatorname{Ker}(\chi^R_C) = 0$ and, as above, that $\operatorname{Ker}(\chi^R_C)=0$.

We conclude the proof by showing that $\operatorname{Ext}^i_R(C,C)=0$ for all $i\geqslant 1$. In the next sequence, the vanishing holds because $C\otimes_R \overline{R}$ is a semidualizing $\overline{R}$-module:

$$\operatorname{Ext}^i_R(C,C/xC) \cong \operatorname{Ext}^i_R(C,C\otimes_R \overline{R}) \cong \operatorname{Ext}^i_{\overline{R}}(C\otimes_R \overline{R}, C\otimes_R \overline{R}) = 0$$

for each $i\geqslant 1$. The first step is straightforward, and the second step is from [**16**, p. 140, Lemma 2]. For $i\geqslant 1$, part of the long exact sequence in $\operatorname{Ext}_R(C,-)$ associated to the sequence (3.4.4.1) has the following form:

$$\operatorname{Ext}^i_R(C,C) \xrightarrow{x} \operatorname{Ext}^i_R(C,C) \to \underbrace{\operatorname{Ext}^i_R(C,C/xC)}_{=0}.$$

It follows that $\operatorname{Ext}^i_R(C,C) = x\operatorname{Ext}^i_R(C,C)$. Since $\operatorname{Ext}^i_R(C,C)$ is finitely generated and x is in $\operatorname{J}(R)$, Nakayama's Lemma implies that $\operatorname{Ext}^i_R(C,C)=0$, as desired. □

Corollary 3.4.5. *Let $\mathbf{x}=x_1,\ldots,x_n\in \operatorname{J}(R)$ be an R-regular sequence. Let C be a finitely generated R-module such that $\mathbf{x}$ is C-regular. If $C/(\mathbf{x})C$ is a (point-wise) dualizing $R/(\mathbf{x})R$-module, then C is a (point-wise) dualizing R-module.*

Proof. Assume that $C/(\mathbf{x})C$ is a point-wise dualizing $R/(\mathbf{x})R$-module. Then Proposition 3.4.4 implies that C is a semidualizing R-module. The proof of Corollary 2.2.7 shows that $\operatorname{id}_{R_\mathfrak{m}}(C_\mathfrak{m})$ is finite for each maximal ideal $\mathfrak{m}\subset R$, so C is point-wise dualizing for R. When $C/(\mathbf{x})C$ is dualizing for $R/(\mathbf{x})$, we similarly conclude that C is dualizing for R. □

We have so far focused on the base change behavior for semidualizing modules. Now we turn our attention to base change properties for Foxby classes.

Proposition 3.4.6. *Let $\varphi\colon R\to S$ be a ring homomorphism, and let M be an S-module. Let C be a semidualizing R-module such that $S\in\mathcal{A}_C(R)$.*

(a) *One has $M\in\mathcal{A}_C(R)$ if and only if $M\in\mathcal{A}_{C\otimes_R S}(S)$.*
(b) *One has $M\in\mathcal{B}_C(R)$ if and only if $M\in\mathcal{B}_{C\otimes_R S}(S)$.*

Proof. (a) Let F be an R-free resolution of C. The assumption $S\in\mathcal{A}_C(R)$ implies that $C\otimes_R S$ is a semidualizing S-module and that $\operatorname{Tor}^R_i(C,S)=0$ for all $i\geqslant 1$; see Theorem 3.4.1. It follows that the complex $F\otimes_R S$ is an S-free resolution of $C\otimes_R S$. This yields the first isomorphism in the next sequence:

$$\operatorname{Tor}^S_i(C\otimes_R S,M)\cong \operatorname{H}_i((F\otimes_R S)\otimes_S M)\cong \operatorname{H}_i(F\otimes_R M)\cong \operatorname{Tor}^R_i(C,M).$$

The second isomorphism comes from tensor cancellation, and the third isomorphism is by definition. In particular, we have $\operatorname{Tor}^S_i(C\otimes_R S,M)=0$ for all $i\geqslant 1$ if and only if $\operatorname{Tor}^R_i(C,M)=0$ for all $i\geqslant 1$.

In the next sequence, the first step is Hom-tensor adjointness

$$\operatorname{Hom}_S(F\otimes_R S, C\otimes_R M)\cong \operatorname{Hom}_R(F,\operatorname{Hom}_S(S,C\otimes_R M))\cong \operatorname{Hom}_R(F,C\otimes_R M)$$

and the second step is induced by Hom cancellation. This provides the third isomorphism in the next sequence:

$$\begin{aligned}\operatorname{Ext}^i_S(C \otimes_R S, (C \otimes_R S) \otimes_S M) &\cong \operatorname{Ext}^i_S(C \otimes_R S, C \otimes_R M)\\ &\cong \mathrm{H}_{-i}(\operatorname{Hom}_S(F \otimes_R S, C \otimes_R M))\\ &\cong \mathrm{H}_{-i}(\operatorname{Hom}_R(F, C \otimes_R M))\\ &\cong \operatorname{Ext}^i_R(C, C \otimes_R M).\end{aligned}$$

The first step is induced by tensor cancellation; and the second and fourth steps are by definition. It follows that $\operatorname{Ext}^i_S(C \otimes_R S, (C \otimes_R S) \otimes_S M) = 0$ for all $i \geqslant 1$ if and only if $\operatorname{Ext}^i_R(C, C \otimes_R M) = 0$ for all $i \geqslant 1$.

Finally, there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\gamma^{C\otimes_R S}_M} & \operatorname{Hom}_S(C \otimes_R S, (C \otimes_R S) \otimes_S M) \\ \Big\downarrow{\scriptstyle \gamma^C_M} & & \\ \operatorname{Hom}_R(C, C \otimes_R M) & & \Big\downarrow{\scriptstyle \cong} \\ \Big\downarrow{\scriptstyle \cong} & & \\ \operatorname{Hom}_R(C, \operatorname{Hom}_S(S, C \otimes_R M)) & \xrightarrow{\cong} & \operatorname{Hom}_S(C \otimes_R S, C \otimes_R M). \end{array}$$

Here the unspecified vertical isomorphisms are induced by Hom cancellation and tensor cancellation, respectively, and the unspecified horizontal isomorphism is Hom-tensor adjointness. It follows that $\gamma^{C\otimes_R S}_M$ is an isomorphism if and only if γ^C_M is an isomorphism. This completes the proof of part (a).

Part (b) is proved similarly. □

Proposition 3.4.7. *Let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension, and let C be a semidualizing R-module. Let M be an R-module such that $\operatorname{Tor}^R_i(M, S) = 0$ for all $i \geqslant 1$. Consider the following conditions:*

(i) *$M \in \mathcal{A}_C(R)$;*
(ii) *$M \otimes_R S \in \mathcal{A}_C(R)$; and*
(iii) *$M \otimes_R S \in \mathcal{A}_{C\otimes_R S}(S)$.*

The implications (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) *always hold, and the conditions* (i)–(iii) *are equivalent when one of the following is satisfied:*

(1) *φ is faithfully flat; or*
(2) *M is finitely generated, and φ is surjective with kernel generated by an R-regular sequence in $\mathrm{J}(R)$.*

Proof. The equivalence (ii) $\Longleftrightarrow$ (iii) is from Proposition 3.4.6(a), and the implication (i) $\Longrightarrow$ (ii) is from Proposition 3.3.14(a). When condition (1) is satisfied, the implication (ii) $\Longrightarrow$ (i) is from Proposition 3.3.2(c). When condition (2) is satisfied, the implication (ii) $\Longrightarrow$ (i) is from Proposition 3.3.14(c). □

The next result is proved like the previous one.

Proposition 3.4.8. *Let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension, and let C be a semidualizing R-module. Let M be an R-module such that $\operatorname{Tor}^R_i(M, S) = 0$ for all $i \geqslant 1$. Consider the following conditions:*

(i) *$M \in \mathcal{B}_C(R)$;*

(ii) $M \otimes_R S \in \mathcal{B}_C(R)$; *and*
(iii) $M \otimes_R S \in \mathcal{B}_{C\otimes_R S}(S)$.

The implications (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) *always hold, and the conditions* (i)–(iii) *are equivalent when one of the following is satisfied:*

(1) φ *is faithfully flat; or*
(2) M *is finitely generated, and* φ *is surjective with kernel generated by an* R*-regular sequence in* $\mathrm{J}(R)$.

The next result compliments Proposition 2.3.6.

Proposition 3.4.9. *Let* k *be a field, and let* R *and* S *be* k*-algebras. Let* B *and* M *be* R*-modules such that* B *is semidualizing, and let* C *and* N *be* S*-modules such that* C *is semidualizing.*

(a) *If* $M \in \mathcal{A}_B(R)$ *and* $N \in \mathcal{A}_C(S)$, *then* $M \otimes_k N \in \mathcal{A}_{B\otimes_k C}(R \otimes_k S)$.
(b) *If* $M \in \mathcal{B}_B(R)$ *and* $N \in \mathcal{B}_C(S)$, *then* $M \otimes_k N \in \mathcal{B}_{B\otimes_k C}(R \otimes_k S)$.

Proof. We prove part (a); the proof of part (b) is similar.

Assume that $M \in \mathcal{A}_B(R)$ and $N \in \mathcal{A}_C(S)$. Note that Proposition 2.3.6 implies that $B\otimes_k C$ is a semidualizing $R\otimes_k S$-module. Proposition A.1.5 yields the following isomorphism for each $i \geqslant 0$:

$$\mathrm{Tor}_i^{R\otimes_k S}(B \otimes_k C, M \otimes_k N) \cong \oplus_{j=0}^i \mathrm{Tor}_j^R(B, M) \otimes_k \mathrm{Tor}_{i-j}^S(C, N).$$

Hence, the conditions $\mathrm{Tor}_i^R(B, M) = 0 = \mathrm{Tor}_i^S(C, N)$ for $i \geqslant 1$ imply that

$$\mathrm{Tor}_i^{R\otimes_k S}(B \otimes_k C, M \otimes_k N) = 0$$

for all $i \geqslant 1$. The case $i = 0$ yields an $R \otimes_k S$-module isomorphism

$$(B \otimes_k C) \otimes_{R\otimes_k S} (M \otimes_k N) \cong (B \otimes_R M) \otimes_k (C \otimes_S N)$$

and thus the first step in the next sequence:

$$\begin{aligned}
&\mathrm{Ext}^i_{R\otimes_k S}(B \otimes_k C, (B \otimes_k C) \otimes_{R\otimes_k S} (M \otimes_k N))\\
&\qquad\cong \mathrm{Ext}^i_{R\otimes_k S}(B \otimes_k C, (B \otimes_R M) \otimes_k (C \otimes_S N))\\
&\qquad\qquad\cong \oplus_{j=0}^i \mathrm{Ext}^j_R(B, B \otimes_R M) \otimes_k \mathrm{Ext}^{i-j}_S(C, C \otimes_S N).
\end{aligned}$$

The second step is from Proposition A.1.5. The conditions $\mathrm{Ext}^i_R(B, B \otimes_R M) = 0 = \mathrm{Ext}^i_S(C, C \otimes_S N)$ for $i \geqslant 1$ imply that

$$\mathrm{Ext}^i_{R\otimes_k S}(B \otimes_k C, (B \otimes_k C) \otimes_{R\otimes_k S} (M \otimes_k N)) = 0$$

for all $i \geqslant 1$. The case $i = 0$ yields an $R \otimes_k S$-module isomorphism

$$\begin{aligned}
&\mathrm{Hom}_R(B, B \otimes_R M) \otimes_k \mathrm{Hom}_S(C, C \otimes_S N)\\
&\qquad\xrightarrow{\cong} \mathrm{Hom}^i_{R\otimes_k S}(B \otimes_k C, (B \otimes_k C) \otimes_{R\otimes_k S} (M \otimes_k N)).
\end{aligned}$$

The proof of Proposition A.1.5 shows that this map is given by the formula $\phi\otimes\psi \mapsto \phi \boxtimes \psi$, where $\phi\colon B \to B \otimes_R M$ and $\psi\colon C \to C \otimes_S N$, and $\phi \boxtimes \psi\colon B \otimes_k C \to (B \otimes_R M) \otimes_k (C \otimes_S N)$ is given by $b \otimes c \mapsto \phi(b) \otimes \psi(c)$. It follows that the next diagram commutes:

$$\begin{array}{ccc}
M \otimes_k N & \xrightarrow[\cong]{\gamma^B_M \otimes_k \gamma^C_N} & \operatorname{Hom}_R(B, B \otimes_R M) \otimes_k \operatorname{Hom}_S(C, C \otimes_S N) \\
\Vert & & \downarrow \cong \\
 & & \operatorname{Hom}_{R\otimes_k S}(B \otimes_k C, (B \otimes_R M) \otimes_k (C \otimes_S N)) \\
\Vert & & \uparrow \cong \\
M \otimes_k N & \xrightarrow[\gamma^{B\otimes_k C}_{M\otimes_k N}]{} & \operatorname{Hom}^i_{R\otimes_k S}(B \otimes_k C, (B \otimes_k C) \otimes_{R\otimes_k S} (M \otimes_k N)).
\end{array}$$

Thus, the map $\gamma^{B\otimes_k C}_{M\otimes_k N}$ is an isomorphism and $M \otimes_k N \in \mathcal{A}_{B\otimes_k C}(R \otimes_k S)$. □

3.5. Local-Global Behavior and Consequences

The next two results are from unpublished notes by Foxby.

Proposition 3.5.1. *Let C and M be R-modules such that M is finitely generated.*

(a) *If there is an R-module isomorphism $\alpha\colon M \xrightarrow{\cong} \operatorname{Hom}_R(C, C \otimes_R M)$, then the natural map $\gamma^C_M\colon M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism.*

(b) *Assume that C is finitely generated. If for every maximal ideal $\mathfrak{m} \subset R$ there is an $R_\mathfrak{m}$-module isomorphism $M \cong \operatorname{Hom}_R(C, C \otimes_R M)_\mathfrak{m}$, then the natural map $\gamma^C_M\colon M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism.*

Proof. (a) It is straightforward to show that the following diagram commutes:

$$\begin{array}{ccc}
\operatorname{Hom}_R(C, C \otimes_R M) & \xrightarrow{\gamma^C_{\operatorname{Hom}_R(C,C\otimes_R M)}} & \operatorname{Hom}_R(C, C \otimes_R \operatorname{Hom}_R(C, C \otimes_R M)) \\
 & \searrow^{\operatorname{id}_{\operatorname{Hom}_R(C,C\otimes_R M)}} & \downarrow \operatorname{Hom}_R(C, \xi^C_{C\otimes_R M}) \\
 & & \operatorname{Hom}_R(C, C \otimes_R M)
\end{array}$$

In particular, the map $\gamma^C_{\operatorname{Hom}_R(C,C\otimes_R M)}$ is a split monomorphism. With $X = \operatorname{Coker}(\gamma^C_{\operatorname{Hom}_R(C,C\otimes_R M)})$, this explains the second isomorphism in the next sequence:

$$\begin{aligned}
M \oplus X &\cong \operatorname{Hom}_R(C, C \otimes_R M) \oplus X \\
&\cong \operatorname{Hom}_R(C, C \otimes_R \operatorname{Hom}_R(C, C \otimes_R M)) \\
&\cong M.
\end{aligned}$$

The other isomorphism are induced by α. Since M is finitely generated, this implies that $X = 0$, that is, that $\gamma^C_{\operatorname{Hom}_R(C,C\otimes_R M)}$ is surjective. Since it is also injective, we have the right-hand vertical isomorphism in the next diagram:

$$\begin{array}{ccc}
M & \xrightarrow[\cong]{\alpha} & \operatorname{Hom}_R(C, C \otimes_R M) \\
\downarrow \gamma^C_M & & \cong \downarrow \gamma^C_{\operatorname{Hom}_R(C,C\otimes_R M)} \\
\operatorname{Hom}_R(C, C \otimes_R M) & \xrightarrow[\cong]{\operatorname{Hom}_R(C,C\otimes_R \alpha)} & \operatorname{Hom}_R(C, C \otimes_R \operatorname{Hom}_R(C, C \otimes_R M)).
\end{array}$$

It follows that δ^C_M is an isomorphism.

(b) This follows from part (a) as in the proof of Proposition 2.2.2. □

The next result is proved like the previous one.

Proposition 3.5.2. *Let C and M be R-modules such that M is finitely generated.*

(a) *If there is an R-module isomorphism $\alpha\colon C \otimes_R \operatorname{Hom}_R(C,M) \xrightarrow{\cong} M$, then the natural evaluation map $\xi^C_M\colon C \otimes_R \operatorname{Hom}_R(C,M) \to M$ is an isomorphism.*

(b) *Assume that C is finitely generated. If for every maximal ideal $\mathfrak{m} \subset R$ there is an $R_\mathfrak{m}$-module isomorphism $(C \otimes_R \operatorname{Hom}_R(C,M))_\mathfrak{m} \cong M_\mathfrak{m}$, then the natural evaluation map $\xi^C_M\colon C \otimes_R \operatorname{Hom}_R(C,M) \to M$ is an isomorphism.*

Here are local-global principals for Foxby classes.

Proposition 3.5.3. *Let C be a semidualizing R-module and M an R-module. The following conditions are equivalent:*

(i) *$M \in \mathcal{A}_C(R)$;*
(ii) *$U^{-1}M \in \mathcal{A}_{U^{-1}C}(U^{-1}R)$ for each multiplicatively closed subset $U \subset R$;*
(iii) *$M_\mathfrak{p} \in \mathcal{A}_{C_\mathfrak{p}}(R_\mathfrak{p})$ for each prime ideal $\mathfrak{p} \subset R$; and*
(iv) *$M_\mathfrak{m} \in \mathcal{A}_{C_\mathfrak{m}}(R_\mathfrak{m})$ for each maximal ideal $\mathfrak{m} \subset R$.*

Proof. The implication (i) $\Longrightarrow$ (ii) is in Proposition 3.4.7, and the implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are straightforward.

(iv) $\Longrightarrow$ (i). For each $i \geqslant 1$ and each each maximal ideal $\mathfrak{m} \subset R$, we have isomorphisms

$$\operatorname{Tor}^R_i(C,M)_\mathfrak{m} \cong \operatorname{Tor}^{R_\mathfrak{m}}_i(C_\mathfrak{m}, M_\mathfrak{m}) = 0$$
$$\operatorname{Ext}^i_R(C, C \otimes_R M)_\mathfrak{m} \cong \operatorname{Ext}^i_{R_\mathfrak{m}}(C_\mathfrak{m}, C_\mathfrak{m} \otimes_{R_\mathfrak{m}} M_\mathfrak{m}) = 0.$$

and a commutative diagram

$$\begin{array}{ccc}
M_\mathfrak{m} & \xrightarrow[\cong]{\gamma^{C_\mathfrak{m}}_{M_\mathfrak{m}}} & \operatorname{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m}, C_\mathfrak{m} \otimes_{R_\mathfrak{m}} M_\mathfrak{m}) \\
\Big\downarrow{\scriptstyle (\gamma^C_M)_\mathfrak{m}} & & \Big\downarrow{\scriptstyle \cong} \\
\operatorname{Hom}_R(C, C \otimes_R M)_\mathfrak{m} & \xrightarrow{\cong} & \operatorname{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m}, (C \otimes_R M)_\mathfrak{m}).
\end{array}$$

Since this is so for each $\mathfrak{m}$ and each $i \geqslant 1$, we conclude that $\operatorname{Tor}^R_i(C,M) = 0 = \operatorname{Ext}^i_R(C, C \otimes_R M)$ for all $i \geqslant 1$ and that γ^C_M is an isomorphism. Hence $M \in \mathcal{A}_C(R)$ as desired. □

Proposition 3.5.4. *Let C be a semidualizing R-module and M an R-module. The following conditions are equivalent:*

(i) *$M \in \mathcal{B}_C(R)$;*
(ii) *$U^{-1}M \in \mathcal{B}_{U^{-1}C}(U^{-1}R)$ for each multiplicatively closed subset $U \subset R$;*
(iii) *$M_\mathfrak{p} \in \mathcal{B}_{C_\mathfrak{p}}(R_\mathfrak{p})$ for each prime ideal $\mathfrak{p} \subset R$; and*
(iv) *$M_\mathfrak{m} \in \mathcal{B}_{C_\mathfrak{m}}(R_\mathfrak{m})$ for each maximal ideal $\mathfrak{m} \subset R$.*

Proof. Similar to Proposition 3.5.3. □

The next result is proved like Proposition 2.3.4, using the previous two results.

Corollary 3.5.5. *Let $R_1, \ldots, R_n$ be noetherian rings, and consider the product $R = R_1 \times \cdots \times R_n$. For $i = 1, \ldots, n$ let C_i be a semidualizing R_i-module, and set $C = C_1 \times \cdots \times C_n$. There are bijections $\mathcal{A}_C(R_1) \times \cdots \times \mathcal{A}_C(R_n) \xrightarrow{\sim} \mathcal{A}_C(R)$ and $\mathcal{B}_C(R_1) \times \cdots \times \mathcal{B}_C(R_n) \xrightarrow{\sim} \mathcal{B}_C(R)$ given by $(M_1, \ldots, M_n) \mapsto M_1 \times \cdots \times M_n$.*

Corollary 3.5.6. *Let C be a semidualizing R-module. Then the Auslander class $\mathcal{A}_C(R)$ contains every R-module locally of finite flat dimension, and the Bass class $\mathcal{B}_C(R)$ contains every R-module locally of finite injective dimension.*

PROOF. If $\mathrm{fd}_{R_\mathfrak{m}}(M_\mathfrak{m})$ is finite for each maximal ideal $\mathfrak{m} \subset R$, then $M_\mathfrak{m} \in \mathcal{A}_{C_\mathfrak{m}}(R_\mathfrak{m})$ for each $\mathfrak{m}$, so $M \in \mathcal{A}_C(R)$ by Proposition 3.5.3. The conclusion for the Bass class holds similarly. $\square$

CHAPTER 4

Relations Between Semidualizing Modules

4.1. First Relations

Proposition 4.1.1. *Let C be a semidualizing R-module, and let M be a finitely generated R-module.*

(a) *The module M is semidualizing and in $\mathcal{A}_C(R)$ if and only if $C \otimes_R M$ is semidualizing and in $\mathcal{B}_C(R)$.*
(b) *The module M is semidualizing and in $\mathcal{B}_C(R)$ if and only if $\operatorname{Hom}_R(C, M)$ is semidualizing and in $\mathcal{A}_C(R)$.*

Proof. (a) Theorem 3.2.1(a) says that $M \in \mathcal{A}_C(R)$ if and only if $C \otimes_R M \in \mathcal{B}_C(R)$, so assume that $M \in \mathcal{A}_C(R)$. Using this assumptions Lemma 3.1.13(a) yields isomorphisms for all $i \geqslant 0$:

$$\operatorname{Ext}^i_R(M, M) \cong \operatorname{Ext}^i_R(C \otimes_R M, C \otimes_R M).$$

Thus $\operatorname{Ext}^i_R(M, M) = 0$ for all $i \geqslant 1$ if and only if $\operatorname{Ext}^i(C \otimes_R M, C \otimes_R M) = 0$ for all $i \geqslant 1$. Furthermore, this implies that $R \cong \operatorname{Hom}_R(M, M)$ if and only if $R \cong \operatorname{Hom}_R(C \otimes_R M, C \otimes_R M)$. Because of Proposition 2.2.2(a) we conclude that the homothety map χ^R_M is an isomorphism if and only if $\chi^R_{C\otimes_R M}$ is an isomorphism. Thus, the R-module M is semidualizing if and only if $C \otimes_R M$ is semidualizing.

The proof of part (b) is similar. $\square$

Corollary 4.1.2. *Let C be a semidualizing R-module and let N be an R-module of finite flat dimension. If $B = C \otimes_R N$ is semidualizing, then N is a finitely generated projective R-module of rank 1 and $N \cong \operatorname{Hom}_R(C, B)$; if furthermore R is local, then $N \cong R$ and $B \cong C$.*

Proof. Assume that $C \otimes_R N$ is semidualizing. Since $\operatorname{fd}_R(N) < \infty$, we have $N \in \mathcal{A}_C(R)$, and hence $N \cong \operatorname{Hom}_R(C, C \otimes_R N) \cong \operatorname{Hom}_R(C, B)$. In particular, since C and $C \otimes_R N$ are finitely generated, we conclude that N is finitely generated. Proposition 4.1.1(a) implies that N is semidualizing. Since N is finitely generated we have $\operatorname{pd}_R(N) = \operatorname{fd}_R(N) < \infty$, so N is a finitely generated projective R-module of rank 1, by Corollary 2.2.8.

When R is local, Corollary 2.2.8 implies that $N \cong R$, so $B = C \otimes_R N \cong C$. $\square$

Corollary 4.1.3. *Let C be a semidualizing R-module and let D be a point-wise dualizing R-module. Then the R-module $\operatorname{Hom}_R(C, D)$ is semidualizing. Furthermore, we have $\operatorname{Ext}^i_R(C, D) = 0 = \operatorname{Tor}^R_i(C, \operatorname{Hom}_R(C, D))$ for all $i \geqslant 1$, and the natural map $\xi^D_C \colon C \otimes_R \operatorname{Hom}_R(C, D) \to D$ is an isomorphism.*

Proof. Since D is locally of finite injective dimension, we have $D \in \mathcal{B}_C(R)$ by Corollary 3.5.6. Proposition 4.1.1(b) implies that $\operatorname{Hom}_R(C, D)$ is semidualizing, and the remaining conclusions follow from the definition of $\mathcal{B}_C(R)$. $\square$

The following result characterizes the semidualizing modules that are locally isomorphic. These are the ones that are homologically indistinguishable in the sense that they determine the same Foxby classes. The symmetry in conditions (iii)–(viii) implies that other conditions are also symmetric.

Proposition 4.1.4. *Let B and C be semidualizing R-modules. The following conditions are equivalent:*

(i) $B \cong C \otimes_R P$ *for some finitely generated projective R-module of rank 1;*
(ii) $B \cong C \otimes_R P$ *for some R-module of finite flat dimension;*
(iii) $B_\mathfrak{p} \cong C_\mathfrak{p}$ *for each prime ideal $\mathfrak{p} \subset R$;*
(iv) $B_\mathfrak{m} \cong C_\mathfrak{m}$ *for each maximal ideal $\mathfrak{m} \subset R$;*
(v) $\mathcal{A}_C(R) = \mathcal{A}_B(R)$*;*
(vi) $\mathcal{B}_C(R) = \mathcal{B}_B(R)$*;*
(vii) $C \in \mathcal{B}_B(R)$ *and* $B \in \mathcal{B}_C(R)$*;*
(viii) $\operatorname{Hom}_R(B, C) \in \mathcal{A}_B(R)$ *and* $\operatorname{Hom}_R(C, B) \in \mathcal{A}_C(R)$*;*
(ix) $\operatorname{Hom}_R(C, B)$ *is a projective R-module of rank 1;*
(x) $\operatorname{fd}_R(\operatorname{Hom}_R(C, B)) < \infty$*; and*
(xi) $B \cong C \otimes_R M$ *and* $C \cong B \otimes_R N$ *for some R-modules M and N with $M \in \mathcal{A}_C(R)$ and $N \in \mathcal{A}_B(R)$.*

When these conditions are satisfied, one has $P \cong \operatorname{Hom}_R(C, B)$ and the module $Q = \operatorname{Hom}_R(P, R) \cong \operatorname{Hom}_R(B, C)$ is a rank 1 projective R-module such that $C \cong B \otimes_R Q$.

Proof. The implications (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (iv) and (ix) $\Longrightarrow$ (x) are straightforward. The implication (ii) $\Longrightarrow$ (i) is from Corollary 4.1.2, and (vii) $\Longleftrightarrow$ (viii) is by Theorem 3.2.1.

(i) $\Longrightarrow$ (iii) Since P is a rank 1 projective R-module, we have $P_\mathfrak{p} \cong R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$, so $B_\mathfrak{p} \cong C_\mathfrak{p} \otimes_{R_\mathfrak{p}} P_\mathfrak{p} \cong C_\mathfrak{p}$ for all such $\mathfrak{p}$.

(iv) $\Longrightarrow$ (v) Assume that $B_\mathfrak{m} \cong C_\mathfrak{m}$ for each maximal ideal $\mathfrak{m} \subset R$. Proposition 3.5.3 implies that an R-module M is in $\mathcal{A}_B(R)$ if and only if $M_\mathfrak{m} \in \mathcal{A}_{B_\mathfrak{m}}(R) = \mathcal{A}_{C_\mathfrak{m}}(R)$ for each such $\mathfrak{m}$, that is, if and only if $M \in \mathcal{A}_C(R)$.

(v) $\Longrightarrow$ (vi) Assume that $\mathcal{A}_B(R) = \mathcal{A}_C(R)$, and let E be a faithfully injective R-module. Proposition 3.3.1 implies that an R-module M is in $\mathcal{B}_B(R)$ if and only if $\operatorname{Hom}_R(M, E) \in \mathcal{A}_B(R) = \mathcal{A}_C(R)$, that is, if and only if $M \in \mathcal{B}_C(R)$.

(vi) $\Longrightarrow$ (vii) Assume that $\mathcal{B}_B(R) = \mathcal{B}_C(R)$. Then Corollary 3.2.2(a) implies that $B \in \mathcal{B}_B(R) = \mathcal{B}_C(R)$ and $C \in \mathcal{B}_C(R) = \mathcal{B}_B(R)$.

(vii) $\Longrightarrow$ (ix) Assume that $C \in \mathcal{B}_B(R)$ and $B \in \mathcal{B}_C(R)$. This implies that

$$C \cong B \otimes_R \operatorname{Hom}_R(B, C) \cong C \otimes_R \operatorname{Hom}_R(C, B) \otimes_R \operatorname{Hom}_R(B, C).$$

From Proposition 4.1.1(b), we conclude that $\operatorname{Hom}_R(C, B)$ is semidualizing, so $\operatorname{Hom}_R(C, B)_\mathfrak{m}$ is a semidualizing $R_\mathfrak{m}$-module for each maximal ideal $\mathfrak{m} \subset R$. Localizing the previous display yields

$$C_\mathfrak{m} \cong C_\mathfrak{m} \otimes_{R_\mathfrak{m}} [\operatorname{Hom}_R(C, B)_\mathfrak{m} \otimes_{R_\mathfrak{m}} \operatorname{Hom}_R(B, C)_\mathfrak{m}].$$

Computing minimal numbers of generators, we find

$$\mu_{R_\mathfrak{m}}(C_\mathfrak{m}) = \mu_{R_\mathfrak{m}}(C_\mathfrak{m})\mu_{R_\mathfrak{m}}(\operatorname{Hom}_R(C, B)_\mathfrak{m})\mu_{R_\mathfrak{m}}(\operatorname{Hom}_R(B, C)_\mathfrak{m})$$

so that $\mu_{R_\mathfrak{m}}(\operatorname{Hom}_R(C, B)_\mathfrak{m}) = 1$. Corollary 2.1.14 implies that $\operatorname{Hom}_R(C, B)_\mathfrak{m} \cong R_\mathfrak{m}$ for each maximal $\mathfrak{m}$, that is, that $\operatorname{Hom}_R(C, B)$ is a rank 1 projective R-module.

(vii) $\Longrightarrow$ (xi) The condition $C \in \mathcal{B}_B(R)$ implies that $N = \operatorname{Hom}_R(B, C) \in \mathcal{A}_B(R)$ and $C \cong B \otimes_R \operatorname{Hom}_R(B, C) \cong B \otimes_R N$, by Theorem 3.2.1. The condition $B \in \mathcal{B}_C(R)$ implies that $M = \operatorname{Hom}_R(C, B) \in \mathcal{A}_B(R)$ and $B \cong C \otimes_R M$.

(xi) $\Longrightarrow$ (vii). Assume that $B \cong C \otimes_R M$ and $C \cong B \otimes_R N$ for some R-modules M and N with $M \in \mathcal{A}_C(R)$ and $N \in \mathcal{A}_B(R)$. The condition $M \in \mathcal{A}_C(R)$ implies that $B \cong C \otimes_R M \in \mathcal{B}_C(R)$, and the condition $N \in \mathcal{A}_B(R)$ implies that $C \in \mathcal{B}_B(R)$ by Theorem 3.2.1.

This proves the equivalence of the conditions (i)–(xi).

Finally, assume that conditions (i)–(xi) are satisfied. Then $P \cong \operatorname{Hom}_R(C, B)$ by Corollary 4.1.2. Proposition 4.1.1(b) implies that By symmetry, we conclude that the module

$$\begin{aligned} Q &= \operatorname{Hom}_R(B, C) \\ &\cong \operatorname{Hom}_R(C \otimes_R \operatorname{Hom}_R(C, B), C) \\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(C, B), \operatorname{Hom}_R(C, C)) \\ &\cong \operatorname{Hom}_R(P, R) \end{aligned}$$

is projective of rank 1 such that $C \cong B \otimes_R Q$, as desired. □

Here is the local case of Proposition 4.1.4.

Corollary 4.1.5. *Assume that R is local, and let B and C be semidualizing R-modules. The following conditions are equivalent:*

(i) $B \cong C$;
(ii) $B \cong C \otimes_R P$ *for some R-module of finite flat dimension;*
(iii) $B_{\mathfrak{p}} \cong C_{\mathfrak{p}}$ *for each prime ideal* $\mathfrak{p} \subset R$;
(iv) $\mathcal{A}_C(R) = \mathcal{A}_B(R)$;
(v) $\mathcal{B}_C(R) = \mathcal{B}_B(R)$;
(vi) $C \in \mathcal{B}_B(R)$ *and* $B \in \mathcal{B}_C(R)$;
(vii) $\operatorname{Hom}_R(B, C) \in \mathcal{A}_B(R)$ *and* $\operatorname{Hom}_R(C, B) \in \mathcal{A}_C(R)$;
(viii) $\operatorname{Hom}_R(C, B) \cong R$;
(ix) $\operatorname{fd}_R(\operatorname{Hom}_R(C, B)) < \infty$; *and*
(x) $B \cong C \otimes_R M$ *and* $C \cong B \otimes_R N$ *for some R-modules M and N with* $M \in \mathcal{A}_C(R)$ *and* $N \in \mathcal{A}_B(R)$.

Proof. Since R is local, every projective R-module is free. Hence, the only finitely generated projective R-module of rank 1 (up to isomorphism) is R. Now apply Proposition 4.1.4. □

Corollary 4.1.6. *Let C be a semidualizing R-module. The following conditions are equivalent:*

(i) *C is a (rank 1) projective R-module;*
(ii) *$\mathcal{A}_C(R)$ contains every R-module;*
(iii) *$\mathcal{B}_C(R)$ contains every R-module;*
(iv) $R/\mathfrak{m} \in \mathcal{A}_C(R)$ *for each maximal ideal* $\mathfrak{m} \subset R$;
(v) $R/\mathfrak{m} \in \mathcal{B}_C(R)$ *for each maximal ideal* $\mathfrak{m} \subset R$;
(vi) $E_R(R/\mathfrak{m}) \in \mathcal{A}_C(R)$ *for each maximal ideal* $\mathfrak{m} \subset R$;
(vii) *$\mathcal{A}_C(R)$ contains a faithfully injective R-module;*
(viii) $R \in \mathcal{B}_C(R)$; *and*
(ix) *$\mathcal{B}_C(R)$ contains a finitely generated projective R-module of rank 1.*

Proof. The equivalence of the conditions (i)–(iii) is from the case $B = R$ of Proposition 4.1.4, using Example 3.1.5. The equivalence (i) $\iff$ (ix) holds similarly. The implications (ii) $\implies$ (iv) and (ii) $\implies$ (vi) and (iii) $\implies$ (v) and (iii) $\implies$ (viii) and (viii) $\implies$ (ix) are routine.

(iv) $\implies$ (i) Assume that $R/\mathfrak{m} \in \mathcal{A}_C(R)$ for each maximal ideal $\mathfrak{m} \subset R$. It follows that $\operatorname{Tor}_i^R(C, R/\mathfrak{m}) = 0$ for all $i \geqslant 1$ and for each $\mathfrak{m}$. Thus, C is projective, and Corollary 2.2.8 implies that C has rank 1.

The implication (v) $\implies$ (i) is verified similarly.

(vi) $\implies$ (vii) If $E_R(R/\mathfrak{m}) \in \mathcal{A}_C(R)$ for each maximal ideal $\mathfrak{m} \subset R$, then the faithfully injective R-module $E = \coprod_{\mathfrak{m}} E_R(R/\mathfrak{m})$ is in $\mathcal{A}_C(R)$ by Proposition 3.1.6(a).

(vii) $\implies$ (viii) Let E be a faithfully injective R-module in $\mathcal{A}_C(R)$. Since $E \cong \operatorname{Hom}_R(R, E) \in \mathcal{A}_C(R)$, Proposition 3.3.1(c) implies that $R \in \mathcal{B}_C(R)$. □

Here is the local case of the previous result.

Corollary 4.1.7. *Assume that $(R, \mathfrak{m}, k)$ is local, and let C be a semidualizing R-module. The following conditions are equivalent:*

(i) $C \cong R$;
(ii) $\mathcal{A}_C(R)$ *contains every R-module;*
(iii) $\mathcal{B}_C(R)$ *contains every R-module;*
(iv) $k \in \mathcal{A}_C(R)$*; and*
(v) $k \in \mathcal{B}_C(R)$.
(vi) $E_R(k) \in \mathcal{A}_C(R)$;
(vii) $\mathcal{A}_C(R)$ *contains a faithfully injective R-module; and*
(viii) $R \in \mathcal{B}_C(R)$.

Corollary 4.1.8. *Let D and D' be point-wise dualizing modules for R.*

(a) *The duals $P = \operatorname{Hom}_R(D', D)$ and $Q = \operatorname{Hom}_R(D, D')$ are rank 1 projective R-modules such that $D \cong D' \otimes_R P$ and $D' \cong D \otimes_R Q$.*
(b) *The R-module D' is dualizing if and only if D is dualizing.*
(c) *If R is local, then $D' \cong D$.*

Proof. Corollary 3.5.6 implies that $D' \in \mathcal{B}_D(R)$ and $D \in \mathcal{B}_{D'}(R)$, so part (a) follows from Proposition 4.1.4. Part (b) is a consequence of Corollary 2.2.5(c), and part (c) follows from Corollary 4.1.5. □

We next show that semidualizing modules over Gorenstein rings are trivial.

Corollary 4.1.9. *Assume that R is (point-wise) Gorenstein, and let C be a semidualizing R-module. Then C is a rank 1 projective R-module and is (point-wise) dualizing for R. If R is local, then C is isomorphic to R and is dualizing.*

Proof. Assume that R is point-wise Gorenstein, that is, that R is locally of finite injective dimension as an R-module. Corollary 3.5.6 implies that $R \in \mathcal{B}_C(R)$. On the other hand, every R-module is in $\mathcal{B}_R(R)$, so $C \in \mathcal{B}_R(R)$. Propositions 4.1.4 implies that $C \cong \operatorname{Hom}_R(R, C)$ is a rank 1 projective R-module. That is, we have $C_{\mathfrak{m}} \cong R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m} \subset R$. Since R is locally of finite injective dimension, the same is true of C, that is C is a point-wise dualizing R-module.

If R is Gorenstein, then we have

$$\operatorname{id}_R(C) = \sup_{\mathfrak{m}}\{\operatorname{id}_{R_{\mathfrak{m}}}(C_{\mathfrak{m}})\} = \sup_{\mathfrak{m}}\{\operatorname{id}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}})\} = \operatorname{id}_R(R) < \infty$$

so C is dualizing for R. If R is local, then both R and C are dualizing for R, so $C \cong R$ by Corollary 4.1.8. □

Corollary 4.1.10. *Assume that D is (point-wise) dualizing for R. Then the following conditions are equivalent:*

(i) *R is (point-wise) Gorenstein;*
(ii) *every semidualizing R-module is projective;*
(iii) *D is a projective R-module;*
(iv) *$\mathcal{A}_D(R)$ contains every R-module;*
(v) *$\mathcal{B}_D(R)$ contains every R-module;*
(vi) *$R/\mathfrak{m} \in \mathcal{A}_D(R)$ for each maximal ideal $\mathfrak{m} \subset R$;*
(vii) *$R/\mathfrak{m} \in \mathcal{B}_D(R)$ for each maximal ideal $\mathfrak{m} \subset R$;*
(viii) *$E_R(R/\mathfrak{m}) \in \mathcal{A}_D(R)$ for each maximal ideal $\mathfrak{m} \subset R$;*
(ix) *$\mathcal{A}_D(R)$ contains a faithfully injective R-module;*
(x) *$R \in \mathcal{B}_D(R)$; and*
(xi) *$\mathcal{B}_D(R)$ contains a finitely generated projective R-module of rank 1.*

Proof. The equivalence of the conditions (iii)–(xi) is from Corollary 4.1.6, since a projective semiduaizing module must have rank 1. The implication (i) $\Longrightarrow$ (ii) is from Corollary 4.1.9. The implication (ii) $\Longrightarrow$ (iii) is routine, and Corollary 2.2.9 justifies the implication (iii) $\Longrightarrow$ (i). □

Here is the local case of the previous result.

Corollary 4.1.11. *Assume that $(R, \mathfrak{m}, k)$ is local and that D is dualizing for R. Then the following conditions are equivalent:*

(i) *R is Gorenstein;*
(ii) *every semidualizing R-module is free;*
(iii) *D is a free R-module;*
(iv) *$\mathcal{A}_D(R)$ contains every R-module;*
(v) *$\mathcal{B}_D(R)$ contains every R-module;*
(vi) *$k \in \mathcal{A}_D(R)$;*
(vii) *$k \in \mathcal{B}_D(R)$;*
(viii) *$E_R(k) \in \mathcal{A}_D(R)$;*
(ix) *$\mathcal{A}_D(R)$ contains a faithfully injective R-module;*
(x) *$R \in \mathcal{B}_D(R)$.*

Corollary 4.1.12. *Let C be a semidualizing R-module. Then C is (point-wise) dualizing for R if and only if R has a (point-wise) dualizing D such that $C \in \mathcal{B}_D(R)$.*

Proof. One implication follows from the condition $C \in \mathcal{B}_C(R)$ found in Corollary 3.2.2(a).

For the converse, assume that R has a (point-wise) dualizing D such that $C \in \mathcal{B}_D(R)$. Corollary 3.5.6 implies that $D \in \mathcal{B}_C(R)$, so Proposition 4.1.4 yields a finitely generated projective R-module P of rank 1 such that $C \cong P \otimes_R D$. Since D is (point-wise) dualizing, it follows from Corollary 2.2.5(c) that C is (point-wise) dualizing. □

4.2. Picard Group Action and Ordering

Definition 4.2.1. The *Picard group* of R is the set $\operatorname{Pic}(R)$ of all isomorphism classes of finitely generated projective R-modules of rank 1. The isomorphism class of a given finitely generated projective R-module P of rank 1 is denoted $[P]$.

Remark 4.2.2. As its name suggests, the set $\mathrm{Pic}(R)$ has the structure of an abelian group, which we write multiplicatively: $[P][Q] = [P \otimes_R Q]$. The identity element of $\mathrm{Pic}(R)$ is $[R]$, and inverses are given by the formula $[P]^{-1} = [\mathrm{Hom}_R(P, R)]$. It follows that $[P]^{-1}[Q] = [\mathrm{Hom}_R(P, Q)]$.

Properties 4.2.3.

4.2.3.1. If R is local, then $\mathrm{Pic}(R) = \{[R]\}$.

4.2.3.2. Corollary 2.2.5(a) implies that $\mathrm{Pic}(R) \subseteq \mathfrak{S}_0(R)$.

4.2.3.3. If R is point-wise Gorenstein, then $\mathrm{Pic}(R) = \mathfrak{S}_0(R)$ by Corollary 4.1.9.

4.2.3.4. Assuming that R has a (point-wise) dualizing module, Corollary 4.1.8 implies that the set of isomorphism classes of (point-wise) dualizing modules is in bijection with $\mathrm{Pic}(R)$.

The next result expands on Corollary 2.2.5(b).

Proposition 4.2.4. *There is a well-defined* action of $\mathrm{Pic}(R)$ on $\mathfrak{S}_0(R)$ *action of the group* $\mathrm{Pic}(R)$ *on the set* $\mathfrak{S}_0(R)$ *given by* $[P][C] = [P \otimes_R C]$.

Proof. Corollary 2.2.5(b) shows that the formula $[P][C] = [P \otimes_R C]$ is well-defined. The identity $[R] \in \mathrm{Pic}(R)$ acts trivially because $R \otimes_R C \cong C$, and the associative law follows from the associativity of tensor product. ☐

The next result says that this group action is free.

Proposition 4.2.5. *For each* $[C] \in \mathfrak{S}_0(R)$*, the stabilizer of* $[C]$ *in* $\mathrm{Pic}(R)$ *is* $\{[R]\}$.

Proof. If $[P] \in \mathrm{Pic}(R)$ is in the stabilizer of $[C]$, then we have $C \cong P \otimes_R C$, so Proposition 4.1.4 implies that $P \cong \mathrm{Hom}_R(C, C) \cong R$. ☐

Definition 4.2.6. Let $\overline{\mathfrak{S}}_0(R)$ denote the set of orbits in $\mathfrak{S}_0(R)$ under the action of $\mathrm{Pic}(R)$. The orbit of a given element $[C] \in \mathfrak{S}_0(R)$ is denoted $\langle C \rangle \in \overline{\mathfrak{S}}_0(R)$.

Remark 4.2.7. If R is local, then the triviality of $\mathrm{Pic}(R)$ implies that the natural map $\mathfrak{S}_0(R) \to \overline{\mathfrak{S}}_0(R)$ is a bijection.

Lemma 4.2.8. *For* $[B], [C] \in \mathfrak{S}_0(R)$ *the following conditions are equivalent:*

(i) $\langle B \rangle = \langle C \rangle$ *in* $\overline{\mathfrak{S}}_0(R)$;
(ii) $B \cong P \otimes_R C$ *for some* $[P] \in \mathrm{Pic}(R)$;
(iii) $B_{\mathfrak{m}} \cong C_{\mathfrak{m}}$ *for each maximal ideal* $\mathfrak{m} \subset R$;
(iv) $\mathcal{A}_B(R) = \mathcal{A}_C(R)$;
(v) $\mathcal{B}_B(R) = \mathcal{B}_C(R)$; *and*
(vi) $C \in \mathcal{B}_B(R)$ *and* $B \in \mathcal{B}_C(R)$.

When $\mathrm{Pic}(R) = \{[R]\}$*, e.g., when* R *is local, these conditions are equivalent to*

(i′) $[B] = [C]$ *in* $\overline{\mathfrak{S}}_0(R)$.

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) is by definition, and the equivalence of the conditions (ii)–(vi) is from Proposition 4.1.4. When $\mathrm{Pic}(R) = \{[R]\}$, the equivalence (i) $\Longleftrightarrow$ (i′) is from Remark 4.2.7. ☐

Definition 4.2.9. For $\langle B \rangle, \langle C \rangle \in \overline{\mathfrak{S}}_0(R)$ we write $\langle B \rangle \trianglelefteq \langle C \rangle$ when $B \in \mathcal{B}_C(R)$.

When $\mathrm{Pic}(R) = \{[R]\}$, e.g., when R is local, we write $[B] \trianglelefteq [C]$ for $[B], [C] \in \mathfrak{S}_0(R)$ when $B \in \mathcal{B}_C(R)$.

Proposition 4.2.10. *The ordering $\trianglelefteq$ on $\overline{\mathfrak{S}}_0(R)$ is well-defined, reflexive, and antisymmetric. When $\operatorname{Pic}(R) = \{[R]\}$, e.g., when R is local, the ordering $\trianglelefteq$ on $\mathfrak{S}_0(R)$ is well-defined, reflexive, and antisymmetric.*

Proof. In view of Remark 4.2.7, it suffices to prove the first statement.

For well-definedness, let $\langle B\rangle = \langle B'\rangle$ and $\langle C\rangle = \langle C'\rangle$ in $\overline{\mathfrak{S}}_0(R)$. Lemma 4.2.8 implies that $\mathcal{B}_C(R) = \mathcal{B}_{C'}(R)$ and that there is an element $[P] \in \operatorname{Pic}(R)$ such that $B' \cong P \otimes_R B$. Thus, if $B \in \mathcal{B}_C(R)$, then Proposition 3.3.2(b) implies that $B' \cong P \otimes_R B \in \mathcal{B}_C(R) = \mathcal{B}_{C'}(R)$, as desired.

Reflexivity follows from the condition $C \in \mathcal{B}_C(R)$ in Corollary 3.2.2(a). For antisymmetry, assume that $\langle B\rangle \trianglelefteq \langle C\rangle$ and $\langle C\rangle \trianglelefteq \langle B\rangle$; that is, we have $C \in \mathcal{B}_B(R)$ and $B \in \mathcal{B}_C(R)$, so Lemma 4.2.8 implies that $\langle B\rangle = \langle C\rangle$. □

Here are some of the big open questions in this area:

Question 4.2.11.

(a) Is the set $\overline{\mathfrak{S}}_0(R)$ finite? If $\operatorname{Pic}(R) = \{[R]\}$, e.g., if R is local, is the set $\mathfrak{S}_0(R)$ finite?
(b) Is there a non-negative integer n such that $\left|\overline{\mathfrak{S}}_0(R)\right| = 2^n$? If $\operatorname{Pic}(R) = \{[R]\}$, e.g., if R is local, is there a non-negative integer n such that $|\mathfrak{S}_0(R)| = 2^n$?
(c) Is the ordering $\trianglelefteq$ on $\overline{\mathfrak{S}}_0(R)$ transitive? If $\operatorname{Pic}(R) = \{[R]\}$, e.g., if R is local, is the ordering $\trianglelefteq$ on $\mathfrak{S}_0(R)$ transitive?

Remark 4.2.12. There exist rings R with infinite Picard group. (Moreover, a theorem of Claiborn says that, for every abelian group G, there is a ring R such that $\mathfrak{S}_0(R) = \operatorname{Pic}(R) \cong G$.) Thus, the versions of Question 4.2.11(a)–(b) for $\mathfrak{S}_0(R)$ are only reasonable when $\operatorname{Pic}(R) = \{[R]\}$.

Property 4.2.3.3 shows that, if R is point-wise Gorenstein then $\overline{\mathfrak{S}}_0(R) = \{\langle R\rangle\}$, so we have $\left|\overline{\mathfrak{S}}_0(R)\right| = 1 = 2^0$. Corollary 4.3.6 gives some motivation for Question 4.2.11(b). The next examples give affirmative answers to the questions in 4.2.11 for some special classes of rings.

Example 4.2.13. Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathfrak{m}^2 = 0$. Let D be a dualizing R-module. (We will prove later that R does in fact admit a dualizing module because it is complete and Cohen-Macaulay.) Then $\mathfrak{S}_0(R) = \{[R], [D]\}$. Indeed, let $[C] \in \mathfrak{S}_0(R)$ such that $C \not\cong R$. Corollary 2.2.8 implies that $\operatorname{pd}_R(C) = \infty$. Let P be a minimal free resolution of C, and let C' be the first syzygy in P. Then there is an exact sequence

$$0 \to C' \to P_0 \to C \to 0.$$

Since P is minimal, we have $C' \subseteq \mathfrak{m}P_0$, so the condition $\mathfrak{m}^2 = 0$ implies that $\mathfrak{m}C' = 0$. That is, we have $C' \cong k^n$ for some $n \geqslant 1$. Since $\operatorname{Ext}^i_R(C, C) = 0 = \operatorname{Ext}^i_R(P_0, C)$ for all $i \geqslant 1$, the long exact sequence in $\operatorname{Ext}_R(-, C)$ associated to the displayed sequence implies that $= 0\operatorname{Ext}^i_R(C', C) \cong \operatorname{Ext}^i_R(k, C)^n$ for all $i \geqslant 1$. Since $n \neq 0$, we have $\operatorname{Ext}^i_R(k, C) = 0$ for all $i \geqslant 1$. Thus, C is injective, i.e., dualizing, so Corollary 4.1.8(c) implies that $C \cong D$.

Example 4.2.14. Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring of minimal multiplicity. This means that there is a flat local ring homomorphism $\varphi\colon (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ such that $\mathfrak{m}R' = \mathfrak{m}'$ where R' has a regular sequence $\mathbf{x} \in \mathfrak{m}'$ such that the local ring $(\overline{R'}, \overline{\mathfrak{m}'}) = (R'/(x), \mathfrak{m}'/(x))$ satisfies $\overline{\mathfrak{m}'}^2 = 0$. We claim that

$$\mathfrak{S}_0(R) = \begin{cases} \{[R],[D]\} & \text{if } R \text{ admits a dualizing module } D \\ \{[R]\} & \text{if } R \text{ does not admit a dualizing module.} \end{cases}$$

Assume that $[C] \in \mathfrak{S}_0(R)$ such that $C \not\cong R$. We show that C is dualizing for R. Corollary 2.1.14 implies that C is not cyclic. Since the homomorphism φ is local, the module $C' = C \otimes_R R'$ is not cyclic. This module is also semidualizing for S by Proposition 2.2.1. Corollary 3.4.3 implies that $\overline{C'} = C'/(x)C'$ is semidualizing for $\overline{R'}$, and $\overline{C'}$ is not cyclic by Nakayama's lemma. Since $\overline{\mathfrak{m}'}^2 = 0$, Example 4.2.13 implies that $\overline{C'}$ is dualizing for $\overline{R'}$. Theorem 2.2.6(a) guarantees that the sequence $\mathbf{x}$ is C'-regular, so we conclude from Corollary 3.4.5 that C' is dualizing for R'. Finally, Proposition 2.2.15 implies that C is dualizing for R, as desired.

Fact 4.2.15. Let $\varphi\colon R \to S$ be a ring homomorphism. If $[P] \in \operatorname{Pic}(R)$, then the S-module $P \otimes_R S$ is finitely generated and projective of rank 1. That is, there is a well-defined map $\operatorname{Pic}(\varphi)\colon \operatorname{Pic}(R) \to \operatorname{Pic}(S)$ given by $[P] \mapsto [P \otimes_R S]$. It is straightforward to show that this map is in fact a group homomorphism; see the proof of Proposition 4.2.17(a).

Definition 4.2.16. Let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension. Define $\mathfrak{S}_0(\varphi)\colon \mathfrak{S}_0(R) \to \mathfrak{S}_0(S)$ by the formula $[C] \mapsto [C \otimes_R S]$. Define $\overline{\mathfrak{S}}_0(\varphi)\colon \overline{\mathfrak{S}}_0(R) \to \overline{\mathfrak{S}}_0(S)$ by the formula $\langle C\rangle \mapsto \langle C \otimes_R S\rangle$.

Proposition 4.2.17. *Let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension.*

(a) *The map $\mathfrak{S}_0(\varphi)\colon \mathfrak{S}_0(R) \to \mathfrak{S}_0(S)$ is well-defined and respects the Picard group actions: $\mathfrak{S}_0(\varphi)([P][C]) = \operatorname{Pic}(\varphi)([P])\mathfrak{S}_0(\varphi)([C])$.*

(b) *There is a commutative diagram*

$$\begin{array}{ccc} \operatorname{Pic}(R) & \xrightarrow{\operatorname{Pic}(\varphi)} & \operatorname{Pic}(S) \\ \big\downarrow & & \big\downarrow \\ \mathfrak{S}_0(R) & \xrightarrow{\mathfrak{S}_0(\varphi)} & \mathfrak{S}_0(S) \end{array}$$

where the vertical maps are the natural inclusions. In particular, if $\mathfrak{S}_0(\varphi)$ is injective, then so is $\operatorname{Pic}(\varphi)$.

(c) *If $\operatorname{Pic}(\varphi)$ is injective, then so is $\mathfrak{S}_0(\varphi)$ when one of the following is satisfied:*
   (1) *φ is faithfully flat; or*
   (2) *φ is surjective with kernel generated by an R-regular sequence in $\operatorname{J}(R)$.*

Proof. (a) The well-definedness of $\mathfrak{S}_0(\varphi)$ is a consequence of Corollary 3.4.2. The fact that $\mathfrak{S}_0(\varphi)$ respects the Picard group actions follows from the S-module isomorphism $(P \otimes_R S) \otimes_S (C \otimes_R S) \cong (P \otimes_R C) \otimes_R S$.

(b) The commutativity of the diagram is by definition, and the second statement follows from the diagram.

(c) Assume that $\operatorname{Pic}(\varphi)$ is injective and that φ satisfies condition (1) or (2). Let $[B],[C] \in \mathfrak{S}_0(R)$ such that $\mathfrak{S}_0(\varphi)([B]) = \mathfrak{S}_0(\varphi)([C])$, that is, such that there is an S-module isomorphism $B \otimes_R S \cong C \otimes_R S$. Corollary 3.2.2(a) implies that $B \otimes_R S \in \mathcal{B}_{B\otimes_R S}(S) = \mathcal{B}_{C\otimes_R S}(S)$, so we conclude from Proposition 3.4.8 that $B \in \mathcal{B}_C(R)$. (Note that this is where we use the assumption that φ satisfies condition (1) or (2).) Similarly, we have $C \in \mathcal{B}_B(R)$, so Proposition 4.1.4 implies that $\mathrm{Hom}_R(B,C)$ is a projective R-module of rank 1 such that

$$C \cong B \otimes_R \mathrm{Hom}_R(B,C). \tag{4.2.17.1}$$

Since $\mathrm{fd}_R(S)$ is finite, tensor evaluation yields the fourth step in the next sequence:

$$\begin{aligned} S &\cong \mathrm{Hom}_S(B \otimes_R S, C \otimes_R S) \cong \mathrm{Hom}_R(B, \mathrm{Hom}_S(S, C \otimes_R S)) \\ &\cong \mathrm{Hom}_R(B, C \otimes_R S) \cong \mathrm{Hom}_R(B,C) \otimes_R S. \end{aligned}$$

The first step is from the assumption $B \otimes_R S \cong C \otimes_R S$, using the fact that $B \otimes_R S$ is a semidualizing S-module. The other steps are Hom-tensor adjointness and Hom-cancellation.

This sequence implies that $\mathrm{Pic}(\varphi)([\mathrm{Hom}_R(B,C)]) = [S]$, so the injectivity of $\mathrm{Pic}(\varphi)$ implies that $\mathrm{Hom}_R(B,C) \cong R$. From (4.2.17.1), we conclude that $C \cong B \otimes_R R \cong B$, as desired. □

Here is an application.

Proposition 4.2.18. *Let B and C be semidualizing R-modules, and let $\mathbf{x} \in \mathrm{J}(R)$ be an R-regular sequence. If $C/\mathbf{x}C \cong C'/\mathbf{x}C'$, then $C \cong C'$.*

Proof. Let $\varphi \colon R \to R/(\mathbf{x})$ denote the natural surjection. It suffices to show that the induced group homomorphism $\mathrm{Pic}(\varphi)\colon \mathrm{Pic}(R) \to \mathrm{Pic}(R/(\mathbf{x}))$ is injective, by Proposition 4.2.17(c). Let $[P] \in \mathrm{Ker}(\mathrm{Pic}(\varphi))$. Then P is a finitely generated rank 1 projective R-module such that $P/\mathbf{x}P \cong R/(\mathbf{x})$. Let $\overline{p} \in P/\mathbf{x}P$ be a generator. Consider the exact sequence

$$R \xrightarrow{\tau} P \to \mathrm{Coker}(\tau) \to 0$$

where $\tau(r) = rp$ for all $r \in R$. This induces an exact sequence

$$[R/(\mathbf{x})] \otimes_R R \xrightarrow[\cong]{[R/(\mathbf{x})]\otimes_R \tau} [R/(\mathbf{x})] \otimes_R P \to [R/(\mathbf{x})] \otimes_R \mathrm{Coker}(\tau) \to 0$$

and hence $[R/(\mathbf{x})] \otimes_R \mathrm{Coker}(\tau) = 0$. Since $\mathbf{x} \in \mathrm{J}(R)$ it follows that $\mathrm{Coker}(\tau) = 0$, that is, the map τ is surjective. Because P is projective, this implies that $R \cong P \oplus \mathrm{Ker}(\tau)$. Since P has rank 1, we have $P_{\mathfrak{m}} \cong R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m} \subset R$, and it follows that $\mathrm{Ker}(\tau)_{\mathfrak{m}} = 0$ for each $\mathfrak{m}$. In other words, we have $\mathrm{Ker}(\tau) = 0$, so τ is an isomorphism. □

Proposition 4.2.19. *Let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension.*

(a) *The map $\overline{\mathfrak{S}}_0(\varphi)\colon \overline{\mathfrak{S}}_0(R) \to \overline{\mathfrak{S}}_0(S)$ is well-defined and respects the orderings on $\overline{\mathfrak{S}}_0(R)$ and $\overline{\mathfrak{S}}_0(S)$: if $\langle B\rangle \trianglelefteq \langle C\rangle$ in $\overline{\mathfrak{S}}_0(R)$, then $\overline{\mathfrak{S}}_0(\varphi)(\langle B\rangle) \trianglelefteq \overline{\mathfrak{S}}_0(\varphi)(\langle C\rangle)$ in $\overline{\mathfrak{S}}_0(S)$.*

(b) *Assume that one of the following is satisfied:*
   (1) *φ is faithfully flat; or*
   (2) *φ is surjective with kernel generated by an R-regular sequence in $\mathrm{J}(R)$.*

   Then the map $\overline{\mathfrak{S}}_0(\varphi)$ is injective and perfectly order-respecting: $\langle B\rangle \trianglelefteq \langle C\rangle$ in $\overline{\mathfrak{S}}_0(R)$ if and only if $\overline{\mathfrak{S}}_0(\varphi)(\langle B\rangle) \trianglelefteq \overline{\mathfrak{S}}_0(\varphi)(\langle C\rangle)$ in $\overline{\mathfrak{S}}_0(S)$.

Proof. (a) The well-definedness of $\overline{\mathfrak{S}}_0(\varphi)$ follows from Proposition 4.2.17(a): if $\langle B\rangle = \langle C\rangle$ in $\overline{\mathfrak{S}}_0(R)$, then there is an element $[P] \in \mathrm{Pic}(R)$ such that $[B] = [P][C]$ in $\mathfrak{S}_0(R)$; this implies that $[B \otimes_R S] = [P \otimes_R S][C \otimes_R S]$ in $\mathfrak{S}_0(S)$, so $\langle B \otimes_R S\rangle = \langle C \otimes_R S\rangle$ in $\overline{\mathfrak{S}}_0(S)$.

To show that $\overline{\mathfrak{S}}_0(\varphi)$ respects the ordering, let $\langle B\rangle, \langle C\rangle \in \overline{\mathfrak{S}}_0(R)$ such that $\langle B\rangle \trianglelefteq \langle C\rangle$. This means that $B \in \mathcal{B}_C(R)$, so Proposition 3.4.8 implies that $B \otimes_R S \in \mathcal{B}_{C\otimes_R S}(S)$. That is $\langle B \otimes_R S\rangle \trianglelefteq \langle C \otimes_R S\rangle$ in $\overline{\mathfrak{S}}_0(S)$.

(b) Assume that φ satisfies condition (1) or (2), and let $\langle B\rangle, \langle C\rangle \in \overline{\mathfrak{S}}_0(R)$. We first prove that $\overline{\mathfrak{S}}_0(\varphi)$ is perfectly order-respecting. One implication is from part (a), so assume that $\overline{\mathfrak{S}}_0(\varphi)(\langle B\rangle) \trianglelefteq \overline{\mathfrak{S}}_0(\varphi)(\langle C\rangle)$ in $\overline{\mathfrak{S}}_0(S)$. This means that $\langle B \otimes_R S\rangle \trianglelefteq \langle C \otimes_R S\rangle$, that is $B \otimes_R S \in \mathcal{B}_{C\otimes_R S}(S)$. As in the proof of Proposition 4.2.17(b), we conclude that $B \in \mathcal{B}_C(R)$, and hence $\langle B\rangle \trianglelefteq \langle C\rangle$.

The injectivity of $\overline{\mathfrak{S}}_0(\varphi)$ now follows. Indeed, assume that $\overline{\mathfrak{S}}_0(\varphi)(\langle B\rangle) = \overline{\mathfrak{S}}_0(\varphi)(\langle C\rangle)$ in $\overline{\mathfrak{S}}_0(S)$, that is, that $\overline{\mathfrak{S}}_0(\varphi)(\langle B\rangle) \trianglelefteq \overline{\mathfrak{S}}_0(\varphi)(\langle C\rangle)$ and $\overline{\mathfrak{S}}_0(\varphi)(\langle C\rangle) \trianglelefteq \overline{\mathfrak{S}}_0(\varphi)(\langle B\rangle)$. It follows that $\langle B\rangle \trianglelefteq \langle C\rangle$ and $\langle C\rangle \trianglelefteq \langle B\rangle$ in $\overline{\mathfrak{S}}_0(R)$, and thus $\langle B\rangle = \langle C\rangle$. □

Here is a compliment to Proposition 2.3.4.

Proposition 4.2.20. *Let $R_1, \ldots, R_n$ be noetherian rings, and set $R = \prod_{i=1}^n R_i$. There is a bijection $\overline{\mathfrak{S}}_0(R_1) \times \cdots \times \overline{\mathfrak{S}}_0(R_n) \xrightarrow{\sim} \overline{\mathfrak{S}}_0(R)$ given by $(\langle C_1\rangle, \ldots, \langle C_n\rangle) \mapsto \langle \prod_{i=1}^n C_i\rangle$. Furthermore, this bijection is perfectly order respecting in the sense that $\langle \prod_{i=1}^n B_i\rangle \trianglelefteq \langle \prod_{i=1}^n C_i\rangle$ in $\overline{\mathfrak{S}}_0(R)$ if and only if $\langle B_i\rangle \trianglelefteq \langle C_i\rangle$ in $\overline{\mathfrak{S}}_0(R_i)$ for each $i = 1, \ldots, n$.*

Proof. Arguing as in the proof of Proposition 2.3.4, one concludes that there is a group isomorphism $\mathrm{Pic}(R_1) \oplus \cdots \oplus \mathrm{Pic}(R_n) \xrightarrow{\cong} \mathrm{Pic}(R)$ given by the formula $([P_1], \ldots, [P_n]) \mapsto [\prod_{i=1}^n P_i]$. Furthermore, this isomorphism respects the appropriate group actions on the sets $\mathfrak{S}_0(R_1) \times \cdots \times \mathfrak{S}_0(R_n) \xrightarrow{\sim} \mathfrak{S}_0(R)$. The fact that the map $\overline{\mathfrak{S}}_0(R_1) \times \cdots \times \overline{\mathfrak{S}}_0(R_n) \xrightarrow{\sim} \overline{\mathfrak{S}}_0(R)$ is well defined and bijective follows from a routine argument. The fact that it is perfectly order respecting follows from Corollary 3.5.5. □

The next result is very helpful for locating semidualizing modules. It requires some background.

Definition 4.2.21. A finitely generated R-module N is *reflexive* if the natural biduality map $\delta_N^R \colon N \to \mathrm{Hom}_R(\mathrm{Hom}_R(N, R), R)$ is an isomorphism.

Let R be a normal domain, that is, an integrally closed integral domain. The *divisor class group* of R, denoted $\mathrm{Cl}(R)$, is the set of isomorphism classes of rank 1 reflexive R-modules. As usual, the isomorphism class of a given rank 1 reflexive R-module $\mathfrak{a}$ is denoted $[\mathfrak{a}]$.

Fact 4.2.22. Let R be a normal domain. Then the set $\mathrm{Cl}(R)$ is a group with operation $[\mathfrak{a}][\mathfrak{b}] = [\mathrm{Hom}_R(\mathrm{Hom}_R(\mathfrak{a} \otimes_R \mathfrak{b}, R), R)]$. The identity element in this group is $[R]$, and inverses are given by the formula $[\mathfrak{a}]^{-1} = [\mathrm{Hom}_R(\mathfrak{a}, R)]$. Moreover, one has $[\mathfrak{a}]^{-1}[\mathfrak{b}] = [\mathrm{Hom}_R(\mathfrak{a}, \mathfrak{b})]$ for all $[\mathfrak{a}], [\mathfrak{b}] \in \mathrm{Cl}(R)$. This is comparable with the operation in $\mathrm{Pic}(R)$. In fact, the group $\mathrm{Pic}(R)$ is a subgroup of $\mathrm{Cl}(R)$. If R is a unique factorization domain (e.g., if R is a regular local ring), then $\mathrm{Cl}(R) = \{[R]\}$.

From [**8**, (1.4.1(a))] we know that a finitely generated R-module N is reflexive if and only if

(1) $N_\mathfrak{p}$ is a reflexive $R_\mathfrak{p}$-module for all prime ideals $\mathfrak{p} \subset R$ such that $\mathrm{depth}(R_\mathfrak{p}) \leqslant 1$, and
(2) $\mathrm{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) \geqslant 2$ for all prime ideals $\mathfrak{p} \subset R$ such that $\mathrm{depth}(R_\mathfrak{p}) \geqslant 2$.

Proposition 4.2.23. *Assume that R is a normal domain. Then every semidualizing R-module is reflexive, so there are containments* $\mathrm{Pic}(R) \subseteq \mathfrak{S}_0(R) \subseteq \mathrm{Cl}(R)$.

PROOF. Let C be a semidualizing R-module, and fix a prime $\mathfrak{p} \in \mathrm{Spec}(R)$. If $\mathrm{depth}(R_\mathfrak{p}) \geqslant 2$, then Proposition 2.2.10 implies that $\mathrm{depth}_{R_\mathfrak{p}}(C_\mathfrak{p}) \geqslant 2$. Assume that $\mathrm{depth}(R_\mathfrak{p}) \leqslant 1$. Since R is a normal domain, it satisfies Serre's conditions (R_1). In particular, the ring $R_\mathfrak{p}$ is regular. Since $C_\mathfrak{p}$ is semidualizing for $R_\mathfrak{p}$, Corollary 4.1.11 implies that $C_\mathfrak{p} \cong R_\mathfrak{p}$, so $C_\mathfrak{p}$ is a reflexive $R_\mathfrak{p}$-module. We conclude from Fact 4.2.22 that C is a reflexive R-module.

The containment $\mathfrak{S}_0(R) \subseteq \mathrm{Cl}(R)$ now follows. The other containment $\mathrm{Pic}(R) \subseteq \mathfrak{S}_0(R)$ is from Property 4.2.3.2. □

Corollary 4.2.24. *Assume that R is a unique factorization domain. If C is a semidualizing R-module, then $C \cong R$. If R has a dualizing module, then R is Gorenstein.*

PROOF. Since R is a unique factorization domain, we have $\{[R]\} \subseteq \mathfrak{S}_0(R) \subseteq \mathrm{Cl}(R) = \{[R]\}$, so $\mathfrak{S}_0(R) = \{[R]\}$. This yields the first of our desired conclusions. The second one follows from Corollary 4.1.11. □

The following example shows the utility of Proposition 4.2.23. It also shows that the set $\mathfrak{S}_0(R)$ does not have the structure of a subgroup of $\mathrm{Cl}(R)$. For details, see [**18**].

Example 4.2.25. Fix integers m, n, r such that $0 \leqslant r < m \leqslant n$ Let k be a field, and let $\mathbf{X} = (X_{i,j})$ be an $m \times n$ matrix of variables. Set $R = k[\mathbf{X}]/I_{r+1}(\mathbf{X})$ where $I_{r+1}(\mathbf{X})$ is the ideal of $k[\mathbf{X}]$ generated by the size $r+1$ minors of the matrix $\mathbf{X}$. Then R is a normal Cohen-Macaulay domain admitting a dualizing module D. It is Gorenstein if and only if either $m = n$ or $r = 0$.

Assume that $r \geqslant 1$, and let $\mathfrak{p} \subset R$ be the ideal generated by the size r minors of the matrix $\mathbf{x}$ of residues in R. Then $\mathrm{Cl}(R) \cong \mathbb{Z}$ with generator $[\mathfrak{p}]$. There is an isomorphism $D \cong \mathfrak{p}^{m-n}$, and we have $\mathfrak{S}_0(R) = \{[R], [D]\} \subseteq \mathrm{Cl}(R) \cong \mathbb{Z}$. In particular, if R is not Gorenstein, then $\mathfrak{S}_0(R)$ is a two element set, so it cannot be isomorphic to a subgroup of $\mathbb{Z}$.

Fact 4.2.26. Let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension between normal domains. Sather-Wagstaff and Spiroff [**19**] it is shown that there is an abelian group homomorphism $\mathrm{Cl}(\varphi)\colon \mathrm{Cl}(R) \to \mathrm{Cl}(S)$ given by the formula $[\mathfrak{a}] \mapsto [\mathrm{Hom}_S(\mathrm{Hom}_S(\mathfrak{a} \otimes_R S, S), S)]$. Furthermore, there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{S}_0(R) & \xrightarrow{\mathfrak{S}_0(\varphi)} & \mathfrak{S}_0(S) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Cl}(R) & \xrightarrow{\mathrm{Cl}(\varphi)} & \mathrm{Cl}(S) \end{array}$$

complimenting the diagram from Proposition 4.2.17(b).

4.3. More Relations

This section contains some results of Frankild and Sather-Wagstaff [**11**].

Proposition 4.3.1. *Let C and N be finitely generated R-modules such that the homothety map $\chi^R_{C \otimes_R C \otimes_R N}\colon R \to \mathrm{Hom}_R(C \otimes_R C \otimes_R N, C \otimes_R C \otimes_R N)$ is an isomorphism. Then C is a rank 1 projective R-module. If R is local, then $C \cong R$.*

Proof. Case 1: $(R,\mathfrak{m},k)$ is local. Let $\theta\colon C\otimes_R C\to C\otimes_R C$ be the commutativity isomorphism, given by the formula $c\otimes c'\mapsto c'\otimes c$. The induced map $\theta\otimes_R N\colon C\otimes_R C\otimes_R N\to C\otimes_R C\otimes_R N$ is also an isomorphism. Since the map $\chi^R_{C\otimes_R C\otimes_R N}$ is an isomorphism, there is an element $r\in R$ such that $\theta\otimes_R N=\chi^R_{C\otimes_R C\otimes_R N}(r)$, that is, such that $(\theta\otimes_R N)(\eta)=r\eta$ for all $\eta\in C\otimes_R C\otimes_R N$. Since $C\otimes_R C\otimes_R N$ is finitely generated and θ is an isomorphism, Nakayama's lemma implies that r is a unit.

Consider minimal finite free presentations

$$R^{b_1}\xrightarrow{\partial}R^{b_0}\xrightarrow{\tau}C\to 0\qquad\text{and}\qquad R^{c_1}\xrightarrow{d}R^{c_0}\xrightarrow{\pi}N\to 0.$$

Since $\operatorname{Hom}_R(C\otimes_R C\otimes_R N,C\otimes_R C\otimes_R N)\cong R\neq 0$, we have $C,N\neq 0$ and thus $b_0,c_0\geqslant 1$.

The right-exactness of tensor product yields the following exact sequence

$$\begin{matrix} R^{b_1}\otimes_R R^{b_0}\otimes_R R^{c_0} \\ \oplus \\ R^{b_0}\otimes_R R^{b_1}\otimes_R R^{c_0} \\ \oplus \\ R^{b_0}\otimes_R R^{b_0}\otimes_R R^{c_1}\end{matrix}\xrightarrow{\delta}R^{b_0}\otimes_R R^{b_0}\otimes_R R^{c_0}\xrightarrow{\tau\otimes\tau\otimes\pi}C\otimes_R C\otimes_R N\to 0$$

where

$$\delta=\begin{pmatrix}\partial\otimes_R R^{b_0}\otimes_R R^{c_0} & R^{b_0}\otimes_R\partial\otimes_R R^{c_0} & R^{b_0}\otimes_R R^{b_0}\otimes_R d\end{pmatrix}.$$

Since $\operatorname{Im}(\partial)\subseteq\mathfrak{m}R^{b_0}$ and $\operatorname{Im}(d)\subseteq\mathfrak{m}R^{c_0}$, we have $\operatorname{Im}(\delta)\subseteq\mathfrak{m}(R^{b_0}\otimes_R R^{b_0}\otimes_R R^{c_0})$.

Let $\theta'\colon R^{b_0}\otimes_R R^{b_0}\to R^{b_0}\otimes_R R^{b_0}$ be the commutativity isomorphism given by $e\otimes e'\mapsto e'\otimes e$. The induced map $\theta'\otimes_R R^{c_0}\colon R^{b_0}\otimes_R R^{b_0}\otimes_R R^{c_0}\to R^{b_0}\otimes_R R^{b_0}\otimes_R R^{c_0}$ is also an isomorphism. Let $\mu^r\colon R^{b_0}\otimes_R R^{b_0}\otimes_R R^{c_0}\to R^{b_0}\otimes_R R^{b_0}\otimes_R R^{c_0}$ be given by multiplication by r. For each $\zeta\in R^{b_0}\otimes_R R^{b_0}\otimes_R R^{c_0}$, the first step in the next sequence follows from the definitions of θ and θ':

$$\begin{aligned}(\tau\otimes\tau\otimes\pi)((\theta'\otimes R^{c_0})(\zeta))&=(\theta\otimes R^{c_0})((\tau\otimes\tau\otimes\pi)(\zeta))\\&=r(\tau\otimes\tau\otimes\pi)(\zeta)\\&=(\tau\otimes\tau\otimes\pi)(r\zeta)\\&=(\tau\otimes\tau\otimes\pi)(\mu^r(\zeta)).\end{aligned}$$

The other steps are by construction. It follows that

$$\operatorname{Im}((\theta'\otimes R^{c_0})-\mu^r)\subseteq\operatorname{Ker}(\tau\otimes\tau\otimes\pi)=\operatorname{Im}(\delta)\subseteq\mathfrak{m}(R^{b_0}\otimes_R R^{b_0}\otimes_R R^{c_0}).\quad(4.3.1.1)$$

We claim that $b_0=1$. Suppose that $b_0\neq 1$. Since $b_0\geqslant 1$, we then have $b_0\geqslant 2$. Consider a basis $e_1,e_2,\ldots,e_{b_0}\in R^{b_0}$. We know that in $(R^{b_0}\otimes_R R^{b_0})\otimes_R k\cong k^{b_0}\otimes_k k^{b_0}$, the vectors $\overline{e_2}\otimes\overline{e_1},\overline{e_1}\otimes\overline{e_2}$ are linearly independent. Letting $f\in R^{c_0}$ be any basis vector, we conclude similarly that the vectors $\overline{e_2}\otimes\overline{e_1}\otimes\overline{f},\overline{e_1}\otimes\overline{e_2}\otimes\overline{f}\in k^{b_0}\otimes_k k^{b_0}\otimes_k k^{c_0}$ are linearly independent.

The display (4.3.1.1) implies that in $R^{b_0}\otimes_R R^{b_0}\otimes_R R^{c_0}$ we have

$$e_2\otimes e_1\otimes f-re_1\otimes e_2\otimes f=((\theta'\otimes R^{c_0})-\mu^r)(e_1\otimes e_2\otimes f)\in\mathfrak{m}(R^{b_0}\otimes_R R^{b_0}\otimes_R R^{c_0}).$$

Reducing modulo $\mathfrak{m}$, this implies that

$$\overline{e_2}\otimes\overline{e_1}\otimes\overline{f}-\overline{r}\,\overline{e_1}\otimes\overline{e_2}\otimes\overline{f}\in k^{b_0}\otimes_k k^{b_0}\otimes_k k^{c_0}.$$

This implies that the vectors $\overline{e_2}\otimes\overline{e_1}\otimes\overline{f},\overline{e_1}\otimes\overline{e_2}\otimes\overline{f}\in k^{b_0}\otimes_k k^{b_0}\otimes_k k^{c_0}$ are linearly dependent. This is a contradiction, establishing the claim $b_0=1$.

It follows that C is cyclic, say $C \cong R/I$. It follows that $I(C \otimes_R C \otimes_R N) = 0$, and thus

$$I \subseteq \mathrm{Ann}_R(C \otimes_R C \otimes_R N) = \mathrm{Ker}(\chi^R_{C \otimes_R}) = 0.$$

We conclude that $C \cong R$, and this concludes the proof when R is local.

Case 2: the general case. As in the proof of Proposition 2.2.2(b), the fact that the homothety map $\chi^R_{C \otimes_R C \otimes_R N}\colon R \to \mathrm{Hom}_R(C \otimes_R C \otimes_R N, C \otimes_R C \otimes_R N)$ is an isomorphism implies that for each maximal ideal $\mathfrak{m} \subset R$ that the homothety map $\chi^{R_\mathfrak{m}}_{C_\mathfrak{m} \otimes_{R_\mathfrak{m}} C_\mathfrak{m} \otimes_{R_\mathfrak{m}} N_\mathfrak{m}}\colon R_\mathfrak{m} \to \mathrm{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m} \otimes_{R_\mathfrak{m}} C_\mathfrak{m} \otimes_{R_\mathfrak{m}} N_\mathfrak{m}, C_\mathfrak{m} \otimes_{R_\mathfrak{m}} C_\mathfrak{m} \otimes_{R_\mathfrak{m}} N_\mathfrak{m})$ is an isomorphism. Thus, Case 1 implies that $C_\mathfrak{m} \cong R_\mathfrak{m}$ for all $\mathfrak{m}$, that is, that C is a projective R-module of rank 1. □

Corollary 4.3.2. *Let C be a semidualizing R-module. Then $C \in \mathcal{A}_C(R)$ if and only if C is projective.*

Proof. If C is projective, then Proposition 4.1.4 implies that every R-module (in particular C) is in $\mathcal{A}_C(R)$. Conversely, if $C \in \mathcal{A}_C(R)$, then we conclude from Proposition 4.1.1(a) that $C \otimes_R C$ is semidualizing, so the desired conclusion follows from Proposition 4.3.1. □

Corollary 4.3.3. *Let $\langle B\rangle, \langle C\rangle \in \overline{\mathfrak{S}}_0(R)$ such that $\langle B\rangle \trianglelefteq \langle C\rangle$.*

(a) *If $\langle C\rangle \trianglelefteq \langle \mathrm{Hom}_R(C,B)\rangle$, then $\langle C\rangle = \langle B\rangle$.*
(b) *If $\langle \mathrm{Hom}_R(C,B)\rangle \trianglelefteq \langle C\rangle$, then $\langle C\rangle = \langle R\rangle$.*
(c) *If $\langle B\rangle \neq \langle C\rangle \neq \langle R\rangle$, then $\langle C\rangle$ and $\langle \mathrm{Hom}_R(C,B)\rangle$ are not comparable under the ordering on $\overline{\mathfrak{S}}_0(R)$.*

Proof. Recall that the condition $\langle B\rangle \trianglelefteq \langle C\rangle$ means that $B \in \mathcal{B}_C(R)$, so Proposition 4.1.1(b) implies that $\mathrm{Hom}_R(C,B)$ is semidualizing.

(a) Assume that $\langle C\rangle \trianglelefteq \langle \mathrm{Hom}_R(C,B)\rangle$, that is, that $C \in \mathcal{B}_{\mathrm{Hom}_R(C,B)}(R)$. Two applications of the defining property for membership in the Bass class imply that

$$\begin{aligned} B &\cong C \otimes_R \mathrm{Hom}_R(C,B) \\ &\cong [\mathrm{Hom}_R(C,B) \otimes_R \mathrm{Hom}_R(\mathrm{Hom}_R(C,B),C)] \otimes_R \mathrm{Hom}_R(C,B) \\ &\cong [\mathrm{Hom}_R(C,B) \otimes_R \mathrm{Hom}_R(C,B)] \otimes_R \mathrm{Hom}_R(\mathrm{Hom}_R(C,B),C). \end{aligned}$$

Since B is semidualizing, Proposition 4.3.1 implies that $\mathrm{Hom}_R(C,B)$ is a rank 1 projective R-module. Thus, the first line of the previous sequence implies that $\langle B\rangle = \langle C\rangle$.

(b) Assume that $\langle \mathrm{Hom}_R(C,B)\rangle \trianglelefteq \langle C\rangle$. As in the proof of part (a), this yields

$$B \cong C \otimes_R \mathrm{Hom}_R(C,B) \cong C \otimes_R C \otimes_R \mathrm{Hom}_R(C, \mathrm{Hom}_R(C,B)).$$

Since B is semidualizing, Proposition 4.3.1 implies that C is a rank 1 projective R-module, that is, that $\langle C\rangle = \langle R\rangle$.

(c) Assume that $\langle B\rangle \neq \langle C\rangle \neq \langle R\rangle$. If $\langle C\rangle \trianglelefteq \langle \mathrm{Hom}_R(C,B)\rangle$, then part (a) implies that $\langle C\rangle = \langle B\rangle$, a contradiction. If $\langle \mathrm{Hom}_R(C,B)\rangle \trianglelefteq \langle C\rangle$, then part (b) implies that $\langle C\rangle = \langle R\rangle$, a contradiction. □

Proposition 4.3.4. *Assume that R admits a point-wise dualizing module D.*

(a) *The operation $\Delta\colon \mathfrak{S}_0(R) \to \mathfrak{S}_0(R)$ given by $[C] \mapsto [\mathrm{Hom}_R(C,D)]$ is an involution (i.e., Δ^2 is the identity map).*
(b) *The operation $\overline{\Delta}\colon \overline{\mathfrak{S}}_0(R) \to \overline{\mathfrak{S}}_0(R)$ given by $\langle C\rangle \mapsto \langle \mathrm{Hom}_R(C,D)\rangle$ is an involution.*

(c) *If R is not point-wise Gorenstein, then $\overline{\Delta}$ has no fixed points.*

PROOF. Since D is a point-wise dualizing module for R, Corollary 2.2.13 implies that R is Cohen-Macaualy and D is a canonical module for R. Corollary 4.1.3 implies that $\operatorname{Hom}_R(C,D)$ is semidualizing for R. From this, we conclude that Δ is well-defined.

To see that $\overline{\Delta}$ is well defined, assume that $\langle B\rangle = \langle C\rangle$ in $\overline{\mathfrak{S}}_0(R)$. Lemma 4.2.8 implies that $B_\mathfrak{m} \cong C_\mathfrak{m}$ for each maximal ideal, so we have

$$\operatorname{Hom}_R(C,D)_\mathfrak{m} \cong \operatorname{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m}, D_\mathfrak{m}) \cong \operatorname{Hom}_{R_\mathfrak{m}}(B_\mathfrak{m}, D_\mathfrak{m}) \cong \operatorname{Hom}_R(C,D)_\mathfrak{m}$$

for each $\mathfrak{m}$. Another application of Lemma 4.2.8 shows that $\langle \operatorname{Hom}_R(C,D)\rangle = \langle \operatorname{Hom}_R(B,D)\rangle$, so $\overline{\Delta}$ is well defined.

Proposition 2.2.3 implies that $C_\mathfrak{m}$ is a semidualizing $R_\mathfrak{m}$-module for each maximal ideal $\mathfrak{m} \subset R$, so Theorem 2.2.6(c) implies that $C_\mathfrak{m}$ is a maximal Cohen-Macaulay $R_\mathfrak{m}$-module for each $\mathfrak{m}$. From [**8**, (3.3.10)] we conclude that the natural biduality map $\delta^{D_\mathfrak{m}}_{C_\mathfrak{m}} \colon C_\mathfrak{m} \to \operatorname{Hom}_{R_\mathfrak{m}}(\operatorname{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m}, D_\mathfrak{m}), D_\mathfrak{m})$ is an isomorphism for each $\mathfrak{m}$, and we conclude that the biduality map $\delta^D_C \colon C \to \operatorname{Hom}_R(\operatorname{Hom}_R(C,D),D)$ is an isomorphism. This shows that Δ and $\overline{\Delta}$ are involutions.

Lastly, assume that R is not point-wise Gorenstein, and suppose that $\langle C\rangle$ is a fixed point for $\overline{\Delta}$. This means that $\langle C\rangle = \langle \operatorname{Hom}_R(C,D)\rangle$ in $\overline{\mathfrak{S}}_0(R)$, that is, that $\langle C\rangle \trianglelefteq \langle \operatorname{Hom}_R(C,D)\rangle \trianglelefteq \langle C\rangle$. Recalling that $\langle D\rangle \trianglelefteq \langle C\rangle$, we conclude from Corollary 4.3.3(a) that $\langle C\rangle = \langle D\rangle$, Corollary 4.3.3(b) implies that $\langle C\rangle = \langle R\rangle$. It follows that $\langle D\rangle = \langle R\rangle$, that is, that there is a projective R-module such that $D \cong P \otimes_R R \cong P$. Corollary 4.1.10 implies that R is point-wise Gorenstein, a contradiction. □

Corollary 4.3.5. *The following conditions are equivalent:*

(i) *R is (point-wise) Gorenstein;*
(ii) *R admits a (point-wise) dualizing module D and a semidualizing module C such that $\langle C\rangle = \langle \operatorname{Hom}_R(C,D)\rangle$ in $\overline{\mathfrak{S}}_0(R)$; and*
(iii) *R admits a (point-wise) dualizing module D, and $\mathfrak{S}_0(R)$ is finite with odd cardinality.*

PROOF. (i) $\Longrightarrow$ (iii) If R is (point-wise) Gorenstein, then R is (point-wise) dualizing for R, and Property (4.2.3.3) implies that $\overline{\mathfrak{S}}_0(R) = \{\langle R\rangle\}$.

(iii) $\Longrightarrow$ (ii) Assume that R admits a (point-wise) dualizing module D, and $\mathfrak{S}_0(R)$ is finite with odd cardinality. Suppose that $\langle C\rangle \neq \langle \operatorname{Hom}_R(C,D)\rangle$ for each $\langle C\rangle \in \overline{\mathfrak{S}}_0(R)$. Since the map $\overline{\Delta} \colon \overline{\mathfrak{S}}_0(R) \to \overline{\mathfrak{S}}_0(R)$ is an involution, this implies that $\overline{\mathfrak{S}}_0(R)$ is a disjoint union of sets of cardinality 2, namely the sets of the form $\{\langle C\rangle, \langle \operatorname{Hom}_R(C,D)\rangle\}$. Since $\overline{\mathfrak{S}}_0(R)$ is finite, it follows that $\overline{\mathfrak{S}}_0(R)$ has even cardinality, a contradiction.

(ii) $\Longrightarrow$ (i) Assume that R admits a point-wise dualizing module D and a semidualizing module C such that $\langle C\rangle = \langle \operatorname{Hom}_R(C,D)\rangle$ in $\overline{\mathfrak{S}}_0(R)$. Then the element $\langle C\rangle$ is a fixed point of $\overline{\Delta}$, so Proposition c implies that R is point-wise Gorenstein.

Assume moreover that D is dualizing for R. Since R is point-wise Gorenstein, Corollary 4.1.10 implies that D is projective. The fact that D is dualizing for R then implies that R is Gorenstein, again by Corollary 4.1.10. □

The next result is a restatement of the equivalence (i) $\Longleftrightarrow$ (iii) from Corollary 4.3.5.

Corollary 4.3.6. *Assume that R admits a point-wise dualizing module D. If R is not point-wise Gorenstein, then $\mathfrak{S}_0(R)$ is either infinite or has even cardinality.*

Corollary 4.3.7. *The following conditions are equivalent:*

(i) *there exist elements of $\overline{\mathfrak{S}}_0(R)$ that are not comparable; and*
(ii) *$\overline{\mathfrak{S}}_0(R)$ has cardinality at least 3.*

Proof. (i) $\Longrightarrow$ (ii) Assume that $\langle C\rangle, \langle B\rangle \in \overline{\mathfrak{S}}_0(R)$ are incomparable elements. It follows that $\langle C\rangle \neq \langle B\rangle$. Since $\langle C\rangle \trianglelefteq \langle R\rangle$ and $\langle B\rangle \trianglelefteq \langle R\rangle$ it follows that $\langle C\rangle \neq \langle R\rangle \neq \langle B\rangle$. Thus, the elements $\langle C\rangle, \langle B\rangle, \langle R\rangle$ are three distinct elements of $\overline{\mathfrak{S}}_0(R)$.

(ii) $\Longrightarrow$ (i) Let $\langle C\rangle, \langle B\rangle, \langle C'\rangle$ be distinct elements of $\overline{\mathfrak{S}}_0(R)$. Assume without loss of generality that $\langle C'\rangle = \langle R\rangle$. Suppose that all elements of $\overline{\mathfrak{S}}_0(R)$ are comparable. Thus, we may assume without loss of generality that $\langle C\rangle \trianglelefteq \langle B\rangle \trianglelefteq \langle C'\rangle$ and $\langle C\rangle \neq \langle B\rangle \neq \langle C'\rangle$. Corollary 4.3.3(c) implies that the elements $\langle C\rangle$ and $\langle \operatorname{Hom}_R(C,B)\rangle$ are incomparable, a contradition. $\square$

CHAPTER 5

Totally C-reflexive Modules

This chapter is about duality.

5.1. Basic Properties of Totally C-reflexive Modules

The term "totally C-reflexive" is defined in 2.1.3.

Proposition 5.1.1. *Let C be a semidualizing R-module, and consider an exact sequence of finitely generated R-modules*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

such that M'' is totally C-reflexive. Then M' is totally C-reflexive if and only if M is totally C-reflexive.

Proof. Set $(-)^\dagger = \mathrm{Hom}_R(-, C)$.

Since M'' is totally C-reflexive, we have $\mathrm{Ext}^i_R(M'', C) = 0$ for all $i \geqslant 1$. Using the long exact sequence in $\mathrm{Ext}^i_R(-, C)$ associated to the given sequence, we conclude that $\mathrm{Ext}^i_R(M', C) = 0$ for all $i \geqslant 1$ if and only if $\mathrm{Ext}^i_R(M, C) = 0$ for all $i \geqslant 1$. Furthermore, the condition $\mathrm{Ext}^1_R(M'', C) = 0$ yields a second exact sequence

$$0 \to (M'')^\dagger \xrightarrow{g^\dagger} M^\dagger \xrightarrow{f^\dagger} (M')^\dagger \to 0. \tag{5.1.1.1}$$

Since M'' is totally C-reflexive, we have $\mathrm{Ext}^i_R((M'')^\dagger, C) = 0$ for all $i \geqslant 1$. Using the long exact sequence in $\mathrm{Ext}^i_R(-, C)$ for the sequence (5.1.1.1), we see that $\mathrm{Ext}^i_R((M')^\dagger, C) = 0$ for all $i \geqslant 2$ if and only if $\mathrm{Ext}^i_R(M^\dagger, C) = 0$ for all $i \geqslant 2$. The naturality of the biduality maps yields a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & & & 0 \\
 & & \downarrow{\scriptstyle \delta^C_{M'}} & & \downarrow{\scriptstyle \delta^C_M} & & \downarrow{\scriptstyle \delta^C_{M''}}\,\cong & & & & \\
0 & \longrightarrow & (M')^{\dagger\dagger} & \xrightarrow{f^{\dagger\dagger}} & M^{\dagger\dagger} & \xrightarrow{g^{\dagger\dagger}} & (M'')^{\dagger\dagger} & \longrightarrow & \mathrm{Ext}^1_R((M')^\dagger, C) & \longrightarrow \mathrm{Ext}^1_R(M^\dagger, C) \longrightarrow & 0.
\end{array}$$

The top row is exact by assumption, and the bottom row is the long exact sequence in $\mathrm{Ext}_R(-, C)$ associated to (5.1.1.1). Since g and $\delta^C_{M''}$ are surjective, we conclude that $g^{\dagger\dagger}$ is surjective as well. It follows that $\mathrm{Ext}^1_R((M')^\dagger, C) \cong \mathrm{Ext}^i_R(M^\dagger, C)$, so $\mathrm{Ext}^1_R((M')^\dagger, C) = 0$ for all $i \geqslant 1$ if and only if $\mathrm{Ext}^i_R(M^\dagger, C) = 0$ for all $i \geqslant 1$. The surjectivity of $g^{\dagger\dagger}$ implies that the bottom row of the next diagram is exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta^C_{M'}} & & \downarrow{\scriptstyle \delta^C_M} & & \downarrow{\scriptstyle \delta^C_{M''}}\,\cong & & \\
0 & \longrightarrow & (M')^{\dagger\dagger} & \xrightarrow{f^{\dagger\dagger}} & M^{\dagger\dagger} & \xrightarrow{g^{\dagger\dagger}} & (M'')^{\dagger\dagger} & \longrightarrow & 0.
\end{array}$$

The snake lemma shows that $\delta^C_{M'}$ is an isomorphism if and only if δ^C_M is an isomorphism. This completes the proof. $\square$

The next example shows that, given an exact sequence $0 \to M' \to M \to M'' \to 0$, if M' and M are totally C-reflexive, then M'' need not be totally C-reflexive. See however Proposition 5.1.3 below.

Example 5.1.2. Let k be a field and $R = k[\![X]\!]$ a formal power series ring in one variable. Consider the exact sequence

$$0 \to R \xrightarrow{X} R \to k \to 0.$$

The module R is totally reflexive, but k is not because $\operatorname{Ext}^1_R(k,R) \cong k \neq 0$.

For the next result, argue as in the proof of Proposition 5.1.1.

Proposition 5.1.3. *Let C be a semidualizing R-module, and consider an exact sequence of finitely generated R-modules*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

such that M' and M are totally C-reflexive. Then M'' is totally C-reflexive if and only if $\operatorname{Ext}^1_R(M'',C) = 0$.

The next result is proved by induction on n. The base case $n = 1$ is in Proposition 5.1.3.

Proposition 5.1.4. *Let C be a semidualizing R-module, and consider an exact sequence of finitely generated R-modules*

$$0 \to G_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} G_0 \to M \to 0$$

such that each module G_i is totally C-reflexive. Then M is totally C-reflexive if and only if $\operatorname{Ext}^i_R(M'',C) = 0$ for $i = 1,\dots,n$.

Proposition 5.1.5. *Let C be a semidualizing R-module, and let M be a finitely generated R-module. Then M is totally C-reflexive if and only if $\operatorname{Hom}_R(M,C)$ is totally C-reflexive and $\operatorname{Ext}^i_R(M,C) = 0$ for all $i \geqslant 1$.*

Proof. For the forward implication, assume that M is totally C-reflexive. Then $\operatorname{Ext}^i_R(M,C) = 0 = \operatorname{Ext}^i_R(\operatorname{Hom}_R(M,C),C)$ for all $i \geqslant 1$. The biduality map δ^C_M is an isomorphism, and hence so is $\operatorname{Hom}_R(\delta^C_M,C)$. From the readily verified equality

$$\operatorname{Hom}_R(\delta^C_M,C) \circ \delta^C_{\operatorname{Hom}_R(M,C)} = \operatorname{id}_{\operatorname{Hom}_R(M,C)} \tag{5.1.5.1}$$

we conclude that $\delta^C_{\operatorname{Hom}_R(M,C)}$ is an isomorphism as well. Furthermore, we have

$$\operatorname{Ext}^i_R(\operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C),C) \cong \operatorname{Ext}^i_R(M,C) = 0$$

for all $i \geqslant 1$, so $\operatorname{Hom}_R(M,C)$ is totally C-reflexive.

For the reverse implication, assume that $\operatorname{Hom}_R(M,C)$ is totally C-reflexive and $\operatorname{Ext}^i_R(M,C) = 0$ for all $i \geqslant 1$. Consider an exact sequence

$$0 \to M_1 \xrightarrow{\epsilon} P \to M \to 0 \tag{5.1.5.2}$$

where P is a finitely generated projective R-module. Since $\operatorname{Ext}^i_R(M,C) = 0 = \operatorname{Ext}^i_R(P,C)$ for all $i \geqslant 1$, the associated long exact sequence in $\operatorname{Ext}_R(-,C)$ shows that $\operatorname{Ext}^i_R(M_1,C) = 0$ for all $i \geqslant 1$. Furthermore, this yields an exact sequence

$$0 \to \operatorname{Hom}_R(M,C) \to \operatorname{Hom}_R(P,C) \to \operatorname{Hom}_R(M_1,C) \to 0.$$

Since $\operatorname{Hom}_R(M,C)$ and $\operatorname{Hom}_R(P,C)$ are totally C-reflexive, Proposition 5.1.3 implies that $\operatorname{Hom}_R(M_1,C)$ is totally C-reflexive.

We claim that M_1 is totally C-reflexive. (Once this is proved, an application of Proposition 5.1.3 to the sequence (5.1.5.2) implies that M is totally C-reflexive. Since $\operatorname{Hom}_R(M_1,C)$ is totally C-reflexive, we have $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M_1,C),C) = 0$ for all $i \geqslant 1$. We have already shown that $\operatorname{Ext}^i_R(M_1,C) = 0$ for all $i \geqslant 1$, so it remains to show that the biduality map $\delta^C_{M_1}\colon M_1 \to \operatorname{Hom}_R(\operatorname{Hom}_R(M_1,C),C)$ is an isomorphism. The fact that $\delta^C_{M_1}$ is injective follows from the next commutative diagram:

$$\begin{array}{ccc}
M_1 & \xhookrightarrow{\quad\epsilon\quad} & P \\
{\scriptstyle \delta^C_{M_1}}\downarrow & & \downarrow{\scriptstyle \delta^C_P}\,\cong \\
\operatorname{Hom}_R(\operatorname{Hom}_R(M_1,C),C) & \xrightarrow{\operatorname{Hom}_R(\operatorname{Hom}_R(\epsilon,C),C)} & \operatorname{Hom}_R(\operatorname{Hom}_R(P,C),C).
\end{array}$$

To show that $\delta^C_{M_1}$ is surjective, set $(-)^\dagger = \operatorname{Hom}_R(-,C)$ and consider the next exact sequence:

$$0 \to M_1 \xrightarrow{\delta^C_{M_1}} M_1^{\dagger\dagger} \to \operatorname{Coker}(\delta^C_{M_1}) \to 0.$$

For each $i \geqslant 1$ we have $\operatorname{Ext}^i_R(M_1^{\dagger\dagger},C) = 0$ since $M_1^\dagger$ is totally C-reflexive. Hence, the long exact sequence in $\operatorname{Ext}_R(-,C)$ shows that, for $i \geqslant 2$ we have

$$\operatorname{Ext}^i_R(\operatorname{Coker}(\delta^C_{M_1}),C) \cong \operatorname{Ext}^{i-1}_R(M_1,C) = 0. \tag{5.1.5.3}$$

The initial piece of this long exact sequence has the form

$$0 \to \operatorname{Coker}(\delta^C_{M_1})^\dagger \to (M_1^{\dagger\dagger})^\dagger \xrightarrow[\cong]{(\delta^C_{M_1})^\dagger} M_1^\dagger \to \operatorname{Ext}^1_R(\operatorname{Coker}(\delta^C_{M_1},C) \to 0.$$

The fact that $(\delta^C_{M_1})^\dagger$ is an isomorphism follows from equation (5.1.5.1) since $\delta^C_{M_1^\dagger}$ is an isomorphism. The exactness of this sequence shows that $\operatorname{Ext}^1_R(\operatorname{Coker}(\delta^C_{M_1},C) = 0 = \operatorname{Coker}(\delta^C_{M_1})^\dagger$. Coupled with (5.1.5.3) this implies that $\operatorname{Ext}^i_R(\operatorname{Coker}(\delta^C_{M_1}),C) = 0$ for all $i \geqslant 0$. Since C and $\operatorname{Coker}(\delta^C_{M_1})$ are finitely generated and $C \neq 0$, we conclude that $\operatorname{Coker}(\delta^C_{M_1}) = 0$; see, e.g. [**16**, (16.6)]. Hence, the map $\delta^C_{M_1}$ is surjective, as desired. □

We next present a result of Holm and Jørgensen [**15**]. It uses the notion of a "trivial extension" popularized by Nagata.

Remark 5.1.6. Let C be a semidualizing R-module. The *trivial extension* of R by C (also known as the *idealization* of C) is denoted $R \ltimes C$. As an R-module, we have $R \ltimes C = R \oplus C$. And we endow $R \ltimes C$ with a ring structure given by $(r,d)(r'd') = (rr', rd' + r'd)$. This makes $R \ltimes C$ into a (commutative noetherian) ring. Furthermore, there is a commutative diagram of ring homomorphisms

$$\begin{array}{ccc}
R & \xrightarrow{f_C} & R \ltimes C \\
& {\scriptstyle \operatorname{id}_R}\searrow & \downarrow{\scriptstyle g_C} \\
& & R
\end{array}$$

where $f(r) = (r,0)$ and $g(r,c) = r$.

Proposition 5.1.7. *Let C be a semidualizing R-module, and let M be a finitely generated R-module. Consider M as an $R \ltimes C$-module via the natural surjection $g_C \colon R \ltimes C \to R$. Then M is totally C-reflexive as an R-module if and only if M is totally reflexive as an $R \ltimes C$-module.*

PROOF. We begin with the following isomorphisms of R-modules:

$$\operatorname{Hom}_R(R\ltimes C, C) \cong \operatorname{Hom}_R(R\oplus C, C) \cong \operatorname{Hom}_R(C, C)\oplus\operatorname{Hom}_R(R, C) \cong R\oplus C \cong R\ltimes C.$$

It is straightforward to show that the R-module isomorphism $\operatorname{Hom}_R(R \ltimes C, C) \cong R \ltimes C$ is in fact an $R \ltimes C$-module isomorphism.

Let I be an injective resolution of C as an R-module. Let $f_C \colon R \to R \ltimes C$ be the natural inclusion. Since R and C are totally C-reflexive, the same is true of $R \oplus C \cong R \ltimes C$. In particular, we have $\operatorname{Ext}^i_R(R \ltimes C, C) = 0$ for all $i \geqslant 1$, so the complex $\operatorname{Hom}_R(R \ltimes C, I)$ is an injective resolution of the $R \ltimes C$-module $\operatorname{Hom}_R(R \ltimes C, C) \cong R \ltimes C$. This explains the first step in the next sequence:

$$\begin{aligned}\operatorname{Ext}^i_{R\ltimes C}(M, R \ltimes C) &\cong \operatorname{H}_{-i}(\operatorname{Hom}_{R\ltimes C}(M, \operatorname{Hom}_R(R \ltimes C, I)))\\ &\cong \operatorname{H}_{-i}(\operatorname{Hom}_R(R \ltimes C \otimes_{R\ltimes C} M, I))\\ &\cong \operatorname{H}_{-i}(\operatorname{Hom}_R(M, I))\\ &\cong \operatorname{Ext}^i_R(M, C).\end{aligned}$$

The second step is Hom-tensor adjointness, the third step is tensor cancellation, and the fourth step is by definition. It follows that $\operatorname{Ext}^i_{R\ltimes C}(M, R \ltimes C) = 0$ for all $i \geqslant 1$ if and only if $\operatorname{Ext}^i_R(M, R) = 0$ for all $i \geqslant 1$. Furthermore, it shows that $\operatorname{Hom}_{R\ltimes C}(M, R \ltimes C) \cong \operatorname{Hom}_R(M, C)$, and hence the first step in the next sequence:

$$\begin{aligned}\operatorname{Ext}^i_{R\ltimes C}(\operatorname{Hom}_{R\ltimes C}(M, R \ltimes C), R \ltimes C) &\cong \operatorname{Ext}^i_{R\ltimes C}(\operatorname{Hom}_R(M, C), R \ltimes C)\\ &\cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(M, C), C).\end{aligned}$$

The second step here follows from the previous display, using $\operatorname{Hom}_R(M, C)$ in place of M. It follows that $\operatorname{Ext}^i_{R\ltimes C}(\operatorname{Hom}_{R\ltimes C}(M, R \ltimes C), R \ltimes C) = 0$ for all $i \geqslant 1$ if and only if $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M, R), R) = 0$ for all $i \geqslant 1$. Furthermore, it shows that there is an $R \ltimes C$-module isomorphism $M \cong \operatorname{Hom}_{R\ltimes C}(\operatorname{Hom}_{R\ltimes C}(M, R \ltimes C), R \ltimes C)$ if and only if there is an R-module isomorphism $M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)$. Using Proposition 2.2.2(a) this means that the biduality map $\delta^{R\ltimes C}_M$ is an isomorphism if and only if δ^R_M is an isomorphism. The desired result now follows. $\square$

The next result is proved like Lemma 3.1.13.

Lemma 5.1.8. *Let C be a semidualizing R-module. Let M and N be R-modules such that $\operatorname{Ext}^i_R(N, C) = 0$ for all $i \geqslant 1$ (e.g., such that N is totally C-reflexive) and M is totally C-reflexive. Then $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M, C), \operatorname{Hom}_R(N, C)) \cong \operatorname{Ext}^i_R(N, M)$ for all $i \geqslant 0$.*

5.2. Complete PP_C-resolutions

Definition 5.2.1. A *complete PP_C resolution* is an exact sequence

$$X = \cdots \to P_1 \to P_0 \xrightarrow{\partial^X_0} C \otimes_R Q_0 \to C \otimes_R Q_1 \to \cdots$$

such that each P_i and Q_j is a finitely generated projective R-module and such that $\operatorname{Hom}_R(X, C)$ is exact. Such a sequence is a *complete PP_C resolution* of an R-module M when $M \cong \operatorname{Im}(\partial^X_0)$.

Remark 5.2.2. Consider a complete PP_C resolution

$$X = \cdots \to P_1 \to P_0 \xrightarrow{\partial_0^X} C \otimes_R Q_0 \to C \otimes_R Q_1 \to \cdots .$$

Proposition 2.1.13 implies that each module P_i and $C \otimes_R Q_j$ is totally C-reflexive. It follows that the natural biduality map $X \to \operatorname{Hom}_R(\operatorname{Hom}_R(X, C), C)$ is an isomorphism. In particular, the sequence $\operatorname{Hom}_R(\operatorname{Hom}_R(X, C), C)$ is exact.

Set $(-)^* = \operatorname{Hom}_R(-, R)$. Note that the modules P_i^* and Q_j^* are finitely generated and projective. (For instance, if P_i is a direct summand of R^n, then P_i^* is a direct summand of $(R^n)^* \cong R^n$.) Since each module P_i is finitely generated and projective, it is totally R-reflexive, so we have $P_i \cong P_i^{**}$. This explains the first step in the next sequence:

$$\operatorname{Hom}_R(P_i, C) \cong \operatorname{Hom}_R(P_i^{**}, C) \cong \operatorname{Hom}_R(R, C) \otimes_R P_i^* \cong C \otimes_R P_i^* \tag{5.2.2.1}$$

$$\operatorname{Hom}_R(C \otimes_R Q_j, C) \cong \operatorname{Hom}_R(Q_j, \operatorname{Hom}_R(C, C)) \cong \operatorname{Hom}_R(Q_j, R) \cong Q_j^*. \tag{5.2.2.2}$$

The second step is Hom-evaluation, and the third step is Hom-cancellation. The fourth step is Hom-tensor adjointness, and the fifth step follows from the fact that C is semidualizing. It follows that we have

$$\operatorname{Hom}_R(X, C) \cong \cdots \to Q_1^* \to Q_0^* \to C \otimes_R P_0^* \to C \otimes_R P_1^* \to \cdots$$

so $\operatorname{Hom}_R(X, C)$ is a complete PP_C resolution.

Here is a result of White [**21**].

Theorem 5.2.3. *For an R-module M, the following conditions are equivalent:*

(i) *M is totally C-reflexive;*
(ii) *M has a complete PP_C resolution; and*
(iii) *M has a complete PP_C resolution of the form*

$$X = \cdots \to R^{b_1} \to R^{b_0} \to C^{a_0} \to C^{a_1} \to \cdots .$$

Proof. Set $(-)^\dagger = \operatorname{Hom}_R(-, C)$. The implication (iii) $\Longrightarrow$ (ii) is routine.

(i) $\Longrightarrow$ (iii) Assume that M is totally C-reflexive, and consider free resolutions

$$P^+ = \cdots \to R^{b_1} \to R^{b_0} \to M \to 0$$

$$Q^+ = \cdots \to R^{a_1} \to R^{a_0} \to M^\dagger \to 0.$$

Since M is totally C-reflexive, we have $\operatorname{Ext}_R^i(M^\dagger, C) = 0$ for all $i \geqslant 1$, so the following sequence is exact:

$$(Q^+)^\dagger \cong \quad 0 \to M^{\dagger\dagger} \to C^{a_0} \to C^{a_1} \to \cdots .$$

Splicing P^+ and $(Q^+)^\dagger$ along the isomorphism $\delta_M^C \colon M \xrightarrow{\cong} M^{\dagger\dagger}$ yields the next commutative diagram with exact row

$$\begin{array}{ccccccccccc} X = & \cdots & \longrightarrow & R^{b_1} & \longrightarrow & R^{b_0} & \xrightarrow{\quad\partial\quad} & C^{a_0} & \longrightarrow & C^{a_1} & \longrightarrow \cdots \\ & & & & & \searrow & & \nearrow & & & \\ & & & & & & M & & & & \end{array}$$

such that $M \cong \operatorname{Im}(\partial)$.

We need to show that the complex $X^\dagger$ in the row of the next commutative diagram is exact:

$$X^\dagger = \quad \cdots \longrightarrow R^{a_1} \longrightarrow R^{a_0} \xrightarrow{\partial^\dagger} C^{b_0} \longrightarrow C^{b_1} \longrightarrow \cdots$$
$$R^{a_0} \twoheadrightarrow M^\dagger \hookrightarrow C^{b_0}.$$

It is straightforward to show that this diagram is isomorphic to the one obtained by splicing Q^+ and $(P^+)^\dagger$ along the identity $M^\dagger \xrightarrow{=} M^\dagger$. The sequence Q^+ is exact by assumption, and the sequence $(P^+)^\dagger$ is exact because $\operatorname{Ext}^i_R(M, C) = 0$ for all $i \geqslant 1$. It follows that $X^\dagger$ is exact as desired.

(ii) $\Longrightarrow$ (i) Assume that M has a complete PP_C resolution

$$X = \quad \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\partial} C \otimes_R Q_0 \longrightarrow C \otimes_R Q_1 \longrightarrow \cdots$$
$$P_0 \twoheadrightarrow M \hookrightarrow C \otimes_R Q_0$$

Since X is exact, it follows that the complex

$$P^+ = \cdots \to R^{b_1} \to R^{b_0} \to M \to 0$$

is exact, that is, that P^+ is an augmented projective resolution of M. Since $X^\dagger$ is exact, it follows that $(P^+)^\dagger$ is exact, so we have $\operatorname{Ext}^i_R(M, C) = 0$ for all $i \geqslant 1$.

As in Remark 5.2.2, the sequence $X^\dagger$ has the following form

$$X^\dagger = \quad \cdots \longrightarrow Q_1^* \longrightarrow Q_0^* \xrightarrow{\partial^\dagger} C \otimes_R P_0^* \longrightarrow C \otimes_R P_1^* \longrightarrow \cdots$$
$$Q_0^* \twoheadrightarrow M^\dagger \hookrightarrow C \otimes_R P_0^*$$

where $(-)^* = \operatorname{Hom}_R(-, R)$. Furthermore, we have $X \cong X^{\dagger\dagger}$. Since $X^\dagger$ is exact, we conclude that the next sequence

$$\tilde{Q}^+ = \cdots \to Q_1^* \to Q_0^* \to M^\dagger \to 0$$

is an augmented projective resolution of $M^\dagger$. Since $X^{\dagger\dagger} \cong X$ is exact, it follows that $\operatorname{Ext}^i_R(M^\dagger, C) = 0$ for all $i \geqslant 1$.

Finally, there is a commutative diagram with exact rows:

$$\begin{array}{llcccccc}
Q^+ = & 0 \longrightarrow & M & \longrightarrow & C \otimes_R Q_0 & \longrightarrow & C \otimes_R Q_1 & \longrightarrow \cdots \\
& & \downarrow{\scriptstyle \delta^C_M} & & \downarrow{\scriptstyle \delta^C_{C\otimes_R Q_0}}\ \cong & & \downarrow{\scriptstyle \delta^C_{C\otimes_R Q_1}}\ \cong & \\
(\tilde{Q}^+)^\dagger & 0 \longrightarrow & M^{\dagger\dagger} & \longrightarrow & (C \otimes_R Q_0)^{\dagger\dagger} & \longrightarrow & (C \otimes_R Q_1)^{\dagger\dagger} & \longrightarrow \cdots .
\end{array}$$

Since the maps $\delta^C_{C\otimes_R Q_i}$ are isomorphisms, so is δ^C_M, so M is totally C-reflexive. $\square$

Proposition 5.2.4. *Let C be a semidualizing R-module, and let P be a finitely generated projective R-module. Let G be a totally C-reflexive R-module with complete PP_C resolution*

$$X = \cdots \to P_1 \to P_0 \xrightarrow{\partial_0^X} C \otimes_R Q_0 \to C \otimes_R Q_1 \to \cdots .$$

(a) *The R-module* $\operatorname{Hom}_R(P,G)$ *is totally C-reflexive with complete PP_C resolution* $\operatorname{Hom}_R(P,X)$.
(b) *The R-module* $P\otimes_R G$ *is totally C-reflexive with complete PP_C resolution* $P\otimes_R X$.
(c) *The R-module* $\operatorname{Hom}_R(G,C\otimes_R P)$ *is totally C-reflexive with complete PP_C resolution* $\operatorname{Hom}_R(X,C\otimes_R P)$. *In particular, the module* $\operatorname{Hom}_R(G,C)$ *is totally C-reflexive with complete PP_C resolution* $\operatorname{Hom}_R(X,C)$.

PROOF. (a) Since P is a finitely generated projective, the modules $\operatorname{Hom}_R(P,P_i)$ and $\operatorname{Hom}_R(P,Q_j)$ are projective. Hence, the sequence

$$\operatorname{Hom}_R(P,X)\cong\cdots\to\operatorname{Hom}_R(P,P_0)\xrightarrow{\operatorname{Hom}_R(P,\partial_0^X)}C\otimes_R(\operatorname{Hom}_R(P,Q_0))\to\cdots$$

has the form of a complete PP_C resolution. (The isomorphisms $\operatorname{Hom}_R(P,C\otimes_R Q_j)\cong C\otimes_R\operatorname{Hom}_R(P,Q_j)$ are by tensor evaluation.) This sequence is exact and has $\operatorname{Hom}_R(P,G)\cong\operatorname{Im}(\operatorname{Hom}_R(P,\partial_0^X))$ because the functor $\operatorname{Hom}_R(P,-)$ is exact. Hom evaluation explains the next isomorphism

$$\operatorname{Hom}_R(\operatorname{Hom}_R(P,X),C)\cong P\otimes_R\operatorname{Hom}_R(X,C)$$

so this sequence is exact because the sequence $\operatorname{Hom}_R(X,C)$ and the functor $P\otimes_R-$ are exact. This concludes the proof of part (a).

Parts (b) and (c) are proved similarly. □

Proposition 5.2.5. *Let C be a semidualizing R-module, and let G be a totally C-reflexive R-module with complete PP_C resolution*

$$X=\cdots\xrightarrow{\partial_2^X}P_1\xrightarrow{\partial_1^X}P_0\xrightarrow{\partial_0^X}C\otimes_R Q_0\xrightarrow{\partial_{-1}^X}C\otimes_R Q_1\xrightarrow{\partial_{-2}^X}\cdots.$$

Then for each $i\in\mathbb{Z}$, the module $\operatorname{Im}(\partial_i^X)$ *is totally C-reflexive.*

PROOF. When $i=0$, this is by assumption since $G\cong\operatorname{Im}(\partial_0^X)$; see Theorem 5.2.3. For $i\geqslant 1$, this follows from an induction argument using Proposition 5.1.1 with the fact that each P_j is totally C-reflexive; see Proposition 2.1.13.

Assume that $i\leqslant 0$. We claim that $\operatorname{Ext}_R^j(\operatorname{Im}(\partial_i^X),C)=0$ for all $j\geqslant 1$. For $i=0$, this is by assumption. For $i<0$, this follows by induction on i using the long exact sequence in $\operatorname{Ext}_R(-,C)$ associated to the sequence

$$0\to\operatorname{Im}(\partial_{i+1}^X)\to C\otimes_R Q_i\to\operatorname{Im}(\partial_i^X)\to 0$$

since $\operatorname{Ext}_R^j(C\otimes_R Q_i,C)=0$ for all $j\geqslant 1$ by Proposition 2.1.13.

Now the desired conclusion follows from Proposition 5.1.4 applied to the exact sequence $0\to G\to C\otimes_R Q_0\to\cdots\to C\otimes_R Q_i\to\operatorname{Im}(\partial_{i+1}^X)\to 0$. □

5.3. Base Change for Totally C-reflexive Modules

This section is similar to section 3.4

Proposition 5.3.1. *Let C be a semidualizing R-module, let $\varphi\colon R\to S$ be a flat ring homomorphism, and Let M be a finitely generated R-module. If M is totally C-reflexive, then the S-module $S\otimes_R M$ is totally $S\otimes_R C$-reflexive; the converse holds when φ is faithfully flat.*

PROOF. Because φ is flat, Corollary 3.4.2 implies that $S\otimes_R C$ is a semidualizing S-module. Since M is finitely generated and φ is flat, we have isomorphisms

$$\operatorname{Ext}^i_S(S\otimes_R M, S\otimes_R C)\cong S\otimes_R \operatorname{Ext}^i_R(M,C)$$
$$\operatorname{Ext}^i_S(\operatorname{Hom}_S(S\otimes_R M, S\otimes_R C), S\otimes_R C)\cong S\otimes_R \operatorname{Ext}^i_R(\operatorname{Hom}_R(M,C),C).$$

Hence, if $\operatorname{Ext}^i_R(M,C)=0=\operatorname{Ext}^i_R(\operatorname{Hom}_R(M,C),C)$ for all $i\geqslant 1$, then we conclude that $\operatorname{Ext}^i_S(S\otimes_R M, S\otimes_R C)=0=\operatorname{Ext}^i_S(\operatorname{Hom}_S(S\otimes_R M, S\otimes_R C), S\otimes_R C)$ for all $i\geqslant 1$; and the converse holds when φ is faithfully flat. We also have a commutative diagram

$$\begin{array}{ccc} S\otimes_R M & \xrightarrow{\delta^{S\otimes_R C}_{S\otimes_R M}} & \operatorname{Hom}_S(\operatorname{Hom}_S(S\otimes_R M, S\otimes_R C), S\otimes_R C) \\ {\scriptstyle S\otimes_R\delta^C_M}\downarrow & & \downarrow\cong \\ S\otimes_R \operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C) & \xrightarrow{\cong} & \operatorname{Hom}_S(S\otimes_R \operatorname{Hom}_S(M,C), S\otimes_R C) \end{array}$$

where the unspecified isomorphisms are the natural ones. Hence, if the biduality map δ^C_M is an isomorphism, then so is $\delta^{S\otimes_R C}_{S\otimes_R M}$; and the converse holds when φ is faithfully flat. □

Proposition 5.3.2. *Let C be a semidualizing R-module, and let M be a finitely generated R-module. Let $\mathbf{x}=x_1,\ldots,x_d\in R$ be a sequence that is R-regular and such that $\mathbf{x}M\neq M$, and set $\overline{R}=R/(\mathbf{x})R$. If M is is totally C-reflexive, then $\mathbf{x}$ is M-regular and the $\overline{R}$-module $\overline{R}\otimes_R M$ is totally $\overline{R}\otimes_R C$-reflexive. The converse holds when $\mathbf{x}$ is in the Jacobson radical $\operatorname{J}(R)$.*

PROOF. We argue by induction on d. We prove the base case $d=1$ and leave the inductive step as a routine exercise.

For notational simplicity, set $x=x_1$ and $\overline{(-)}=\overline{R}\otimes_R -$. Recall that if N is a finitely generated R-module such that x is N-regular, then we have

$$\operatorname{Ext}^i_R(N,\overline{C})\cong \operatorname{Ext}^i_{\overline{R}}(\overline{N},\overline{C}) \tag{5.3.2.1}$$

for all $i\geqslant 1$; see, e.g. [**16**, p. 140, Lemma 2]. Also, there is an exact sequence

$$0\to C\xrightarrow{x} C\to \overline{C}\to 0. \tag{5.3.2.2}$$

since x is C-regular.

Step 1: We show that if M is totally C-reflexive, then x is M-regular. For this, we use the following sequence:

$$\begin{aligned}\operatorname{Ass}_R(M)&=\operatorname{Ass}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C))\\ &=\operatorname{Supp}_R(\operatorname{Hom}_R(M,C))\cap \operatorname{Ass}_R(C)\\ &\subseteq \operatorname{Ass}_R(C)\\ &=\operatorname{Ass}(R).\end{aligned}$$

The first step is by assumption, and the second step is a result of Bourbaki. The third step is routine, and the fourth step is from Proposition 2.1.16(a). Since x is R-regular, it is not in any associated prime ideal of R. Thus, the previous sequence shows that x is not in any associated prime ideal of M, so x is M-regular. This completes Step 1.

Because of Step 1, we may assume without loss of generality for the rest of the proof that x is M-regular.

Step 2: We show that (a) if $\operatorname{Ext}^i_R(M,C)=0$ for all $i\geqslant 1$, then $\operatorname{Ext}^i_{\overline{R}}(\overline{M},\overline{C})=0$ for all $i\geqslant 1$, and (b) the converse holds when $x\in \mathrm{J}(R)$.

(a) If $\operatorname{Ext}^i_R(M,C)=0$ for all $i\geqslant 1$, then the long exact sequence in $\operatorname{Ext}^i_R(M,-)$ associated to the sequence (5.3.2.2) shows that $0=\operatorname{Ext}^i_R(M,\overline{C})\cong\operatorname{Ext}^i_{\overline{R}}(\overline{M},\overline{C})$ for all $i\geqslant 1$; the isomorphism is from (5.3.2.1).

(b) Conversely, assume that $\operatorname{Ext}^i_{\overline{R}}(\overline{M},\overline{C})=0$ for all $i\geqslant 1$ and that $x\in \mathrm{J}(R)$. From (5.3.2.1) we conclude that $\operatorname{Ext}^i_R(M,\overline{C})=0$ for all $i\geqslant 1$. Hence, part of the the long exact sequence in $\operatorname{Ext}^i_R(M,-)$ associated to (5.3.2.2) has the form

$$\operatorname{Ext}^i_R(M,C)\xrightarrow{x}\operatorname{Ext}^i_R(M,C)\to 0$$

for all $i\geqslant 1$. This implies that $\operatorname{Ext}^i_R(M,C)=x\operatorname{Ext}^i_R(M,C)$, so Nakayama's Lemma implies that $\operatorname{Ext}^i_R(M,C)=0$. This completes Step 2.

Because of Step 2, we may assume without loss of generality for the rest of the proof that $\operatorname{Ext}^i_R(M,C)=0$ for all $i\geqslant 1$.

Step 3: We show that $\overline{\operatorname{Hom}_R(M,C)}\cong\operatorname{Hom}_{\overline{R}}(\overline{M},\overline{C})$ and that x is $\operatorname{Hom}_R(M,C)$-regular.

Because we have $\operatorname{Ext}^i_R(M,C)=0$, the long exact sequence in $\operatorname{Ext}^i_R(M,-)$ associated to (5.3.2.2) begins as

$$0\to\operatorname{Hom}_R(M,C)\xrightarrow{x}\operatorname{Hom}_R(M,C)\to\operatorname{Hom}_R(M,\overline{C})\to 0.$$

This shows that x is a non-zero-divisor on $\operatorname{Hom}_R(M,C)$. It also explains the first isomorphism in the next display

$$\overline{\operatorname{Hom}_R(M,C)}\cong\operatorname{Hom}_R(M,\overline{C})\cong\operatorname{Hom}_{\overline{R}}(\overline{M},\overline{C})$$

while the second isomorphism is from (5.3.2.1).

Thus, to complete Step 3, it remains to show that $\overline{\operatorname{Hom}_R(M,C)}\neq 0$, that is, that $\operatorname{Hom}_{\overline{R}}(\overline{M},\overline{C})\neq 0$. Suppose by way of contradiction that $\operatorname{Hom}_{\overline{R}}(\overline{M},\overline{C})=0$. Since $\overline{C}$ is a semidualizing $\overline{R}$-module, we have $\operatorname{Supp}_{\overline{R}}(\overline{C})=\operatorname{Spec}(\overline{R})$ by Proposition 2.1.16(a). Step 2 shows that $\operatorname{Ext}^i_{\overline{R}}(\overline{M},\overline{C})=0$ for all $i\geqslant 0$. We conclude that $\overline{M}=0$, contradicting the assumption that x is M-regular. This completes Step 3.

Step 4: We show that (a) if $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M,C),C)=0$ for all $i\geqslant 1$, then $\operatorname{Ext}^i_{\overline{R}}(\operatorname{Hom}_{\overline{R}}(\overline{M},\overline{C}),\overline{C})=0$ for all $i\geqslant 1$, (b) the converse holds when $x\in \mathrm{J}(R)$.

From Step 3, we know that x is $\operatorname{Hom}_R(M,C)$-regular, and that $\overline{\operatorname{Hom}_R(M,C)}\cong\operatorname{Hom}_{\overline{R}}(\overline{M},\overline{C})$. Thus, Step 2 shows that (a) if $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M,C),C)=0$ for all $i\geqslant 1$, then $0=\operatorname{Ext}^i_{\overline{R}}(\overline{\operatorname{Hom}_R(M,C)},\overline{C})\cong\operatorname{Ext}^i_{\overline{R}}(\operatorname{Hom}_{\overline{R}}(\overline{M},\overline{C}),\overline{C})=0$ for all $i\geqslant 1$, and (b) the converse holds when $x\in \mathrm{J}(R)$. This concludes Step 4.

Because of Step 4, we may assume that $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M,C),C)=0$ for $i\geqslant 1$.

Step 5: We show that $\overline{\operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C)}\cong\operatorname{Hom}_{\overline{R}}(\operatorname{Hom}_{\overline{R}}(\overline{M},\overline{C}),\overline{C})$ and that x is $\operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C)$-regular. As in Step 4, this follows from an application of Step 3 to the module $\operatorname{Hom}_R(M,C)$.

Step 6: We show that (a) if the biduality map δ^C_M is an isomorphism, then the biduality map $\delta^{\overline{C}}_{\overline{M}}$ is an isomorphism, and (b) the converse holds when $x \in \mathrm{J}(R)$.

For this, consider the following commutative diagram:

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\delta^{\overline{C}}_{\overline{M}}} & \mathrm{Hom}_{\overline{R}}(\mathrm{Hom}_{\overline{R}}(\overline{M},\overline{C}),\overline{C}) \\ \Big\downarrow{\scriptstyle \overline{\delta^M_C}} & & \Big\downarrow{\scriptstyle \cong} \\ \overline{\mathrm{Hom}_R(\mathrm{Hom}_R(M,C),C)} & \xrightarrow{\cong} & \mathrm{Hom}_{\overline{R}}(\overline{\mathrm{Hom}_R(M,C)},\overline{C}). \end{array}$$

The unspecified isomorphisms are from Steps 3 and 5.

(a) If δ^C_M is an isomorphism, then so is $\overline{\delta^C_M}$, and the diagram shows that $\delta^{\overline{C}}_{\overline{M}}$ is an isomorphism.

(b) Assume that $\delta^{\overline{C}}_{\overline{M}}$ is an isomorphism and $x \in \mathrm{J}(R)$. The diagram above shows that $\overline{\delta^C_M}$ is an isomorphism. Consider the exact sequence

$$M \xrightarrow{\delta^C_M} \mathrm{Hom}_R(\mathrm{Hom}_R(M,C),C) \to \mathrm{Coker}(\delta^C_M) \to 0$$

and apply the right-exact functor $\overline{(-)}$ to obtain the exact sequence

$$\overline{M} \xrightarrow[\cong]{\overline{\delta^C_M}} \overline{\mathrm{Hom}_R(\mathrm{Hom}_R(M,C),C)} \to \overline{\mathrm{Coker}(\delta^C_M)} \to 0.$$

Since $\overline{\delta^C_M}$ is an isomorphism, it follows that $\overline{\mathrm{Coker}(\delta^C_M)} = 0$. Nakayama's Lemma implies that $\mathrm{Coker}(\delta^C_M) = 0$, that is, that δ^C_M is surjective.

Now, consider the exact sequence

$$0 \to \mathrm{Ker}(\delta^C_M) \to M \xrightarrow{\delta^C_M} \mathrm{Hom}_R(\mathrm{Hom}_R(M,C),C) \to 0.$$

Since x is $\mathrm{Hom}_R(\mathrm{Hom}_R(M,C),C)$-regular, we have

$$\mathrm{Tor}^R_1(\overline{R}, \mathrm{Hom}_R(\mathrm{Hom}_R(M,C),C)) = 0$$

so the following sequence is exact:

$$0 \to \overline{\mathrm{Ker}(\delta^C_M)} \to \overline{M} \xrightarrow[\cong]{\overline{\delta^C_M}} \overline{\mathrm{Hom}_R(\mathrm{Hom}_R(M,C),C)} \to 0.$$

Since $\overline{\delta^C_M}$ is an isomorphism, it follows that $\overline{\mathrm{Ker}(\delta^C_M)} = 0$. Nakayama's Lemma implies that $\mathrm{Ker}(\delta^C_M) = 0$, that is, that δ^C_M is injective. This completes Step 6 and the proof of the result. □

The next result is proved like Proposition 3.4.9.

Proposition 5.3.3. *Let k be a field, and let R and S be k-algebras. Let B and M be R-modules such that B is semidualizing, and let C and N be S-modules such that C is semidualizing. If M is totally B-reflexive and N is totally C-reflexive, then $M \otimes_k N$ is totally $B \otimes_k C$-reflexive.*

5.4. Local-Global Principle for Totally C-reflexive Modules

The next result is from unpublished notes by Foxby. See also Avramov, Iyengar, and Lipman [**7**].

Proposition 5.4.1. *Let C and M be R-modules such that M is finitely generated.*

(a) *If there is an R-module isomorphism $\alpha\colon M \xrightarrow{\cong} \operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C)$, then the natural biduality map $\delta^C_M\colon M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C)$ is an isomorphism.*

(b) *Assume that C is finitely generated. If for every maximal ideal $\mathfrak{m} \subset R$ there is an $R_\mathfrak{m}$-module isomorphism $M_\mathfrak{m} \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C)_\mathfrak{m}$, then the natural biduality map $\delta^C_M\colon M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C)$ is an isomorphism.*

Proof. (a) It is straightforward to show that the following diagram commutes

$$\begin{array}{ccc}
\operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C) & \xrightarrow{\delta'} & \operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C),C),C) \\
& \searrow^{\operatorname{id}_{\operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C)}} & \downarrow{\scriptstyle \operatorname{Hom}_R(\delta^C_{\operatorname{Hom}_R(M,C)},C)} \\
& & \operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C)
\end{array}$$

where $\delta' = \delta^C_{\operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C)}$. In particular, the map δ' is a split monomorphism. With $X = \operatorname{Coker}(\delta')$, this explains the second isomorphism in the next sequence:

$$\begin{aligned}
M \oplus X &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C) \oplus X \\
&\cong \operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C),C),C) \\
&\cong M.
\end{aligned}$$

The other isomorphism are induced by α. Since M is finitely generated, this implies that $X = 0$, that is, that δ' is surjective. Since it is also injective, we have the right-hand vertical isomorphism in the next diagram:

$$\begin{array}{ccc}
M & \xrightarrow[\cong]{\alpha} & \operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C) \\
{\scriptstyle \delta^C_M}\downarrow & & \cong\downarrow{\scriptstyle \delta'} \\
\operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C) & \xrightarrow[\cong]{\alpha'} & \operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C),C),C).
\end{array}$$

Here $\alpha' = \operatorname{Hom}_R(\operatorname{Hom}_R(\alpha,C),C)$, and it follows that δ^C_M is an isomorphism.

(b) This follows from part (a) as in the proof of Proposition 2.2.2. □

Here is a local global principal for totally reflexive modules. Its proof is similar to that of Proposition 3.5.3.

Proposition 5.4.2. *Let C be a semidualizing R-module, and let M be a finitely generated R-module. The following conditions are equivalent:*

(i) *M is a totally C-reflexive R-module;*

(ii) *$U^{-1}M$ is a totally $U^{-1}C$-reflexive $U^{-1}R$-module for each multiplicatively closed subset $U \subset R$;*

(iii) *$M_\mathfrak{p}$ is a totally $C_\mathfrak{p}$-reflexive $R_\mathfrak{p}$-module for each prime ideal $\mathfrak{p} \subset R$; and*

(iv) *$M_\mathfrak{m}$ is a totally $C_\mathfrak{m}$-reflexive $R_\mathfrak{m}$-module for each maximal ideal $\mathfrak{m} \subset R$.*

If X is a complete PP_C resolution of M over R, then $U^{-1}X$ is a complete $PP_{U^{-1}C}$ resolution of $U^{-1}M$ over $U^{-1}R$ for each multiplicatively closed subset $U \subset R$.

The next result contains partial converses of Proposition 5.2.4(a)–(b).

Proposition 5.4.3. *Let C be a semidualizing R-module, let P be a finitely generated faithfully projective R-module, and let M be a finitely generated R-module. If $M \otimes_R P$ or $\operatorname{Hom}_R(P, M)$ is totally C-reflexive, then M is is totally C-reflexive.*

Proof. The assumption that P is a finitely generated faithfully projective R-module implies that, for each maximal ideal $\mathfrak{m} \subset R$, there is an integer $e_\mathfrak{m} \geqslant 1$ such that $P_\mathfrak{m} \cong R_\mathfrak{m}^{e_\mathfrak{m}}$. If $M \otimes_R P$ is totally C-reflexive, then Proposition 5.4.2 implies that the $R_\mathfrak{m}$-module

$$(M \otimes_R P)_\mathfrak{m} \cong M_\mathfrak{m} \otimes_{R_\mathfrak{m}} P_\mathfrak{m} \cong M_\mathfrak{m} \otimes_{R_\mathfrak{m}} R_\mathfrak{m}^{e_\mathfrak{m}} \cong M_\mathfrak{m}^{e_\mathfrak{m}}$$

is totally $C_\mathfrak{m}$-reflexive for each $\mathfrak{m}$. Since $e_\mathfrak{m} \geqslant 1$, Proposition 2.1.4 implies that $M_\mathfrak{m}$ is totally $C_\mathfrak{m}$-reflexive for each $\mathfrak{m}$, so we conclude from Proposition 5.4.2 that M is is totally C-reflexive.

The proof is similar when $\operatorname{Hom}_R(P, M)$ is assume to be totally C-reflexive. □

The next result is proved like Proposition 2.3.4, using Proposition 5.4.2.

Corollary 5.4.4. *Let $R_1, \ldots, R_n$ be noetherian rings, and consider the product $R = R_1 \times \cdots \times R_n$. For $i = 1, \ldots, n$ let C_i be a semidualizing R_i-module, and set $C = C_1 \times \cdots \times C_n$. There is a bijection $\mathcal{G}_C(R_1) \times \cdots \times \mathcal{G}_C(R_n) \xrightarrow{\sim} \mathcal{G}_C(R)$ given by $(M_1, \ldots, M_n) \mapsto M_1 \times \cdots \times M_n$.*

The following example shows why we need P to be faithfully projective in Proposition 5.4.3.

Example 5.4.5. Let $(R_1, \mathfrak{m}_1, k_1)$ be an artinian local ring that has a semidualizing module C_1 that is not dualizing; see Example 2.3.1. Let R_2 be a field and set $R = R_1 \times R_2$. The module $C = C_1 \times R_2$ is semidualizing for R. Set $P = 0 \times R_2$ and $M = k \times 0$. It follows that $P \otimes_R M = 0 = \operatorname{Hom}_R(P, M)$, and thus $P \otimes_R M$ and $\operatorname{Hom}_R(P, M)$ are totally C-reflexive. However, the module M is not totally C-reflexive. Indeed, we have $\operatorname{Ext}^i_{R_1}(k_1, C_1) \neq 0$ for all $i \geqslant 0$ since R_1 is artinian and $\operatorname{id}_{R_1}(C_1) = \infty$. Thus, the R_1-module k_1 is not totally C_1-reflexive, so Corollary 5.4.4 implies that M is not totally C-reflexive.

The next result is like [**8**] and has a similar proof. See also Proposition 2.2.10.

Proposition 5.4.6. *Let C be a semidualizing R-module, and let M be a finitely generated R-module. Then M is totally C-reflexive if and only if the following conditions are satisfied:*

(1) *For each $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\operatorname{depth}(R_\mathfrak{p}) \geqslant 2$, one has $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geqslant 2$;*
(2) *For each $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\operatorname{depth}(R_\mathfrak{p}) \leqslant 1$, there is an $R_\mathfrak{p}$-module isomorphism $M_\mathfrak{p} \cong \operatorname{Hom}_{R_\mathfrak{p}}(\operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, C_\mathfrak{p}), C_\mathfrak{p})$; and*
(3) *One has $\operatorname{Ext}^i_R(M, C) = 0 = \operatorname{Ext}^i(R(\operatorname{Hom}_R(M, C), C)$ for all $i \geqslant 1$.*

Corollary 5.4.7. *Assume that R is a normal domain. Let C be a semidualizing R-module, and let M be a finitely generated R-module. Then the biduality map $\delta^C_M \colon M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)$ is an isomorphism if and only if the biduality map $\delta^R_M \colon M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), R)$ is an isomorphism, that is, if and only if M is reflexive.*

Proof. The fact that R is a normal domain implies that R satisfies Serre's conditions (R_1) and (S_2). Thus, for each $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{depth}(R_\mathfrak{p}) \leqslant 1$, the ring $R_\mathfrak{p}$ is regular. In particular, for these primes we have $C_\mathfrak{p} \cong R_\mathfrak{p}$ by Corollary 4.1.9.

The proof of Proposition 5.4.6 shows that the biduality map δ^C_M is an isomorphism if and only if the following conditions are satisfied:

(C1) For each $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{depth}(R_\mathfrak{p}) \geqslant 2$, one has $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geqslant 2$; and
(C2) For each $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{depth}(R_\mathfrak{p}) \leqslant 1$, there is an $R_\mathfrak{p}$-module isomorphism $M_\mathfrak{p} \cong \operatorname{Hom}_{R_\mathfrak{p}}(\operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, C_\mathfrak{p}), C_\mathfrak{p})$.

Similarly, (or using [**8**]) the biduality map δ^R_M is an isomorphism if and only if the following conditions are satisfied:

(R1) For each $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{depth}(R_\mathfrak{p}) \geqslant 2$, one has $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geqslant 2$; and
(R2) For each $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{depth}(R_\mathfrak{p}) \leqslant 1$, there is an $R_\mathfrak{p}$-module isomorphism $M_\mathfrak{p} \cong \operatorname{Hom}_{R_\mathfrak{p}}(\operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}), R_\mathfrak{p})$.

It is clear that (C1)=(R1). From the first paragraph of this proof, we have (C2) $\iff$ (R2), hence the desired equivalence. □

Proposition 5.4.8. *Let C be a semidualizing R-module, and let G be a totally C-reflexive R-module. If N is an R-module locally of finite flat dimension, then*

$$\operatorname{Tor}^R_i(G, N) = 0 = \operatorname{Ext}^i_R(G, C \otimes_R N)$$

for all $i \geqslant 1$.

Proof. Case 1: assume that R is local. Since N is locally of finite flat dimension, this implies that $f = \operatorname{fd}_R(N) < \infty$. Fix a complete PP_C resolution of G:

$$X = \cdots \xrightarrow{\partial^X_2} P_1 \xrightarrow{\partial^X_1} P_0 \xrightarrow{\partial^X_0} C \otimes_R Q_0 \xrightarrow{\partial^X_{-1}} C \otimes_R Q_1 \xrightarrow{\partial^X_{-2}} \cdots .$$

Since Q_i is projective, we have $\operatorname{Tor}^R_i(C \otimes_R Q_i, N) \cong \operatorname{Tor}^R_i(C, N) \otimes_R Q_i$. As $N \in \mathcal{A}_C(R)$ by Corollary 3.5.6, we have $\operatorname{Tor}^R_i(C, N) = 0$ for all $i \geqslant 1$, and thus $\operatorname{Tor}^R_i(C \otimes_R Q_i, N) = 0$ for all $i \geqslant 1$.

For each $i \geqslant 0$ set $G_i = \operatorname{Im}(\partial^X_{-i})$. Then we have $G \cong G_0$, and there are exact sequences

$$0 \to G_i \to C \otimes_R Q_i \to G_{i+1} \to 0$$

for each $i \geqslant 0$. Using the vanishing from the previous paragraph, a dimension-shifting argument yields the isomorphism $\operatorname{Tor}^R_i(G, N) \cong \operatorname{Tor}^R_{i+f}(G_f, N) = 0$ for $i \geqslant 1$, while the vanishing is from the condition $i + f > f = \operatorname{fd}_R(N)$. This justifies the first of our desired vanishing conclusions.

For the second desired vanishing conclusion, we argue by induction on f.

Base case: $f = 0$. In this case the R-module N is flat. Hence, using tensor evaluation, we have the isomorphism in the next sequence

$$\operatorname{Ext}^i_R(G, C \otimes_R N) \cong \operatorname{Ext}^i_R(G, C) \otimes_R N = 0$$

for all $i \geqslant 1$; the vanishing follows because G is totally C-reflexive.

Induction step: Assume that $f \geqslant 1$ and that the result holds for modules of flat dimension $f - 1$. Consider an exact sequence

$$0 \to N' \to F \to N \to 0$$

wherein F is flat and $\mathrm{fd}_R(N') = f - 1$. The condition $N \in \mathcal{A}_C(R)$ implies that $\mathrm{Tor}_1^R(C, N) = 0$, so the induced sequence

$$0 \to C \otimes_R N' \to C \otimes_R F \to C \otimes_R N \to 0$$

is exact. The base case implies that $\mathrm{Ext}_R^i(G, C \otimes_R F) = 0$ for all $i \geqslant 1$, so a dimension-shifting argument implies that

$$\mathrm{Ext}_R^i(G, C \otimes_R N) \cong \mathrm{Ext}_R^{i+1}(G, C \otimes_R N') = 0$$

for all $i \geqslant 1$; the vanishing is from our induction hypothesis.

Case 2: the general case. For each maximal ideal $\mathfrak{m} \subset R$, the $R_\mathfrak{m}$-module $G_\mathfrak{m}$ is totally $C_\mathfrak{m}$-reflexive, and we have $\mathrm{fd}_{R_\mathfrak{m}}(N_\mathfrak{m}) < \infty$. This explains the vanishing in the next sequence for $i \geqslant 1$:

$$\mathrm{Tor}_i^R(G, N)_\mathfrak{m} \cong \mathrm{Tor}_i^{R_\mathfrak{m}}(G_\mathfrak{m}, N_\mathfrak{m}) = 0$$
$$\mathrm{Ext}_R^i(G, C \otimes_R N)_\mathfrak{m} \cong \mathrm{Ext}_{R_\mathfrak{m}}^i(G_\mathfrak{m}, C_\mathfrak{m} \otimes_{R_\mathfrak{m}} N_\mathfrak{m}) = 0.$$

Since we have $\mathrm{Tor}_i^R(G, N)_\mathfrak{m} = 0 = \mathrm{Ext}_R^i(G, C \otimes_R N)_\mathfrak{m}$ for each $\mathfrak{m}$, the desired conclusion $\mathrm{Tor}_i^R(G, N) = 0 = \mathrm{Ext}_R^i(G, C \otimes_R N)$ follows. □

The next result is proved like the previous one.

Proposition 5.4.9. *Let C be a semidualizing R-module, and let G be a totally C-reflexive R-module. If N is an R-module locally of finite injective dimension, then*

$$\mathrm{Ext}_R^i(G, N) = 0 = \mathrm{Tor}_i^R(G, \mathrm{Hom}_R(C, N))$$

for all $i \geqslant 1$.

Here is where we put the "dual" in a dualizing module.

Proposition 5.4.10. *Let D be a point-wise dualizing R-module, and let M be a finitely generated R-module.*

(a) *If for each maximal ideal $\mathfrak{m} \subset R$ the $R_\mathfrak{m}$-module $M_\mathfrak{m}$ is either 0 or maximal Cohen-Macaulay, then M is totally D-reflexive.*
(b) *For each resolution P of M by finitely generated projective R-modules, the ith syzygy* $\mathrm{Coker}(\partial_{i+1}^P)$ *is totally D-reflexive for each $i > \dim(R) + 1$.*

Proof. (a) Corollary 2.2.13 says that R is Cohen-Macaulay and D is a canonical module for R, that is, that $R_\mathfrak{m}$ is Cohen-Macaulay and $D_\mathfrak{m}$ is a canonical $R_\mathfrak{m}$-module for each maximal ideal $\mathfrak{m} \subset R$. Since $M_\mathfrak{m}$ is either 0 or a maximal Cohen-Macaulay $R_\mathfrak{m}$-module, we conclude from [**8**, (3.3.10)] that $M_\mathfrak{m}$ is totally $D_\mathfrak{m}$-reflexive. As this is so for each maximal ideal $\mathfrak{m}$, Proposition 5.4.2 implies that M is totally D-reflexive.

(b) Assume without loss of generality that $d = \dim(R) < \infty$. For each $\mathfrak{m}$, the complex $P_\mathfrak{m}$ is a resolution of $M_\mathfrak{m}$ by finitely generated free $R_\mathfrak{m}$-modules. Hence, for each $i > \dim(R) + 1 \geqslant \dim(R_\mathfrak{m}) + 1$, the $R_\mathfrak{m}$-module $\mathrm{Coker}(\partial_{i+1}^{P_\mathfrak{m}}) \cong \mathrm{Coker}(\partial_{i+1}^P)_\mathfrak{m}$ is either 0 or maximal Cohen-Macaulay; see, e.g., [**4**, (1.2.8)]. Part (a) implies that $\mathrm{Coker}(\partial_{i+1}^P)$ is totally D-reflexive. □

Corollary 5.4.11. *Assume that R is point-wise Gorenstein, and let M be a finitely generated R-module.*

(a) *If for each maximal ideal $\mathfrak{m} \subset R$ the $R_\mathfrak{m}$-module $M_\mathfrak{m}$ is either 0 or maximal Cohen-Macaulay, then M is totally reflexive.*

(b) *For each resolution F of M by finitely generated free R-modules, the ith syzygy* $\operatorname{Coker}(\partial_{i+1}^F)$ *is totally reflexive for each $i > \dim(R) + 1$.*

PROOF. This is the case $D = R$ of Proposition 5.4.10. □

Corollary 5.4.12. *Let D be a point-wise dualizing R-module. If C is a semidualizing R-module, then C is totally D-reflexive and $C \in \mathcal{A}_{\operatorname{Hom}_R(C,D)}(R)$.*

PROOF. The ring $R_{\mathfrak{m}}$ is Cohen-Macaulay for each maximal ideal $\mathfrak{m} \subset R$ by Corollary 2.2.13. Hence, we conclude that $C_{\mathfrak{m}}$ is a maximal Cohen-Macaulay $R_{\mathfrak{m}}$-module for each $\mathfrak{m}$; see Propositions 2.2.3 and 2.1.16(b) and Theorem 2.2.6(c). The fact that C is totally D-reflexive now follows from Proposition 5.4.10(a).

Set $C^\dagger = \operatorname{Hom}_R(C, D)$, which is semidualizing by Corollary 4.1.3. Corollary 3.5.6 implies that $D \in \mathcal{B}_{C^\dagger}(R)$, so we have $\operatorname{Hom}_R(C^\dagger, D) \in \mathcal{A}_{C^\dagger}(R)$ by Proposition 4.1.1(b). Since C is totally D-reflexive, we have $C \cong \operatorname{Hom}_R(C^\dagger, D) \in \mathcal{A}_{C^\dagger}(R)$. □

The next result augments Propositions 5.3.1 and 5.3.2.

Theorem 5.4.13. *Let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension. Let C be a semidualizing R-module, and let G be a totally C-reflexive R-module with complete PP_C resolution*

$$X = \cdots \to P_1 \to P_0 \xrightarrow{\partial_0^X} C \otimes_R Q_0 \to C \otimes_R Q_1 \to \cdots .$$

Then the S-module $S \otimes_R G$ is totally $S \otimes_R C$-reflexive with complete $PP_{S\otimes_R C}$-resolution $S \otimes_R X$, and $\operatorname{Tor}_i^R(S, G) = 0$ for all $i \geqslant 1$.

PROOF. Write

$$X = \cdots \xrightarrow{\partial_{i+1}^X} X_i \xrightarrow{\partial_i^X} X_{i-1} \xrightarrow{\partial_{i-1}^X} \cdots .$$

For each integer j, set $G_j = \operatorname{Im}(\partial_j^X)$. In particular, each G_j is totally C-reflexive, and we have $G_0 \cong G$. Proposition 5.4.8 implies that $\operatorname{Tor}_i^R(S, G_j) = 0$ for each $j \in \mathbb{Z}$ and all $i \geqslant 1$. Hence, we have $\operatorname{Tor}_i^R(S, G) = 0$ for all $i \geqslant 1$.

Because of the exact sequence

$$0 \to G_{j+1} \to X_j \to G_j \to 0$$

the Tor-vanishing $\operatorname{Tor}_i^R(S, G_j) = 0$ implies that the induced sequence

$$0 \to S \otimes_R G_{j+1} \to S \otimes_R X_j \to S \otimes_R G_j \to 0$$

is exact for each j, and it follows from a standard argument that $S \otimes_R X$ is exact.

Proposition 5.2.4 shows that the complex $\operatorname{Hom}_R(X, C)$ is a complete PP_C resolution of $\operatorname{Hom}_R(G, C)$ over R, so it also follows that the complex $S \otimes_R \operatorname{Hom}_R(X, C)$ is exact.

Since each P_i and Q_j is a finitely generated projective R-module, each $S \otimes_R P_i$ and $S \otimes_R Q_j$ is a finitely generated projective R-module. Thus, the sequence

$$S \otimes_R X = \cdots \to S \otimes_R P_0 \xrightarrow{S \otimes_R \partial_0^X} (S \otimes_R C) \otimes_S (S \otimes_R Q_0) \to \cdots$$

has the form of a complete $PP_{S\otimes_R C}$-resolution $S\otimes_R G$. To complete the proof, it remains to observe that the following complex is exact:

$$\begin{aligned}\operatorname{Hom}_S(S\otimes_R X, S\otimes_R C) &\cong \operatorname{Hom}_R(X, \operatorname{Hom}_S(S, S\otimes_R C))\\ &\cong \operatorname{Hom}_R(X, S\otimes_R C)\\ &\cong S\otimes_R \operatorname{Hom}_R(X, C).\end{aligned}$$

The first isomorphism is Hom tensor adjointness, and the second isomorphism is induced by Hom cancellation. The third isomorphism is tensor evaluation, using the finiteness of $\operatorname{fd}_R(S)$, and the exactness is from the previous paragraph. □

The next example shows that, in Theorem 5.4.13 one cannot replace the finite flat dimension hypothesis with the assumption $S \in \mathcal{A}_C(R)$.

Example 5.4.14. Let k be a field, and set $R = k[\![X, Y]\!]/(XY)$. We work with the semidualizing R-module R. It is straightforward to show that the R-module $M = R/XR$ with complete PP_R-resolutions

$$Z = \cdots \xrightarrow{X} R \xrightarrow{Y} R \xrightarrow{X} R \xrightarrow{Y} \cdots .$$

Consider the natural surjection $\varphi\colon R \to S = R/YR \cong k[\![X]\!]$. Then we have $S \in \mathcal{A}_R(R)$ by Example 3.1.5, and we have

$$S\otimes_R M \cong R/YR\otimes_R R/XR \cong R/(X,Y) \cong k.$$

This module is not totally reflexive as an S-module (using the semidualizing S-module $S \cong S\otimes_R S$) because $\operatorname{Ext}^1_S(k, S) \cong \operatorname{Ext}^1_{k[\![X]\!]}(k, k[\![X]\!]) \cong k \neq 0$. Furthermore, the complex

$$S\otimes_R Z = \cdots \xrightarrow{X} S \xrightarrow{0} S \xrightarrow{X} S \xrightarrow{0} \cdots$$

is not a complete resolution since it is not exact. (Every other homology module is non-zero.) Since the left-most half of Z is a free resolution of M, this also shows that $\operatorname{Tor}^R_i(S, M) \neq 0$ for infinitely many values of $i \geqslant 1$.

5.5. Co-base Change

The next result augments Proposition 2.2.14.

Proposition 5.5.1. *Let $\varphi\colon R \to S$ be a ring homomorphism such that S is finitely generated as an R-module, and let C be a semidualizing R-module. Then S is totally C-reflexive as an R-module if and only if* $\operatorname{Hom}_R(S, C)$ *is a semidualizing S-module and* $\operatorname{Ext}^i_R(S, C) = 0$ *for all $i \geqslant 1$.*

Proof. Since one of the defining conditions for S to be totally C-reflexive is $\operatorname{Ext}^i_R(S, C) = 0$ for all $i \geqslant 1$, we assume this condition for the rest of the proof. Also, because C is finitely generated, the module $\operatorname{Hom}_R(S, C)$ is finitely generated over R. As the S-module structure on $\operatorname{Hom}_R(S, C)$ is compatible with the R-module structure via φ, it follows that $\operatorname{Hom}_R(S, C)$ is finitely generated over S.

Let I be an injective resolution of C over R. It is straightforward to show that $\operatorname{Hom}_R(S, I_j)$ is an injective S-module for each j. Since $\operatorname{Ext}^i_R(S, C) = 0$ for all $i \geqslant 1$, the complex $\operatorname{Hom}_R(S, I)$ is an injective resolution of $\operatorname{Hom}_R(S, C)$ as an S-module.

This yields the first isomorphism in the next sequence:

$$\begin{aligned}\mathrm{Ext}^i_S(\mathrm{Hom}_R(S,C),\mathrm{Hom}_R(S,C)) &\cong \mathrm{H}_{-i}(\mathrm{Hom}_S(\mathrm{Hom}_R(S,C),\mathrm{Hom}_R(S,I))\\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(S\otimes_S \mathrm{Hom}_R(S,C),I)\\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(\mathrm{Hom}_R(S,C),I)\\ &\cong \mathrm{Ext}^i_R(\mathrm{Hom}_R(S,C),C).\end{aligned}$$

The second isomorphism is induced by Hom-tensor adjointness. The third isomorphism is induced by tensor-cancellation. The fourth isomorphism is by definition. We conclude that $\mathrm{Ext}^i_S(\mathrm{Hom}_R(S,C),\mathrm{Hom}_R(S,C))=0$ for all $i\geqslant 1$ if and only if $\mathrm{Ext}^i_R(\mathrm{Hom}_R(S,C),C)=0$ for all $i\geqslant 1$.

In the next commutative diagram, the unspecified isomorphisms are by tensor-cancellation and Hom-tensor adjointness:

$$\begin{array}{ccc} S & \xrightarrow{\delta^C_S} & \mathrm{Hom}_R(\mathrm{Hom}_R(S,C),C) \\ \downarrow{\scriptstyle \chi^S_{\mathrm{Hom}_R(S,C)}} & & \downarrow{\scriptstyle\cong} \\ \mathrm{Hom}_R(S\otimes_S \mathrm{Hom}_R(S,C),C) & \xrightarrow{\cong} & \mathrm{Hom}_S(\mathrm{Hom}_R(S,C),\mathrm{Hom}_R(S,C)). \end{array}$$

It is straightforward to show that these maps are S-linear. In particular, the map δ^C_S is an isomorphism if and only if $\chi^S_{\mathrm{Hom}_R(S,C)}$ is an isomorphism. □

Proposition 5.5.2. *Let* $\varphi\colon R\to S$ *be a ring homomorphism, and let* C *be a semidualizing* R*-module. Assume that* S *is totally* C*-reflexive as an* R*-module, and let* M *be a finitely generated* S*-module. Then* M *is totally* $\mathrm{Hom}_R(S,C)$*-reflexive over* S *if and only if it is totally* C*-reflexive over* R*.*

Proof. Let I be an R-injective resolution of C. As in the proof of Proposition 5.5.1, the assumption $\mathrm{Ext}^i_R(S,C)=0$ for all $i\geqslant 1$ implies that $\mathrm{Hom}_R(S,I)$ is an S-injective resolution of the semidualizing S-module $\mathrm{Hom}_R(S,C)$. This explains the first isomorphism in the next sequence:

$$\begin{aligned}\mathrm{Ext}^i_S(M,\mathrm{Hom}_R(S,C)) &\cong \mathrm{H}_{-i}(\mathrm{Hom}_S(M,\mathrm{Hom}_R(S,I)))\\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(S\otimes_S M,I))\\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(M,I))\\ &\cong \mathrm{Ext}^i_R(M,C).\end{aligned}$$

The other isomorphism are by Hom-tensor adjointness, tensor cancellation, and definition. It follows that $\mathrm{Ext}^i_S(M,\mathrm{Hom}_R(S,C))=0$ for all $i\geqslant 1$ if and only if $\mathrm{Ext}^i_R(M,C)=0$ for all $i\geqslant 1$. Furthermore, the case $i=0$ explains the second step in the next sequence

$$\begin{aligned}\mathrm{Ext}^i_S(\mathrm{Hom}_S(M,\mathrm{Hom}_R(S,C)),\mathrm{Hom}_R(S,C)) &\cong \mathrm{Ext}^i_R(\mathrm{Hom}_S(M,\mathrm{Hom}_R(S,C)),C)\\ &\cong \mathrm{Ext}^i_R(\mathrm{Hom}_R(M,C),C).\end{aligned}$$

The first step follows from an application of the previous sequence to the S-module $\mathrm{Hom}_S(M,\mathrm{Hom}_R(S,C))$. Thus $\mathrm{Ext}^i_S(\mathrm{Hom}_S(M,\mathrm{Hom}_R(S,C)),\mathrm{Hom}_R(S,C))=0$ for all $i\geqslant 1$ if and only if $\mathrm{Ext}^i_R(\mathrm{Hom}_R(M,C),C)=0$ for all $i\geqslant 1$.

Finally, there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\delta_M^{\operatorname{Hom}_R(S,C)}} & \operatorname{Hom}_S(\operatorname{Hom}_S(M,\operatorname{Hom}_R(S,C)),\operatorname{Hom}_R(S,C)) \\ {\scriptstyle \delta_M^C}\downarrow & & \downarrow{\scriptstyle \cong} \\ \operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C) & \xrightarrow{\cong} & \operatorname{Hom}_R(\operatorname{Hom}_S(M,\operatorname{Hom}_R(S,C)),C) \end{array}$$

where the unspecified isomorphisms are combinations of Hom tensor adjointness and tensor cancellation. It follows that $\delta_M^{\operatorname{Hom}_R(S,C)}$ is an isomorphism if and only if δ_M^C is an isomorphism. $\square$

CHAPTER 6

G_C-Dimension

6.1. Definitions and Basic Properties of G_C-dimension

Definition 6.1.1. Let C be a semidualizing R-module and M a finitely generated R-module. An *augmented G_C-resolution* of M is an exact sequence

$$G^+ = \cdots \xrightarrow{\partial^G_{i+1}} G_i \xrightarrow{\partial^G_i} G_{i-1} \xrightarrow{\partial^G_{i-1}} \cdots G_1 \xrightarrow{\partial^G_1} G_0 \xrightarrow{\tau} M \to 0$$

wherein each G_i is totally C-reflexive. The *G_C-resolution* of M associated to G^+ is the sequence obtained by truncating:

$$G^+ = \cdots \xrightarrow{\partial^G_{i+1}} G_i \xrightarrow{\partial^G_i} G_{i-1} \xrightarrow{\partial^G_{i-1}} \cdots G_1 \xrightarrow{\partial^G_1} G_0 \to 0$$

Definition 6.1.2. Let C be a semidualizing R-module and M a finitely generated R-module. If M admits a G_C-resolution G such that $G_i = 0$ for $i \gg 0$, then we say that M has finite G_C-dimension. More specifically, the *G_C-dimension* of M is the shortest such resolution:

$$\mathrm{G}_C\text{-}\dim_R(M) = \inf\{\sup\{n \geqslant 0 \mid G_n \neq 0\} \mid G \text{ is a } \mathrm{G}_C\text{-resolution of } M\}.$$

Example 6.1.3. Let C be a semidualizing R-module. A non-zero finitely generated R-module is totally reflexive if and only if it has G_C-dimension 0. By definition, we have $\mathrm{G}_C\text{-}\dim_R(M) = -\infty$ if and only if $M = 0$.

Since every finitely generated projective R-module is totally C-reflexive, it follows that every (augmented) resolution by finitely generated projective R-modules is an (augmented) G_C-resolution.

Proposition 6.1.4. *If D is a dualizing R-module, then $\mathrm{G}_D\text{-}\dim_R(M) < \infty$ for each finitely generated R-module M.*

Proof. Example 2.1.11 implies that $d = \dim(R) < \infty$. Let F be a resolution of M by finitely generated free R-module,. and let $i > d+1$. Proposition 5.4.10(b) implies that $\mathrm{Coker}(\partial^F_{i+1})$ is totally D-reflexive. It follows that the exact sequence

$$0 \to \mathrm{Coker}(\partial^F_{i+1}) \to F_{i-1} \to \cdots \to F_0 \to M \to 0$$

is an augmented G_D-resolution of length i, so we have $\mathrm{G}_D\text{-}\dim_R(M) \leqslant i < \infty$. □

Corollary 6.1.5. *If R is Gorenstein, then $\mathrm{G}\text{-}\dim_R(M) < \infty$ for each finitely generated R-module M.*

Proof. This is the case $D = R$ of Proposition 6.1.4. □

Here is a version of Schanuel's Lemma for G_C-resolutions.

Lemma 6.1.6. *Let C be a semidualizing R-module, and let M be a finitely generated R-module. Assume that there are exact sequences*

$$0 \longrightarrow K_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow L_n \longrightarrow H_{n-1} \longrightarrow \cdots \longrightarrow H_0 \longrightarrow M \longrightarrow 0$$

such that each G_i, H_i is totally C-reflexive. Then K_n is totally C-reflexive if and only if L_n is totally C-reflexive.

Proof. The case $n = 0$ is straightforward, so we assume that $n \geqslant 1$.

Assume first that each H_i is finitely generated and projective. The proof of Schanuel's Lemma [**17**] shows that there is an exact sequence

$$0 \to L_n \to K_n \oplus H_{n-1} \xrightarrow{\partial} G_{n-1} \oplus H_{n-2} \to \cdots \to G_1 \oplus H_0 \to G_0 \to 0.$$

Set $N = \operatorname{Im}(\partial)$ and consider the exact sequence

$$0 \to N \to G_{n-1} \oplus H_{n-2} \to \cdots \to G_1 \oplus H_0 \to G_0 \to 0.$$

Since each of the modules $G_{n-1}, H_{n-2}, \ldots, G_1, H_0, G_0$ is totally C-reflexive, Proposition 5.1.1 implies that N is totally C-reflexive. Now, using the exact sequence

$$0 \to L_n \to K_n \oplus H_{n-1} \to N \to 0$$

another application of Proposition 5.1.1 shows that L_n is totally C-reflexive if and only if $K_n \oplus H_{n-1}$ is totally C-reflexive, that is, if and only if K_n is totally C-reflexive; see Proposition 2.1.4.

Now, for the general case. Consider an exact sequence

$$0 \to Z_n \to P_{n-1} \to \cdots P_0 \to M \to 0$$

obtained from an augmented resolution of M by finitely generated projective R-modules. The previous paragraph shows that L_n is totally C-reflexive if and only if Z_n is totally C-reflexive if and only if K_n is totally C-reflexive, as desired. □

Proposition 6.1.7. *Let C be a semidualizing R-module, and let M be a finitely generated R-module. For each integer $n \geqslant 0$, the following conditions are equivalent:*

(i) $\mathrm{G}_C\text{-}\dim_R(M) \leqslant n$;
(ii) *M has a G_C-resolution G such that $G_i = 0$ for all $i > n$;*
(iii) *M has a G_C-resolution G such that $G_{n+1} = 0$;*
(iv) *for each G_C-resolution G of M, the module $\operatorname{Coker}(\partial_i^G)$ is totally C-reflexive for each $i > n$;*
(v) *there is a resolution F of M by finitely generated free R-modules such that $\operatorname{Coker}(\partial_{n+1}^F)$ is totally C-reflexive; and*
(vi) $\mathrm{G}_C\text{-}\dim_R(M) < \infty$ *and* $\operatorname{Ext}_R^m(M, C) = 0$ *for all $m > n$.*

In particular, if $\mathrm{G}_C\text{-}\dim_R(M) < \infty$, then

$$\mathrm{G}_C\text{-}\dim_R(M) = \sup\{i \geqslant 0 \mid \operatorname{Ext}_R^i(M, C) \neq 0\}.$$

Proof. The equivalence (i) $\iff$ (ii) is by definition, and the equivalence (ii) $\iff$ (iii) is routine. The implication (iv) $\implies$ (v) follows from the fact that M has a resolution by finitely generated free R-modules and that every such resolution is a G_C-resolution. For the implication (v) $\implies$ (ii), argue as in the proof of Proposition 6.1.4. Also, once the equivalence (i) $\iff$ (vi) is shown, the displayed equality follows directly.

For the remainder of the proof, let G be a G_C-resolution of M. For each $i \geqslant 0$, set $M_i = \mathrm{Coker}(\partial^G_{i+1})$, and consider the exact sequences

$$0 \to M_i \to G_{i-1} \to \cdots \to G_0 \to M \to 0. \tag{6.1.7.1}$$

Since $\mathrm{Ext}^m_R(G_j, C) = 0$ for each j and each $m \geqslant 1$, a standard dimension shifting argument implies that

$$\mathrm{Ext}^m_R(M, C) \cong \mathrm{Ext}^{m-i}_R(M_i, C) \tag{6.1.7.2}$$

for each $m > i$.

(iii) $\implies$ (vi) Assume that $G_{n+1} = 0$. It follows that $M_n \cong G_n$. With $i = n$, the exact sequence (6.1.7.1) has the form

$$0 \to G_n \to \cdots \to G_0 \to M \to 0.$$

so we have $\mathrm{G}_C\text{-}\dim_R(M) \leqslant n$. Furthermore, the isomorphism (6.1.7.2) implies that

$$\mathrm{Ext}^m_R(M, C) \cong \mathrm{Ext}^{m-n}_R(M_n, C) \cong \mathrm{Ext}^{m-n}_R(G_n, C) = 0$$

for all $m > n$.

(vi) $\implies$ (iv) Assume that $g = \mathrm{G}_C\text{-}\dim_R(M) < \infty$ and $\mathrm{Ext}^m_R(M, C) = 0$ for all $m > n$. Then M has a G_C-resolution H such that $H_i = 0$ for all $i > g$.

We claim that $g \leqslant n$. Suppose by way of contradiction that $g > n$. Let $M' = \mathrm{Coker}(\partial^H_g)$. As we noted above, the exact sequence

$$0 \to M' \to H_{g-2} \to \cdots \to H_0 \to M \to 0. \tag{6.1.7.3}$$

implies that $\mathrm{Ext}^m_R(M', C) = 0$ for all $m \geqslant 1$. Hence, because of the exact

$$0 \to H_g \to H_{g-1} \to M' \to 0$$

Proposition 5.1.3 implies that M' is totally C-reflexive. The sequence (6.1.7.3) is thus an augmented G_C-resolution of length $g-1$. This implies that $\mathrm{G}_C\text{-}\dim_R(M) \leqslant g - 1 = \mathrm{G}_C\text{-}\dim_R(M) - 1$, a contradiction.

We now show that M_g is totally C-reflexive. Indeed, we have exact sequences

$$0 \longrightarrow M_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow H_n \longrightarrow H_{n-1} \longrightarrow \cdots \longrightarrow H_0 \longrightarrow M \longrightarrow 0$$

such that the modules G_i, H_i are totally C reflexive for $i = 0, \ldots, g-1$. Since H_g is also totally C-reflexive, Lemma 6.1.6 implies that M_g is also totally C-reflexive.

Finally, for each $i > n$, we have $i > g$. Because of the exact sequence

$$0 \to M_i \to G_{i-1} \to \cdots \to G_g \to M_g \to 0$$

the fact that the modules $G_{i-1}, \ldots, G_g, M_g$ are totally C-reflexive implies that M_i is totally C-reflexive by Proposition 5.1.1. □

The next result says that the class of R-modules of finite G_C-dimension satisfies the two-of-three property.

Proposition 6.1.8. *Let C be a semidualizing R-module, and consider an exact sequence of finitely generated R-modules:*

$$0 \to M^1 \to M^2 \to M^3 \to 0.$$

If two of the M_i have finite G_C-dimension, then so does the third.

Proof. Let P^1 and P^3 be resolutions of M_1 and M_3, respectively, by finitely generated projective R-modules. For each $n \geqslant 0$, the horseshoe lemma yields a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_n^1 & \longrightarrow & M_n^2 & \longrightarrow & M_n^3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{n-1}^1 & \longrightarrow & P_{n-1}^1 \oplus P_{n-1}^3 & \longrightarrow & P_{n-1}^3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0^1 & \longrightarrow & P_0^1 \oplus P_0^3 & \longrightarrow & P_0^3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^1 & \longrightarrow & M^2 & \longrightarrow & M^3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact columns and rows.

Assume that $\mathrm{G}_C\text{-}\dim_R(M^2), \mathrm{G}_C\text{-}\dim_R(M^3) \leqslant n$. Then Proposition 6.1.7 implies that M_n^2 and M_n^3 are totally C-reflexive. From the top row of the diagram, we conclude that M_n^1 is totally C-reflexive; see Proposition 5.1.1. Hence, the first column of the diagram is a bounded augmented G_C resolution of M^1, so we have $\mathrm{G}_C\text{-}\dim_R(M^1) \leqslant n$.

A similar argument shows that, if $\mathrm{G}_C\text{-}\dim_R(M^1), \mathrm{G}_C\text{-}\dim_R(M^3) \leqslant n$, then $\mathrm{G}_C\text{-}\dim_R(M^2) \leqslant n$.

Assume that $\mathrm{G}_C\text{-}\dim_R(M^1), \mathrm{G}_C\text{-}\dim_R(M^2) \leqslant n$. Again, it follows that M_n^1 and M_n^2 are totally C-reflexive. Thus, the top row of the diagram shows that $\mathrm{G}_C\text{-}\dim_R(M_n^3) \leqslant 1$. Furthermore, by combining the top row and the right-most column of this diagram, we obtain an exact sequence

$$0 \to M_n^1 \to M_n^2 \to P_{n-1}^3 \to \cdots \to P_0^3 \to M^3 \to 0.$$

This is an augmented G_C resolution of M^3 of length $n+1$, so we conclude that $\mathrm{G}_C\text{-}\dim_R(M^3) \leqslant n+1$. □

Remark 6.1.9. Let C be a semidualizing R-module, and consider an exact sequence of finitely generated R-modules:

$$0 \to M^1 \to M^2 \to M^3 \to 0.$$

The proof of Proposition 6.1.8 shows the following:

$$\begin{aligned}
\mathrm{G}_C\text{-}\dim_R(M^1) &\leqslant \sup\{\mathrm{G}_C\text{-}\dim_R(M^2), \mathrm{G}_C\text{-}\dim_R(M^3)\}\\
\mathrm{G}_C\text{-}\dim_R(M^2) &\leqslant \sup\{\mathrm{G}_C\text{-}\dim_R(M^1), \mathrm{G}_C\text{-}\dim_R(M^3)\}\\
\mathrm{G}_C\text{-}\dim_R(M^3) &\leqslant \sup\{\mathrm{G}_C\text{-}\dim_R(M^1), \mathrm{G}_C\text{-}\dim_R(M^2)\} + 1.
\end{aligned}$$

The next result shows how the third displayed can be improved.

Proposition 6.1.10. *Let C be a semidualizing R-module, and consider an exact sequence of finitely generated R-modules of finite G_C-dimension:*

$$0 \to M^1 \to M^2 \to M^3 \to 0.$$

(a) *If* $\mathrm{G}_C\text{-}\dim_R(M^1) > \mathrm{G}_C\text{-}\dim_R(M^3)$ *or* $\mathrm{G}_C\text{-}\dim_R(M^2) > \mathrm{G}_C\text{-}\dim_R(M^3)$, *then* $\mathrm{G}_C\text{-}\dim_R(M^1) = \mathrm{G}_C\text{-}\dim_R(M^2)$.

(b) *If* $\mathrm{G}_C\text{-}\dim_R(M^3) \geqslant 1$ *and M^2 is totally C-reflexive, then* $\mathrm{G}_C\text{-}\dim_R(M^1) = \mathrm{G}_C\text{-}\dim_R(M^3) - 1$.

PROOF. (a) Assume that $\mathrm{G}_C\text{-}\dim_R(M^1) > \mathrm{G}_C\text{-}\dim_R(M^3)$. Suppose by way of contradiction that $\mathrm{G}_C\text{-}\dim_R(M^3) > \mathrm{G}_C\text{-}\dim_R(M^2)$. Remark 6.1.9 then yields

$$\begin{aligned}
\mathrm{G}_C\text{-}\dim_R(M^1) &\leqslant \sup\{\mathrm{G}_C\text{-}\dim_R(M^2), \mathrm{G}_C\text{-}\dim_R(M^3)\}\\
&= \mathrm{G}_C\text{-}\dim_R(M^3)\\
&< \mathrm{G}_C\text{-}\dim_R(M^1)
\end{aligned}$$

which is a contradiction.

Hence, we have $\mathrm{G}_C\text{-}\dim_R(M^3) \leqslant \mathrm{G}_C\text{-}\dim_R(M^2)$. This yields the next sequence

$$\begin{aligned}
\mathrm{G}_C\text{-}\dim_R(M^2) &\leqslant \sup\{\mathrm{G}_C\text{-}\dim_R(M^1), \mathrm{G}_C\text{-}\dim_R(M^3)\}\\
&= \mathrm{G}_C\text{-}\dim_R(M^1)\\
&\leqslant \sup\{\mathrm{G}_C\text{-}\dim_R(M^2), \mathrm{G}_C\text{-}\dim_R(M^3)\}\\
&= \mathrm{G}_C\text{-}\dim_R(M^2)
\end{aligned}$$

and hence the equality $\mathrm{G}_C\text{-}\dim_R(M^1) = \mathrm{G}_C\text{-}\dim_R(M^2)$.

The case where $\mathrm{G}_C\text{-}\dim_R(M^2) > \mathrm{G}_C\text{-}\dim_R(M^3)$ is handled similarly.

(b) Assume that $\mathrm{G}_C\text{-}\dim_R(M^3) \geqslant 1$. We then have $\mathrm{Ext}^i_R(M^3, C) \neq 0$ for some $i \geqslant 1$, specifically for $i = \mathrm{G}_C\text{-}\dim_R(M^3)$, by Proposition 6.1.7.

By assumption, we have $\mathrm{Ext}^i_R(M^2, C) = 0$ for all $i \geqslant 1$. Hence, a dimension shifting argument shows that

$$\mathrm{Ext}^i_R(M^1, C) \cong \mathrm{Ext}^{i+1}_R(M^3, C)$$

for all $i \geqslant 1$. Proposition 6.1.7 explains the first and last steps in the next sequence

$$\begin{aligned}
\mathrm{G}_C\text{-}\dim_R(M^1) &= \sup\{i \geqslant 0 \mid \mathrm{Ext}^i_R(M^1, C) \neq 0\}\\
&= \sup\{i \geqslant 0 \mid \mathrm{Ext}^{i+1}_R(M^3, C) \neq 0\}\\
&= \sup\{i \geqslant 1 \mid \mathrm{Ext}^i_R(M^3, C) \neq 0\} - 1\\
&= \sup\{i \geqslant 0 \mid \mathrm{Ext}^i_R(M^3, C) \neq 0\} - 1\\
&= \mathrm{G}_C\text{-}\dim_R(M^3) - 1.
\end{aligned}$$

The fourth step is due to the previous paragraph. □

6.2. Stability Results

The next two results compliment Propositions 5.4.8 and 5.4.9.

Proposition 6.2.1. *Let C be a semidualizing R-module, and let M be a finitely generated R-module. If N is an R-module locally of finite flat dimension, then*

$$\mathrm{Tor}_i^R(M,N) = 0 = \mathrm{Ext}_R^i(M, C \otimes_R N)$$

for all $i > \mathrm{G}_C\text{-}\dim_R(M)$.

Proof. Assume without loss of generality that $g = \mathrm{G}_C\text{-}\dim_R(M) < \infty$. We proceed by induction on g. The case $g = 0$ is in Proposition 5.4.8. For the inductive step, assume that $g \geqslant 1$ and that the result holds for all R-modules of G_C-dimension $n-1$. There is an exact sequence

$$0 \to M' \to G \to M \to 0$$

such that G is totally C-reflexive and $\mathrm{G}_C\text{-}\dim_R(M') = g-1$. The base case implies that $\mathrm{Tor}_i^R(G,N) = 0 = \mathrm{Ext}_R^i(G, C \otimes_R N)$ for all $i \geqslant 1$. Hence, a dimension-shifting argument yields the following isomorphisms for $i \geqslant 2$:

$$\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_{i-1}^R(M',N)$$
$$\mathrm{Ext}_R^i(M, C \otimes_R N) \cong \mathrm{Ext}_R^{i-1}(M', C \otimes_R N)$$

The induction hypothesis implies that $\mathrm{Tor}_{i-1}^R(M',N) = 0 = \mathrm{Ext}_R^{i-1}(M', C \otimes_R N)$ when $i-1 > g-1$. For $i > g$, we have $i \geqslant g+1 \geqslant 2$, so the displayed isomorphisms imply that $\mathrm{Tor}_i^R(M,N) = 0 = \mathrm{Ext}_R^i(M, C \otimes_R N)$ when $i > g$. □

The next result is proved like the previous one.

Proposition 6.2.2. *Let C be a semidualizing R-module, and let M be a finitely generated R-module. If N is an R-module locally of finite injective dimension, then*

$$\mathrm{Ext}_R^i(M,N) = 0 = \mathrm{Tor}_i^R(M, \mathrm{Hom}_R(C,N))$$

for all $i > \mathrm{G}_C\text{-}\dim_R(M)$.

Proposition 6.2.3. *Let C be a semidualizing R-module, and let N be an R-module. Let G be a G_C-resolution of a finitely generated R-module M.*

(a) *If $\mathrm{Ext}_R^i(H,N) = 0$ for all $i \geqslant 1$ and for each totally C-reflexive R-module H, then $\mathrm{Ext}_R^i(M,N) \cong \mathrm{H}_{-i}(\mathrm{Hom}_R(G,N))$ for each $i \geqslant 0$.*
(b) *If $\mathrm{Tor}_i^R(H,N) = 0$ for all $i \geqslant 1$ and for each totally C-reflexive R-module H, then $\mathrm{Tor}_i^R(M,N) \cong \mathrm{H}_i(G \otimes_R N)$ for each $i \geqslant 0$.*

Proof. (a) The assumption $\mathrm{Ext}_R^i(H,N) = 0$ for all $i \geqslant 1$ and for each totally C-reflexive R-module H says that the totally C-reflexive R-modules are $\mathrm{Ext}_R(-,N)$-acyclic. So the desired result follows from standard homological nonsense.

The proof of part (b) is similar. □

Corollary 6.2.4. *Let C be a semidualizing R-module, and let N be an R-module. Let G be a G_C-resolution of a finitely generated R-module M.*

(a) *If N is locally of finite flat dimension, then $\mathrm{Tor}_i^R(M,N) \cong \mathrm{H}_i(G \otimes_R N)$ and $\mathrm{Ext}_R^i(M, C \otimes_R N) \cong \mathrm{H}_{-i}(\mathrm{Hom}_R(G, C \otimes_R N))$ for each $i \geqslant 0$.*
(b) *One has $\mathrm{Ext}_R^i(M,C) \cong \mathrm{H}_{-i}(\mathrm{Hom}_R(G,C))$ for each $i \geqslant 0$.*

(c) *If N is locally of finite injective dimension, then one has* $\mathrm{Ext}^i_R(M,N) \cong \mathrm{H}_{-i}(\mathrm{Hom}_R(G,N))$ *and* $\mathrm{Tor}^R_i(M,\mathrm{Hom}_R(C,N)) \cong \mathrm{H}_i(G\otimes_R(\mathrm{Hom}_R(C,N)))$ *for* $i \geqslant 0$.

PROOF. Combine Propositions 6.2.1–6.2.3. □

Proposition 6.2.5. *Let C be a semidualizing R-module. Let M and N be finitely generated R-modules such that the quantities $g = \mathrm{G}_C\text{-}\mathrm{dim}_R(M)$ and $p = \mathrm{pd}_R(N)$ are finite. If $\mathrm{Tor}^R_i(N,M) = 0$ for $i = 1,\ldots,g$, then $\mathrm{G}_C\text{-}\mathrm{dim}_R(N\otimes_R M) \leqslant p+g$, with equality when R is local. Also, there is an isomorphism*

$$\mathrm{Ext}^{p+g}_R(N\otimes_R M,C) \cong \mathrm{Ext}^p_R(N,R)\otimes_R \mathrm{Ext}^g_R(M,C). \qquad (6.2.5.1)$$

PROOF. Our assumptions imply that $g,p > -\infty$, and hence $M \neq 0 \neq N$. Assuming that $\mathrm{Tor}^R_i(N,M) = 0$ for $i = 1,\ldots,g$, Proposition 6.2.1 implies that $\mathrm{Tor}^R_i(N,M) = 0$ for all $i \geqslant 1$.

Let P be a resolution of N by finitely generated projective R-modules, and let G be a G_C-resolution of M such that $P_i = 0 = G_j$ for $i > p$ and for $j > g$. The Tor-vanishing assumption implies that the complex $P\otimes_R G$ is acyclic, that is, it is a G_C-resolution of $N\otimes_R M$. Since this resolution has length at most $g+p$, we have the inequality $\mathrm{G}_C\text{-}\mathrm{dim}_R(N\otimes_R M) \leqslant p+g$.

Furthermore, Corollary 6.2.4 yields the first step in the next sequence:

$$\begin{aligned}\mathrm{Ext}^i_R(N\otimes_R M,C) &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(P\otimes_R G,C))\\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(P,\mathrm{Hom}_R(G,C)))\\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(\mathrm{Hom}_R(\mathrm{Hom}_R(P,R),R),\mathrm{Hom}_R(G,C)))\\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(P,R)\otimes_R \mathrm{Hom}_R(R,\mathrm{Hom}_R(G,C)))\\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(P,R)\otimes_R \mathrm{Hom}_R(G,C)).\end{aligned}$$

The second step is Hom-tensor adjointness. The third step follows from the fact that finitely generated projective R-modules are reflexive. The fourth step is Hom-evaluation, and the fifth step is Hom-cancellation.

Note that the complex $\mathrm{Hom}_R(P,R)$ lives in homological degrees 0 to $-p$, and $\mathrm{Hom}_R(M,C)$ lives in homological degrees 0 to $-g$. It follows that the complex $\mathrm{Hom}_R(P,R)\otimes_R \mathrm{Hom}_R(G,C)$ lives in homological degrees 0 to $-(p+g)$. Hence, the second step in the next sequence is from the right-exactness of tensor product:

$$\begin{aligned}\mathrm{Ext}^{p+g}_R(N\otimes_R M,C) &\cong \mathrm{H}_{-(p+g)}(\mathrm{Hom}_R(P,R)\otimes_R \mathrm{Hom}_R(G,C))\\ &\cong \mathrm{H}_{-p}(\mathrm{Hom}_R(P,R))\otimes_R \mathrm{H}_{-g}(\mathrm{Hom}_R(G,C))\\ &\cong \mathrm{Ext}^p_R(N,R)\otimes_R \mathrm{Ext}^g_R(M,C).\end{aligned}$$

The previous display explains the first step, and the third step is from Corollary 6.2.4. This explains the isomorphism (6.2.5.1).

Finally, assume that R is local. To prove that $\mathrm{G}_C\text{-}\mathrm{dim}_R(N\otimes_R M) = p+g$, we need to show that $\mathrm{Ext}^{p+g}_R(N\otimes_R M,C) \neq 0$; see Proposition 6.1.7. Using a minimal free resolution of N, we know that $\mathrm{Ext}^p_R(N,R) \neq 0$. Since $\mathrm{Ext}^g_R(M,C) \neq 0$ by Proposition 6.1.7, Nakayama's Lemma implies that $\mathrm{Ext}^p_R(N,R)\otimes_R \mathrm{Ext}^g_R(M,C) \neq 0$, so the condition $\mathrm{Ext}^{p+g}_R(N\otimes_R M,C) \neq 0$ follows from (6.2.5.1). □

The proof of the next result is similar.

Proposition 6.2.6. *Let C be a semidualizing R-module. Let M and N be finitely generated R-modules such that the quantities $g = \mathrm{G}_C\text{-}\dim_R(M)$ and $p = \mathrm{pd}_R(N)$ are finite. If $\mathrm{Ext}^i_R(N,M) = 0$ for all $i < p$, then $\mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}^p_R(N,M)) \leqslant p+g$, with equality when R is local. Also, there is an isomorphism*

$$\mathrm{Ext}^{p+g}_R(\mathrm{Ext}^p_R(N,M),C) \cong N \otimes_R \mathrm{Ext}^g_R(M,C).$$

Proposition 6.2.7. *Let C be a semidualizing R-module. Let M be a finitely generated R-module such that $g = \mathrm{G}_C\text{-}\dim_R(M)$ is finite. If $\mathrm{Ext}^i_R(M,C) = 0$ for all $i < g$, then there is an equality $\mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}^g_R(M,C)) = g$, and one has*

$$\mathrm{Ext}^i_R(\mathrm{Ext}^g_R(M,C),C) \cong \begin{cases} 0 & \text{if } i \neq g \\ M & \text{if } i = g. \end{cases}$$

Proof. Again note that our assumptions imply that $M \neq 0$.

Let G be a G_C-resolution of M such that $G_i = 0$ for all $i > g$. The assumption $\mathrm{Ext}^i_R(M,C) = 0$ for all $i < g$ implies that the induced complex $\Sigma^g \mathrm{Hom}_R(G,C)$ is a G_C-resolution of $\mathrm{Ext}^g_R(M,C)$; see Corollary 6.2.4(b). (Here the operator Σ^g shifts a complex g steps to the left. That is, we have $(\Sigma^g X)_i = X_{i-g}$ for each integer i.) Since this resolution has length g, we have $\mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}^g_R(M,C)) \leqslant g$. Once we show that $\mathrm{Ext}^g_R(\mathrm{Ext}^g_R(M,C),C) \cong M$, we have $\mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}^g_R(M,C)) = g$ since $M \neq 0$; see Proposition 6.1.7.

Corollary 6.2.4(b) provides the first step in the next sequence:

$$\begin{aligned} \mathrm{Ext}^i_R(\mathrm{Ext}^g_R(M,C),C) &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(\Sigma^g \mathrm{Hom}_R(G,C),C)) \\ &\cong \mathrm{H}_i(\Sigma^{-g} \mathrm{Hom}_R(\mathrm{Hom}_R(G,C),C)) \\ &\cong \mathrm{H}_{g-i}(\mathrm{Hom}_R(\mathrm{Hom}_R(G,C),C)) \\ &\cong \mathrm{H}_{g-i}(G) \\ &\cong \begin{cases} 0 & \text{if } i \neq g \\ M & \text{if } i = g. \end{cases} \end{aligned}$$

The second and third steps are straightforward. The fourth step is due to the fact that each G_i is totally C-reflexive, and the fifth step follows because G is a G_C-resolution of M. This proves the desired isomorphisms and hence the result. □

Proposition 6.2.8. *Let C be a semidualizing R-module. Let M and N be finitely generated R-modules such that the quantities $g = \mathrm{G}_C\text{-}\dim_R(M)$ and $p = \mathrm{pd}_R(N)$ are finite. If $\mathrm{Ext}^i_R(M, C \otimes_R N) = 0$ for all $i < g$, then there is an inequality $\mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}^g_R(M, C \otimes_R N)) \leqslant p+g$, with equality when R is local. Also, there is an isomorphism*

$$\mathrm{Ext}^{p+g}_R(\mathrm{Ext}^g_R(M, C \otimes_R N),C) \cong \mathrm{Ext}^p_R(N,M). \tag{6.2.8.1}$$

Proof. Let P be a resolution of N by finitely generated projective R-modules, and let G be a G_C-resolution of M such that $P_i = 0 = G_j$ for $i > p$ and for $j > g$.

Step 1: We prove that the complex $\mathrm{Hom}_R(P,G)$ has homology

$$\mathrm{H}_{-i}(\mathrm{Hom}_R(P,G)) \cong \mathrm{Ext}^i_R(N,M) \tag{6.2.8.2}$$

for all $i \in \mathbb{Z}$. (This is somewhat routine. However, it is good preparation for the next step.) The augmented resolution G^+ is essentially a mapping cone: $G^+ \cong \Sigma^{-1}\mathrm{Cone}(G \xrightarrow{\alpha} M)$ where α is the quasi-isomorphism induced by the augmentation map $G_0 \to M$. The complex G^+ is exact. Since P is a bounded below complex of projective R-modules, it follows that the complex $\operatorname{Hom}_R(P, G^+)$ is exact. There are isomorphisms

$$\operatorname{Hom}_R(P, G^+) \cong \operatorname{Hom}_R(P, \Sigma^{-1} \operatorname{Cone}(\alpha)) \cong \Sigma^{-1} \operatorname{Cone}(\operatorname{Hom}_R(P, \alpha))$$

and it follows that $\operatorname{Hom}_R(P, \alpha)$ is an isomorphism. In particular, we have

$$\mathrm{H}_{-i}(\operatorname{Hom}_R(P, G)) \cong \mathrm{H}_{-i}(\operatorname{Hom}_R(P, M)) \cong \operatorname{Ext}^i_R(N, M)$$

as desired.

Step 2: We prove that the complex $\operatorname{Hom}_R(G, C \otimes_R P)$ has homology

$$\mathrm{H}_{-i}(\operatorname{Hom}_R(G, C \otimes_R P)) \cong \operatorname{Ext}^i_R(M, C \otimes_R N) \tag{6.2.8.3}$$

for all $i \in \mathbb{Z}$. Let $\beta \colon P \xrightarrow{\simeq} N$ be the quasi-isomorphism induced by the augmentation map $P_0 \to N$. Proposition 3.1.9 implies that $N \in \mathcal{A}_C(R)$, and hence $\operatorname{Tor}^R_i(C, N) = 0$ for all $i \geqslant 1$. It follows that the morphism

$$C \otimes_R \beta \colon C \otimes_R P \to C \otimes_R N$$

is a quasiisomorphism. Thus, the complex

$$(C \otimes_R P)^+ \cong \Sigma^{-1} \operatorname{Cone}(C \otimes_R \beta)$$

is exact.

For each i, set $N_i = \operatorname{Coker}(\partial^P_{i+1})$. Thus, we have $N \cong N_0$, and the exactness of P^+ yields exact sequences

$$0 \to N_{i+1} \to P_i \to N_i \to 0$$

for each i. Each module N_i has finite flat dimension, and hence $\operatorname{Tor}^R_1(C, N_i) = 0$ for each i. Thus, the induced sequence

$$0 \to C \otimes_R N_{i+1} \to C \otimes_R P_i \to C \otimes_R N_i \to 0$$

is exact. Corollary 6.2.4(a) implies that $\operatorname{Ext}^1_R(G_j, C \otimes_R N_{i+1}) = 0$ for each i and j. Hence, the sequence

$$0 \to \operatorname{Hom}_R(G_j, C \otimes_R N_{i+1}) \to \operatorname{Hom}_R(G_j, C \otimes_R P_i) \to \operatorname{Hom}_R(G_j, C \otimes_R N_i) \to 0$$

is exact for each i and each j. It follows that the sequence $\operatorname{Hom}_R(G_j, (C \otimes_R P)^+)$ is exact for each j, and hence that the following sequence is exact:

$$\begin{aligned} \operatorname{Hom}_R(G, (C \otimes_R P)^+) &\cong \operatorname{Hom}_R(G, \Sigma^{-1} \operatorname{Cone}(C \otimes_R \beta)) \\ &\cong \Sigma^{-1} \operatorname{Hom}_R(G, \operatorname{Cone}(C \otimes_R \beta)) \\ &\cong \Sigma^{-1} \operatorname{Cone}(\operatorname{Hom}_R(G, C \otimes_R \beta)). \end{aligned}$$

It follows that the morphism

$$\operatorname{Hom}_R(G, C \otimes_R \beta) \colon \operatorname{Hom}_R(G, C \otimes_R P) \to \operatorname{Hom}_R(G, C \otimes_R N)$$

is a quasi-isomorphism, and hence the first step in the next sequence:

$$\begin{aligned} \mathrm{H}_{-i}(\operatorname{Hom}_R(G, C \otimes_R P)) &\cong \mathrm{H}_{-i}(\operatorname{Hom}_R(G, C \otimes_R N)) \\ &\cong \operatorname{Ext}^i_R(M, C \otimes_R N). \end{aligned}$$

The second step is from Corollary 6.2.4(a).

Step 3: We verify the isomorphism (6.2.8.1). The complex P lives in homological degrees p to 0, hence so does $C \otimes_R P$. Since G lives in homological degrees g to 0, it follows that $\operatorname{Hom}_R(G, C \otimes_R P)$ lives in homological degrees p to $-g$. Since each module $\mathrm{Hom}_R(G_i, C \otimes_R P_j)$ is totally C-reflexive by Proposition 5.2.4(c), it follows from (6.2.8.3) that the complex $\Sigma^g \mathrm{Hom}_R(G, C \otimes_R P)$ is a G_C-resolution of $\mathrm{Ext}^g_R(M, C \otimes_R N)$ of length at most $p+g$. This explains the inequality

$$\mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}^g_R(M, C \otimes_R N)) \leqslant p+g.$$

We compute:

$$\begin{aligned}
\mathrm{Ext}^{p+g}_R(\mathrm{Ext}^g_R(M, C \otimes_R N), C) &\cong \mathrm{H}_{-(p+g)}(\mathrm{Hom}_R(\Sigma^g \mathrm{Hom}_R(G, C \otimes_R P), C)) \\
&\cong \mathrm{H}_{-p}(\mathrm{Hom}_R(\mathrm{Hom}_R(G, C \otimes_R P), C)) \\
&\cong \mathrm{H}_{-p}(\mathrm{Hom}_R(\mathrm{Hom}_R(G, C) \otimes_R P, C)) \\
&\cong \mathrm{H}_{-p}(\mathrm{Hom}_R(P, \mathrm{Hom}_R(\mathrm{Hom}_R(G, C), C))) \\
&\cong \mathrm{H}_{-p}(\mathrm{Hom}_R(P, G)) \\
&\cong \mathrm{H}_{-p}(\mathrm{Hom}_R(P, M)) \\
&\cong \mathrm{Ext}^p_R(N, M).
\end{aligned}$$

The first step is by Corollary 6.2.4(b), and the second step is routine. The third step is tensor-evaluation, and the fourth step is Hom-tensor adjointness. The fifth step is due to the fact that each G_i is totally C-reflexive, and the sixth step is from (6.2.8.2).

Step 4: We complete the proof. Assume that R is local. Since $\mathrm{pd}_R(N) = p$ and $M \neq 0$, it can be shown using a minimal free resolution of N that $\mathrm{Ext}^p_R(N, M) \neq 0$. Thus, the equality $\mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}^g_R(M, C \otimes_R N)) = p+g$ follows from the isomorphism (6.2.8.1) and Proposition 6.1.7. □

Remark 6.2.9. The astute reader will note that Proposition 6.2.7 is a special case of Proposition 6.2.8. We include a separate proof of Proposition 6.2.7 because it is slightly simpler than the proof of Proposition 6.2.8.

6.3. Base Change for G_C-dimension

The Tor-vanishing hypothesis in the next result is automatic when φ is flat.

Proposition 6.3.1. *Let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension, and let C be a semidualizing R-module. Let M be a finitely generated R-module such that $\mathrm{Tor}^R_i(S, M) = 0$ for all $i \geqslant 1$. Then one has*

$$\mathrm{G}_{S\otimes_R C}\text{-}\dim_S(S \otimes_R M) \leqslant \mathrm{G}_C\text{-}\dim_R(M)$$

with equality when φ is faithfully flat or when φ is surjective with kernel generated by an R-regular sequence.

Proof. Let P be a resolution of M by finitely generated free R-modules. The Tor-vanishing hypothesis implies that the complex $S \otimes_R P$ is a resolution of $S \otimes_R M$ by finitely generated free S-modules. For each $n \geqslant 0$, there is an isomorphism

$$\mathrm{Coker}(\partial^{S\otimes_R P}_{n+1}) \cong S \otimes_R \mathrm{Coker}(\partial^P_{n+1}).$$

Using Theorem 5.4.13, we conclude that if $\mathrm{Coker}(\partial^P_{n+1})$ is totally C-reflexive, then $\mathrm{Coker}(\partial^{S\otimes_R P}_{n+1})$ is totally $S \otimes_R C$-reflexive; the converse holds when φ is faithfully flat by Proposition 5.3.1; the converse holds when φ is surjective with kernel generated by an R-regular sequence by Proposition 5.3.2. The desired conclusions now follow from Proposition 6.1.7. □

Here is a local-global principle.

Proposition 6.3.2. *Let C be a semidualizing R-module, and let M be a finitely generated R-modules. For an integer $n \geqslant 0$, the following conditions are equivalent:*

(i) $\mathrm{G}_C\text{-}\dim_R(M) \leqslant n$;
(ii) $\mathrm{G}_{U^{-1}C}\text{-}\dim_{U^{-1}R}(U^{-1}M) \leqslant n$ *for each multiplicatively closed subset* $U \subset R$;
(iii) $\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) \leqslant n$ *for each prime ideal* $\mathfrak{p} \subset R$; *and*
(iv) $\mathrm{G}_{C_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(M_\mathfrak{m}) \leqslant n$ *for each maximal ideal* $\mathfrak{m} \subset R$.

In particular, there are equalities

$$\begin{aligned}\mathrm{G}_C\text{-}\dim_R(M) &= \sup\{\mathrm{G}_{C_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \subset R \textit{ is maximal}\} \\ &= \sup\{\mathrm{G}_{C_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Supp}_R(M) \textit{ is maximal}\}.\end{aligned}$$

Proof. As usual, the implications (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) are routine. Furthermore, once we verify the implication (iv) $\implies$ (i), the displayed equality follows from a routine argument.

(iv) $\implies$ (i) Assume that $\mathrm{G}_{C_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(M_\mathfrak{m}) \leqslant n$ for each maximal ideal $\mathfrak{m} \subset R$. Let P be a resolution of M by finitely generated free R-modules. for each maximal ideal $\mathfrak{m} \subset R$, the complex $P_\mathfrak{m}$ is a resolution of $M_\mathfrak{m}$ by finitely generated free $R_\mathfrak{m}$-modules such that

$$\mathrm{Coker}(\partial^{P_\mathfrak{m}}_{n+1}) \cong \mathrm{Coker}(\partial^P_{n+1})_\mathfrak{m}.$$

Proposition 6.1.7 then implies that $\mathrm{Coker}(\partial^P_{n+1})_\mathfrak{m}$ is totally $C_\mathfrak{m}$-reflexive for each $\mathfrak{m}$, so we conclude from Proposition 5.4.2 that $\mathrm{Coker}(\partial^P_{n+1})$ is totally C-reflexive. Another application of Proposition 6.1.7 implies that $\mathrm{G}_C\text{-}\dim_R(M) \leqslant n$. □

Corollary 6.3.3. *Let C be a semidualizing R-module, and let M be a finitely generated R-modules. For an integer $n \geqslant 0$, we have* $\mathrm{G}_C\text{-}\dim_R(M) \leqslant n$ *if and only if* $\mathrm{G}_{C_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(M_\mathfrak{m}) < \infty$ *for each maximal ideal* $\mathfrak{m} \subset R$ *and* $\mathrm{Ext}^i_R(M,C) = 0$ *for all* $i > n$.

Proof. The forward implication follows from Propositions 6.1.7 and 6.3.2. For the reverse implication, assume that $\mathrm{G}_{C_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(M_\mathfrak{m}) < \infty$ for each maximal ideal $\mathfrak{m} \subset R$ and $\mathrm{Ext}^i_R(M,C) = 0$ for all $i > n$. As M is finitely generated, we have

$$0 = \mathrm{Ext}^i_R(M,C)_\mathfrak{m} = \mathrm{Ext}^i_{R_\mathfrak{m}}(M_\mathfrak{m}, C_\mathfrak{m})$$

for each $\mathfrak{m}$ and each $i > m$. Proposition 6.1.7 implies that $\mathrm{G}_{C_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(M_\mathfrak{m}) \leqslant n$ for each $\mathfrak{m}$, so the inequality $\mathrm{G}_C\text{-}\dim_R(M) \leqslant n$ follows from Proposition 6.3.2. □

Remark 6.3.4. Avramov, Iyengar, and Lipman [**7**, (3.3)] prove the following result that is stronger than Corollary 6.3.3: Let C be a semidualizing R-module, and let M be a finitely generated R-modules. Then $\mathrm{G}_C\text{-}\dim_R(M) < \infty$ if and only if $\mathrm{G}_{C_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(M_\mathfrak{m}) < \infty$ for each maximal ideal $\mathfrak{m} \subset R$. Unfortunately, the proof of this result is beyond the scope of this manuscript.

6.4. The AB-formula and Some Consequences

The next result is the case $\mathrm{depth}(R) = 0$ of the AB-formula for G_C-dimension.

Proposition 6.4.1. *Assume that R is local with* $\mathrm{depth}(R) = 0$*, and let C be a semidualizing R-module. If M is a finitely generated R-module of finite G_C-dimension, then M is totally C-reflexive.*

Proof. Set $(-)^\dagger = \operatorname{Hom}_R(-,C)$. We have $\operatorname{depth}_R(C) = \operatorname{depth}(R) = 0$. Hence, a routine argument shows that for a finitely generated R-module N one has $N^\dagger = 0$ if and only if $N = 0$.

Let $n \geqslant 1$ such that $\mathrm{G}_C\text{-}\dim_R(M) \leqslant n$. We prove by induction on n that M is totally C-reflexive.

Base case: $n = 1$. This implies that there is an exact sequence

$$0 \to G_1 \to G_0 \to M \to 0$$

such that G_1 and G_0 are totally C-reflexive. The associated long exact sequence in $\operatorname{Ext}_R(-,C)$ begins as follows:

$$0 \to M^\dagger \to G_0^\dagger \to G_0^\dagger \to \operatorname{Ext}^1_R(M,C) \to 0.$$

Applying the left exact functor $(-)^\dagger$ to this sequence yields the exact sequence in the bottom row of the next commutative diagram:

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & G_1 & \longrightarrow & G_0 \\ & & & & \cong \big\downarrow \delta^C_{G_1} & & \cong \big\downarrow \delta^C_{G_0} \\ 0 & \longrightarrow & \operatorname{Ext}^1_R(M,C)^\dagger & \longrightarrow & G_1^{\dagger\dagger} & \longrightarrow & G_0^{\dagger\dagger}. \end{array}$$

It follows that $\operatorname{Ext}^1_R(M,C)^\dagger = 0$, so the first paragraph of this proof shows that $\operatorname{Ext}^1_R(M,C) = 0$. Since $\mathrm{G}_C\text{-}\dim_R(M) \leqslant 1$, we also have $\operatorname{Ext}^i_R(M,C) = 0$ for all $i \geqslant 2$, hence Proposition 6.3.2 implies that $\mathrm{G}_C\text{-}\dim_R(M) = 0$, that is, that M is totally C-reflexive.

Induction step: Assume that $n > 1$ and that every finitely generated R-module N with $\mathrm{G}_C\text{-}\dim_R(N) < n$ is totally C-reflexive. The assumption $\mathrm{G}_C\text{-}\dim_R(M) \leqslant n$ yields an exact sequence

$$0 \to G_n \to \cdots \to G_1 \xrightarrow{\partial} G_0 \to M \to 0.$$

Setting $M' = \operatorname{Im}(\partial)$ this yields two exact sequences

$$0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_1 \longrightarrow M' \longrightarrow 0$$

$$0 \longrightarrow M' \longrightarrow G_0 \longrightarrow M \longrightarrow 0.$$

The first of these sequences implies that $\mathrm{G}_C\text{-}\dim_R(M') < n$, so our induction hypothesis implies that M' is totally C-reflexive. Thus, the second sequence implies that $\mathrm{G}_C\text{-}\dim_R(M) \leqslant 1$, so the base case implies that M is totally C-reflexive. $\square$

Here is the AB-formula for G_C-dimension.

Proposition 6.4.2. *Assume that R is local, and let C be a semidualizing R-module. If M is a finitely generated R-module of finite G_C-dimension, then*

$$\mathrm{G}_C\text{-}\dim_R(M) = \operatorname{depth}(R) - \operatorname{depth}_R(M).$$

Proof. Let M be a finitely generated R-module of finite G_C-dimension. (In particular, this implies that $M \neq 0$.) We prove the result by induction on $d = \operatorname{depth}(R)$.

Base case: $d = 0$. Proposition 6.4.1 implies that M is totally C-reflexive, hence the first equality in the next sequence:

$$\mathrm{G}_C\text{-}\dim_R(M) = 0 = \operatorname{depth}(R) - \operatorname{depth}_R(M).$$

For the second equality, we need to show that $\operatorname{depth}_R(M) = 0$, that is, that the maximal ideal $\mathfrak{m}$ of R is an associated prime of M. We compute:

$$\operatorname{Ass}_R(M) = \operatorname{Ass}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C) = \operatorname{Supp}_R(\operatorname{Hom}_R(M,C)) \cap \operatorname{Ass}_R(C).$$

Since $M \neq 0$ is totally C-reflexive, we have $\operatorname{Hom}_R(M,C) \neq 0$, and thus $\mathfrak{m} \in \operatorname{Supp}_R(\operatorname{Hom}_R(M,C))$. Because $\operatorname{depth}(R) = 0$, we have $\mathfrak{m} \in \operatorname{Ass}(R) = \operatorname{Ass}_R(C)$; see Proposition 2.1.16(a). Hence, the displayed sequence implies that $\mathfrak{m} \in \operatorname{Ass}_R(M)$, as desired.

Induction step: Assume that $d \geqslant 1$ and that the result holds for local rings S with $\operatorname{depth}(S) < d$.

Case 1: $\operatorname{depth}_R(M) \geqslant 1$. Since $d = \operatorname{depth}(R) \geqslant 1$, a prime avoidance argument yields an element $x \in \mathfrak{m}$ that is R-regular and M-regular. In particular, the natural map $R \to R/xR$ has finite flat dimension and $\operatorname{Tor}_i^R(R/xR, M) = 0$ for all $i \geqslant 1$. Thus, Proposition 6.3.1 yields the first equality in the next sequence

$$\begin{aligned}
\mathrm{G}_C\text{-}\dim_R(M) &= \mathrm{G}_{R/xR\otimes_R C}\text{-}\dim_{R/xR}(R/xR \otimes_R M) \\
&= \operatorname{depth}(R/xR) - \operatorname{depth}_{R/xR}(R/xR \otimes_R M) \\
&= (\operatorname{depth}(R) - 1) - (\operatorname{depth}_R(M) - 1) \\
&= \operatorname{depth}(R) - \operatorname{depth}(M).
\end{aligned}$$

The second equality is from our induction hypothesis since

$$\mathrm{G}_{R/xR\otimes_R C}\text{-}\dim_{R/xR}(R/xR \otimes_R M) = \mathrm{G}_C\text{-}\dim_R(M) < \infty.$$

The third equality follows from the fact that x is R-regular and M-regular.

Case 2: $\operatorname{depth}_R(M) = 0$. We first note that M is not totally C-reflexive. Indeed, from a previous computation, if M were totally C-reflexive, we would have $\operatorname{Ass}_R(M) \subseteq \operatorname{Ass}_R(C) = \operatorname{Ass}(R)$. Since $\operatorname{depth}_R(M) = 0$, we have $\mathfrak{m} \in \operatorname{Ass}_R(M)$, and hence $\mathfrak{m} \in \operatorname{Ass}(R)$. This contradicts the fact that $\operatorname{depth}(R) \geqslant 1$.

Consider an exact sequence

$$0 \to M' \to P \to M \to 0$$

wherein P is a finitely generated free R-module. In particular, the R-module P is totally C-reflexive. Furthermore, since $\operatorname{depth}_R(P) = \operatorname{depth}(R) \geqslant 1$, we have $\operatorname{depth}_R(M') \geqslant 1$. Since $\operatorname{depth}_R(M) = 0$ a standard argument using the long exact sequence in $\operatorname{Ext}_R(R/\mathfrak{m}, -)$ shows that $\operatorname{depth}_R(M') = 1 = 1 + \operatorname{depth}_R(M)$. This explains the third step in the next sequence:

$$\begin{aligned}
\mathrm{G}_C\text{-}\dim_R(M) &= \mathrm{G}_C\text{-}\dim_R(M') + 1 \\
&= [\operatorname{depth}(R) - \operatorname{depth}_R(M')] + 1 \\
&= [\operatorname{depth}(R) - (\operatorname{depth}_R(M) + 1)] + 1 \\
&= \operatorname{depth}(R) - \operatorname{depth}_R(M).
\end{aligned}$$

The first step is by Proposition 6.1.10(b), and Step 1 explains the second step. □

Corollary 6.4.3. *Let C be a semidualizing R-module. If M is a finitely generated R-module of finite G_C-dimension, then* $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leqslant \operatorname{depth}(R_\mathfrak{p})$ *for each* $\mathfrak{p} \in \operatorname{Supp}_R(M)$.

PROOF. Assume that $\mathrm{G}_C\text{-}\dim_R(M) < \infty$. For each $\mathfrak{p} \in \mathrm{Supp}_R(M)$, we have $\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ by Proposition 6.3.2, so the AB-formula explains the second step in the next sequence:

$$0 \leqslant \mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{depth}(R_\mathfrak{p}) - \mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}).$$

The desired inequality now follows. □

Corollary 6.4.4. *Let C be a semidualizing R-module. If M is a finitely generated R-module of finite G_C-dimension, then* $\mathrm{G}_C\text{-}\dim_R(M) \leqslant \dim(R)$.

PROOF. For each maximal ideal, the first step in the next sequence is by the AB-formula 6.4.2:

$$\begin{aligned} \mathrm{G}_{C_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(M_\mathfrak{m}) &= \mathrm{depth}(R_\mathfrak{m}) - \mathrm{depth}_{R_\mathfrak{m}}(M_\mathfrak{m}) \\ &\leqslant \mathrm{depth}(R_\mathfrak{m}) \\ &\leqslant \dim(R_\mathfrak{m}) \\ &\leqslant \dim(R). \end{aligned}$$

The remaining steps are standard. Hence, the inequality $\mathrm{G}_C\text{-}\dim_R(M) \leqslant \dim(R)$ follows from Proposition 6.3.2. □

Corollary 6.4.5. *Let C be a semidualizing R-module, and let M be a finitely generated R-module. There is an inequality*

$$\mathrm{G}_C\text{-}\dim_R(M) \leqslant \mathrm{pd}_R(M)$$

with equality when $\mathrm{pd}_R(M) < \infty$.

PROOF. Assume without loss of generality that $\mathrm{pd}_R(M) < \infty$. For each maximal ideal $\mathfrak{m} \subset R$, a bounded resolution of $M_\mathfrak{m}$ by finitely generated projective $R_\mathfrak{m}$-modules is also a bounded $\mathrm{G}_{C_\mathfrak{m}}$-resolution of $M_\mathfrak{m}$. Thus, we have

$$\mathrm{G}_{C_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(M_\mathfrak{m}) \leqslant \mathrm{pd}_{R_\mathfrak{m}}(M_\mathfrak{m}) \leqslant \mathrm{pd}_R(M) < \infty$$

for each maximal ideal $\mathfrak{m} \subset R$. Thus, the AB-formulas for G_C-dimension and projective dimension explain the second and third equality in the next sequence:

$$\begin{aligned} \mathrm{G}_C\text{-}\dim_R(M) &= \sup\{\mathrm{G}_{C_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \subset R \text{ is maximal}\} \\ &= \sup\{\mathrm{depth}(R_\mathfrak{m}) - \mathrm{depth}_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \subset R \text{ is maximal}\} \\ &= \sup\{\mathrm{pd}_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \subset R \text{ is maximal}\} \\ &= \mathrm{pd}_R(M). \end{aligned}$$

The first step is from Proposition 6.3.2, and the last step is standard. □

Proposition 6.4.6. *Let C be a semidualizing R-module, and let $n \geqslant 0$. The following conditions are equivalent:*

(i) *C is dualizing for R and* $\dim(R) \leqslant n$;
(ii) *every finitely generated R-module M has* $\mathrm{G}_C\text{-}\dim_R(M) \leqslant n$;
(iii) *for each prime ideal $\mathfrak{p} \subset R$, one has* $\mathrm{G}_C\text{-}\dim_R(R/\mathfrak{p}) \leqslant n$;
(iv) *for each maximal ideal $\mathfrak{m} \subset R$, one has* $\mathrm{G}_C\text{-}\dim_R(R/\mathfrak{m}) \leqslant n$.

PROOF. The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are straightforward.

(i) $\Longrightarrow$ (ii) If C is dualizing, then Proposition 6.1.4 says that $\mathrm{G}_C\text{-}\dim_R(M) < \infty$ for each finitely generated M. So, Corollary 6.4.4 implies that $\mathrm{G}_C\text{-}\dim_R(M) \leqslant \dim(R) \leqslant n$.

(iv) $\Longrightarrow$ (i) Assume that $\mathrm{G}_C\text{-}\dim_R(R/\mathfrak{m}) \leqslant n$ for each maximal ideal $\mathfrak{m} \subset R$. Proposition 6.1.7 implies that $\mathrm{Ext}^i_R(R/\mathfrak{m}, C) = 0$ for all $i > n$ and for each $\mathfrak{m}$. It follows that $\mathrm{id}_R(C) \leqslant n$, so C is dualizing. Since $\mathrm{Supp}_R(C) = \mathrm{Spec}(R)$ and $\mathrm{id}_R(C) < \infty$, it follows that $\dim(R) = \mathrm{id}_R(C) \leqslant n$. □

Corollary 6.4.7. *Let C be a semidualizing R-module. The following conditions are equivalent:*

(i) *C is dualizing for R;*
(ii) *every finitely generated R-module has finite G_C-dimension;*
(iii) *for each prime ideal $\mathfrak{p} \subset R$, one has $\mathrm{G}_C\text{-}\dim_R(R/\mathfrak{p}) < \infty$;*
(iv) *for each maximal ideal $\mathfrak{m} \subset R$, one has $\mathrm{G}_C\text{-}\dim_R(R/\mathfrak{m}) < \infty$; and*
(v) *C is point-wise dualizing for R.*

The implications (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longleftrightarrow$ (v) *always hold. When $\dim(R) < \infty$, the conditions* (i)–(v) *are equivalent.*

Proof. The implication (i) $\Longrightarrow$ (ii) is from Proposition 6.4.6. This uses the fact that, when C admits a dualizing module, one has $\dim(R) < \infty$. The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are routine.

(iii) $\Longrightarrow$ (ii) Assume that for each prime ideal $\mathfrak{p} \subset R$, one has $\mathrm{G}_C\text{-}\dim_R(R/\mathfrak{p}) < \infty$. Let M be a finitely generated R-module. Then M has a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that for each $i = 1, \ldots, n$ there is a prime $\mathfrak{p}_i \in \mathrm{Spec}(R)$ such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$. Since each $R/\mathfrak{p}_i$ has finite G_C-dimension by assumption, we may use induction on n with Proposition 6.1.8 to conclude that $\mathrm{G}_C\text{-}\dim_R(M) < \infty$.

(iv) $\Longrightarrow$ (v) Assume that $\mathrm{G}_C\text{-}\dim_R(R/\mathfrak{m}) < \infty$ for each maximal ideal $\mathfrak{m} \subset R$. Proposition 6.3.2 implies that $\mathrm{G}_{C_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) < \infty$ for each m. Proposition 6.4.6 implies that $C_\mathfrak{m}$ is dualizing for $R_\mathfrak{m}$, for each $\mathfrak{m}$, that is, that C is point-wise dualizing for R.

(v) $\Longrightarrow$ (iv) Assume that C is point-wise dualizing for R, that is $C_\mathfrak{m}$ is dualizing for $R_\mathfrak{m}$, for each maximal ideal $\mathfrak{m} \subset R$. It follows that $\mathrm{G}_{C_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) < \infty$ for each m. Since $\mathfrak{m}$ is the only maximal ideal in $\mathrm{Supp}_R(R/\mathfrak{m})$, we conclude from Proposition 6.3.2 that $\mathrm{G}_C\text{-}\dim_R(R/\mathfrak{m}) < \infty$.

Finally, if $\dim(R) < \infty$, then we have (v) $\Longrightarrow$ (i) by Example 2.1.11. □

The next two results are the cases $C = R$ of the previous two results.

Proposition 6.4.8. *Let $n \geqslant 0$. The following conditions are equivalent:*

(i) *R is Goreinstein and $\dim(R) \leqslant n$;*
(ii) *every finitely generated R-module M has $\mathrm{G}\text{-}\dim_R(M) \leqslant n$;*
(iii) *for each prime ideal $\mathfrak{p} \subset R$, one has $\mathrm{G}\text{-}\dim_R(R/\mathfrak{p}) \leqslant n$;*
(iv) *for each maximal ideal $\mathfrak{m} \subset R$, one has $\mathrm{G}\text{-}\dim_R(R/\mathfrak{m}) \leqslant n$.*

Corollary 6.4.9. *The following conditions are equivalent:*

(i) *R is Goreinstein;*
(ii) *every finitely generated R-module has finite G-dimension;*
(iii) *for each prime ideal $\mathfrak{p} \subset R$, one has $\mathrm{G}\text{-}\dim_R(R/\mathfrak{p}) < \infty$;*
(iv) *for each maximal ideal $\mathfrak{m} \subset R$, one has $\mathrm{G}\text{-}\dim_R(R/\mathfrak{m}) < \infty$; and*
(v) *R is point-wise Goreinstein.*

The implications (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longleftrightarrow$ (v) *always hold. When $\dim(R) < \infty$, the conditions* (i)–(v) *are equivalent.*

Proposition 6.4.10. *Assume that R is local, and let C be a semidualizing R-module. Let M and N be finitely generated R-modules such that the quantities $g = \mathrm{G}_C\text{-}\dim_R(M)$ and $p = \mathrm{pd}_R(N)$ are finite.*

(a) *If $\mathrm{Tor}_i^R(N, M) = 0$ for $i = 1, \ldots, g$, then*

$$\mathrm{depth}_R(N \otimes_R M) = \mathrm{depth}_R(M) + \mathrm{depth}_R(N) - \mathrm{depth}(R).$$

(b) *If $\mathrm{Ext}_R^i(N, M) = 0$ for all $i < p$, then*

$$\mathrm{depth}_R(\mathrm{Ext}_R^p(N, M)) = \mathrm{depth}_R(M) + \mathrm{depth}_R(N) - \mathrm{depth}(R).$$

(c) *If $\mathrm{Ext}_R^i(M, C) = 0$ for all $i < g$, then*

$$\mathrm{depth}_R(\mathrm{Ext}_R^g(M, C)) = \mathrm{depth}_R(M).$$

(d) *If $\mathrm{Ext}_R^i(M, C \otimes_R N) = 0$ for all $i < g$, then*

$$\mathrm{depth}_R(\mathrm{Ext}_R^g(M, C \otimes_R N)) = \mathrm{depth}_R(M) + \mathrm{depth}_R(N) - \mathrm{depth}(R).$$

In particular, in cases (a), (b), *and* (d), *we have*

$$\mathrm{depth}(R) \leqslant \mathrm{depth}_R(M) + \mathrm{depth}_R(N).$$

Proof. (a) The AB-formula yields the second and fourth steps in the next sequence, and the third step is from Proposition 6.2.5:

$$\begin{aligned} 0 &\leqslant \mathrm{depth}_R(N \otimes_R M) \\ &= \mathrm{depth}(R) - \mathrm{G}_C\text{-}\dim(N \otimes_R M) \\ &= \mathrm{depth}(R) - (p + g) \\ &= \mathrm{depth}(R) - (\mathrm{depth}(R) - \mathrm{depth}_R(N) + \mathrm{depth}(R) - \mathrm{depth}_R(M)) \\ &= \mathrm{depth}_R(M) + \mathrm{depth}_R(N) - \mathrm{depth}(R). \end{aligned}$$

The proofs for (b)–(d) are similar. □

6.5. More Relations between Semidualizing Modules

The next result compares to Proposition 4.1.1.

Proposition 6.5.1. *Let C be a semidualizing R-module, and let M be a finitely generated R-module. Then M is semidualizing and totally C-reflexive if and only if $\mathrm{Ext}_R^i(M, C) = 0$ for all $i \geqslant 1$ and $\mathrm{Hom}_R(M, C)$ is semidualizing and totally C-reflexive.*

Proof. Proposition 5.1.5 says that M is totally C-reflexive if and only if $\mathrm{Ext}_R^i(M, C) = 0$ for all $i \geqslant 1$ and $\mathrm{Hom}_R(M, C)$ is totally C-reflexive. Thus we may assume that M is totally C-reflexive and prove that M is semidualizing if and only if $\mathrm{Hom}_R(M, C)$ is semidualizing.

Lemma 5.1.8 implies that

$$\mathrm{Ext}_R^i(\mathrm{Hom}_R(M, C), \mathrm{Hom}_R(M, C)) \cong \mathrm{Ext}_R^i(M, M)$$

for all $i \geqslant 0$. It follows that $\mathrm{Ext}_R^i(\mathrm{Hom}_R(M, C), \mathrm{Hom}_R(M, C)) = 0$ for all $i \geqslant 1$ if and only if $\mathrm{Ext}_R^i(M, M) = 0$ for all $i \geqslant 1$. Also, we have an isomorphism $R \cong \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), \mathrm{Hom}_R(M, C))$ if and only if $R \cong \mathrm{Hom}_R(M, M)$. By Proposition 2.2.2(a), this means that the homothety map $\chi^R_{\mathrm{Hom}_R(M,C)}$ is an isomorphism if and only if χ^R_M is an isomorphism, hence the desired result. □

Here is a companion for Proposition 4.1.4. also discuss order reversing for generalized dagger duality in $\mathfrak{S}_0(R)$ and $\overline{\mathfrak{S}}_0(R)$. companion to 4.3.4.

Proposition 6.5.2. *Let B and C be a semidualizing R-modules. The following conditions are equivalent:*

(i) *B is totally C-reflexive;*
(ii) $\mathrm{G}_C\text{-}\dim_R(B)$ *is finite;*
(iii) *$C \in \mathcal{B}_B(R)$; and*
(iv) $\mathrm{Hom}_R(B,C)$ *is a semidualizing R-module and* $\mathrm{Ext}^i_R(B,C) = 0$ *for all $i \geqslant 1$.*

When these conditions are satisfied, the module $\mathrm{Hom}_R(B,C)$ *is totally C-reflexive, and $B \in \mathcal{A}_{\mathrm{Hom}_R(B,C)}(R)$ and* $\mathrm{Hom}_R(B,C) \in \mathcal{A}_B(R)$ *and* $C \cong B \otimes_R \mathrm{Hom}_R(B,C)$.

PROOF. $\square$

APPENDIX A

Some Aspects of Homological Algebra

A.1. Natural Maps

Definition A.1.1. Let L, M, N be R-modules.

The *tensor evaluation* homomorphism

$$\omega_{LMN} \colon \operatorname{Hom}_R(L, M) \otimes_R N \to \operatorname{Hom}_R(L, M \otimes_R N)$$

is given by $\omega_{LMN}(\psi \otimes n)(l) = \psi(l) \otimes n$.

The *Hom evaluation* homomorphism

$$\theta_{LMN} \colon L \otimes_R \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(\operatorname{Hom}_R(L, M), N)$$

is given by $\theta_{LMN}(l \otimes \psi)(\phi) = \psi(\phi(l))$.

The next two lemmata are from Ishikawa.

Lemma A.1.2. *Let L, M, N be R-modules. The tensor evaluation homomorphism $\omega_{LMN} \colon \operatorname{Hom}_R(L, M) \otimes_R N \to \operatorname{Hom}_R(L, M \otimes_R N)$ is an isomorphism under either of the following conditions:*

(1) *L is finitely generated and projective; or*
(2) *L is finitely generated and N is flat.*

Proof. First observe that, for R-modules L', L'', the following commutative diagram shows that the map $\omega_{(L' \oplus L'')MN}$ is an isomorphism if and only if $\omega_{L'MN}$ and $\omega_{L''MN}$ are isomorphisms:

$$\begin{array}{ccc} \operatorname{Hom}_R(L' \oplus L'', M) \otimes_R N & \xrightarrow{\cong} & (\operatorname{Hom}_R(L', M) \otimes_R N) \oplus (\operatorname{Hom}_R(L'', M) \otimes_R N) \\ \downarrow{\scriptstyle \omega_{(L' \oplus L'')MN}} & & \downarrow{\scriptstyle \omega_{L'MN} \oplus \omega_{L''MN}} \\ \operatorname{Hom}_R(L' \oplus L'', M \otimes_R N) & \xrightarrow{\cong} & \operatorname{Hom}_R(L', M \otimes_R N) \oplus \operatorname{Hom}_R(L'', M \otimes_R N). \end{array}$$

(1) It is straightforward to show that ω_{LMN} is an isomorphism when $L = R$. Hence, an induction argument using the above observation shows that ω_{LMN} is an isomorphism when $L = R^n$ for some n. When L is finitely generated and projective, there is an R-module L'' such that $L \oplus L'' \cong R^n$ for some n, so again the previous paragraph implies that ω_{LMN} is an isomorphism in this case.

(2) Assume that L is finitely generated and N is flat. Since R is noetherian, there exists an exact sequence

$$R^m \xrightarrow{f} R^n \xrightarrow{g} L \to 0.$$

The left-exactness of $\operatorname{Hom}_R(-, M)$ implies that the next sequence is exact

$$0 \to \operatorname{Hom}_R(L, M) \xrightarrow{g^M} \operatorname{Hom}_R(R^n, M) \xrightarrow{f^M} \operatorname{Hom}_R(R^m, M)$$

where $f^M = \operatorname{Hom}_R(f, M)$ and $g^M = \operatorname{Hom}_R(g, M)$. Since N is flat, the top row of the next commutative diagram is also exact

$$\begin{array}{ccccccc}
0 \longrightarrow & \operatorname{Hom}(L, M) \otimes N & \xrightarrow{g^M \otimes N} & \operatorname{Hom}(R^n, M) \otimes N & \xrightarrow{f^M \otimes N} & \operatorname{Hom}(R^m, M) \otimes N \\
& \downarrow{\scriptstyle \omega_{LMN}} & & \downarrow{\scriptstyle \omega_{R^n MN}} \cong & & \downarrow{\scriptstyle \omega_{R^m MN}} \cong \\
0 \longrightarrow & \operatorname{Hom}(L, M \otimes N) & \xrightarrow{g^{M \otimes N}} & \operatorname{Hom}(R^n, M \otimes N) & \xrightarrow{f^{M \otimes N}} & \operatorname{Hom}(R^m, M \otimes N).
\end{array}$$

The bottom row is exact because $\operatorname{Hom}_R(-, M \otimes N)$ is left-exact. The maps $\omega_{R^m MN}$ and $\omega_{R^n MN}$ are isomorphisms by case (1), so a diagram chase shows that ω_{LMN} is an isomorphism as well. □

Lemma A.1.3. *Let L, M, N be R-modules. The Hom evaluation homomorphism $\theta_{LMN} \colon L \otimes_R \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(\operatorname{Hom}_R(L, M), N)$ is an isomorphism under either of the following conditions:*

(1) *L is finitely generated and projective; or*
(2) *L is finitely generated and N is injective.*

Proof. Similar to the proof of Lemma A.1.2. □

Here is a version that we did not know about before.

Lemma A.1.4. *Let L, M, N be R-modules. The tensor evaluation homomorphism $\omega_{LMN} \colon \operatorname{Hom}_R(L, M) \otimes_R N \to \operatorname{Hom}_R(L, M \otimes_R N)$ is an isomorphism under either of the following conditions:*

(1) *N is finitely generated and projective; or*
(2) *L is projective and N is finitely generated.*

Proof. The proof is similar to the proof of Lemma A.1.2. We provide a sketch.

First observe that, for R-modules N', N'', the map $\omega_{LM(N' \oplus N'')}$ is an isomorphism if and only if $\omega_{LMN'}$ and $\omega_{LMN'}$ are isomorphisms.

(1) It is straightforward to show that ω_{LMR} is an isomorphism. Hence, the map ω_{LMN} is an isomorphism when $N = R^n$ for some n, and thus whenever N is a finitely generated projective.

(2) When L is projective and N is finitely generated, use a presentation

$$R^m \xrightarrow{f} R^n \xrightarrow{g} L \to 0.$$

with the fact that ω_{LMR^m} and ω_{LMR^n} are isomorphisms to conclude that ω_{LMN} is an isomorphism as well. □

Here is a consequence of the Künneth formula.

Proposition A.1.5. *Let k be a field, and let R and S be k-algebras. Let B and B' be R-modules such that B is finitely generated, and let C and C' be S-modules such that C is finitely generated. For each $i \geqslant 0$, there are $R \otimes_k S$-module isomorphisms*

$$\operatorname{Tor}_i^{R \otimes_k S}(B \otimes_k C, B' \otimes_k C') \cong \oplus_{j=0}^{i} \operatorname{Tor}_j^R(B, B') \otimes_k \operatorname{Tor}_{i-j}^S(C, C')$$
$$\operatorname{Ext}^i_{R \otimes_k S}(B \otimes_k C, B' \otimes_k C') \cong \oplus_{j=0}^{i} \operatorname{Ext}^j_R(B, B') \otimes_k \operatorname{Ext}^{i-j}_S(C, C').$$

Proof. First, let X be a complex of R-modules, and let Y be a complex of S-modules. The Künneth formula [**17**, (10.81)] implies that there is a k-isomorphism

$$\oplus_{p+q=i} \operatorname{H}_p(X) \otimes_k \operatorname{H}_q(Y) \xrightarrow{\cong} \operatorname{H}_i(X \otimes_k Y) \tag{A.1.5.1}$$

given by $\sum_p \overline{x_p} \otimes \overline{y_{i-p}} \mapsto \sum_p \overline{x_p \otimes y_{i-p}}$. (Here $X \otimes_k Y$ is the total complex, not the double complex.) It is straightforward to show that α is $R \otimes_k S$-linear.

Let P be a resolution of B by finitely generated free R-modules, and let Q be a resolution of C by finitely generated free S-modules. Since k is a field, we have $\operatorname{Tor}_i^k(B, C) = 0$ for all $i \geqslant 1$. Hence the complex $P \otimes_k Q$ is a resolution of $B \otimes_k C$ by finitely generated free $R \otimes_k S$-modules.

It is straightforward to show that there is an isomorphism of complexes

$$(P \otimes_k Q) \otimes_{R \otimes_k S} (B' \otimes_k C') \xrightarrow{\cong} (P \otimes_R B') \otimes_k (Q \otimes_S C')$$

given by $(p \otimes q) \otimes (b' \otimes c') \mapsto (p \otimes b') \otimes (q \otimes c')$. Furthermore, this map is $R \otimes_k S$-linear. This explains the second step in the next sequence:

$$\begin{aligned}
\operatorname{Tor}_i^{R \otimes_k S}(B \otimes_k C, B' \otimes_k C') &\cong \mathrm{H}_i((P \otimes_k Q) \otimes_{R \otimes_k S} (B' \otimes_k C')) \\
&\cong \mathrm{H}_i((P \otimes_R B') \otimes_k (Q \otimes_S C')) \\
&\cong \oplus_{p+q=i} \mathrm{H}_p(P \otimes_R B') \otimes_k \mathrm{H}_q(Q \otimes_S C') \\
&\cong \oplus_{j=0}^{i} \operatorname{Tor}_j^R(B, B') \otimes_k \operatorname{Tor}_{i-j}^S(C, C')
\end{aligned}$$

The first step comes from the fact that $P \otimes_k Q$ is a resolution of $B \otimes_k C$ by free $R \otimes_k S$-modules. The third step is from the Künneth formula (A.1.5.1), and the fourth step is by definition. This yields the desired isomorphism for Tor.

The isomorphism for Ext is verified similarly using the isomorphism

$$\operatorname{Hom}_R(P, B') \otimes_k \operatorname{Hom} S(Q, C') \xrightarrow{\cong} \operatorname{Hom}_{R \otimes_k S}(P \otimes_k Q, B' \otimes_k C')$$

given by the formula $\phi \otimes \psi \mapsto \phi \boxtimes \psi$: here $\phi \colon P_m \to B'$ and $\psi \colon Q_n \to C'$, and $\phi \boxtimes \psi \colon P_m \otimes_k Q_n \to B' \otimes_k C'$ is given by $p_m \otimes q_n \mapsto \phi(p_m) \otimes \psi(q_n)$. □

A.2. Fidelity

Lemma A.2.1. *Let M and N be non-zero R-modules. If M is finitely generated and* $\operatorname{Supp}_R(M) = \operatorname{Spec}(R)$, *then* $\operatorname{Hom}_R(M, N) \neq 0$ *and* $M \otimes_R N \neq 0$.

Proof. If R is local with maximal ideal $\mathfrak{m}$, then $\operatorname{Hom}_R(M, R/\mathfrak{m}) \neq 0$. Indeed, Nakayama's Lemma implies that $M/\mathfrak{m}M$ is a non-zero vector space over $R/\mathfrak{m}$ and so any composition $M \twoheadrightarrow M/\mathfrak{m}M \twoheadrightarrow R/\mathfrak{m}$ gives a non-zero element of $\operatorname{Hom}_R(M, R/\mathfrak{m})$. It follows that, for each $\mathfrak{p} \in \operatorname{Spec}(R)$, we have

$$\operatorname{Hom}_R(M, R/\mathfrak{p})_\mathfrak{p} \cong \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \neq 0$$

so $\operatorname{Hom}_R(M, R/\mathfrak{p}) \neq 0$.

Use the fact that R is noetherian to conclude that N has an associated prime $\mathfrak{p}$, and hence a monomorphism $R/\mathfrak{p} \hookrightarrow N$. Apply $\operatorname{Hom}_R(M, -)$ to find a monomorphism $0 \neq \operatorname{Hom}_R(M, R/\mathfrak{p}) \hookrightarrow \operatorname{Hom}_R(M, N)$. It follows that $\operatorname{Hom}_R(M, N) \neq 0$.

For the tensor product, note that the identity $N \to N$ is a non-zero element of $\operatorname{Hom}_R(N, N)$. Therefore, the previous paragraph provides the nonvanishing in the next sequence while the isomorphism is by Hom-tensor adjointness:

$$\operatorname{Hom}_R(M \otimes N, N) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_R(N, N)) \neq 0.$$

It follows that $M \otimes N \neq 0$.

Here is an alternate proof for the tensor product. Choose a maximal ideal $\mathfrak{m} \in \operatorname{Supp}_R(N)$. Nakayama's Lemma yields an epimorphism $N_\mathfrak{m} \twoheadrightarrow R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$. The right-exactness of $M_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} -$ yields the epimorphism in the middle of the next display:

$$(M \otimes_R N)_{\mathfrak{m}} \cong M_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} N_{\mathfrak{m}} \twoheadrightarrow M_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}} \cong M_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}} M_{\mathfrak{m}} \neq 0.$$

The isomorphisms are standard. The non-vanishing is from Nakayama's Lemma since we have $\mathfrak{m} \in \operatorname{Spec}(R) = \operatorname{Supp}_R(M)$. It follows that $(M \otimes_R N)_{\mathfrak{m}} \neq 0$, so $M \otimes_R N \neq 0$. $\square$

Definition A.2.2. An injective R-module E is *faithfully injective* if the functor $\operatorname{Hom}_R(-, E)$ is faithfully exact, that is, if it satisfies the following condition: a sequence S of R-module homomorphisms is exact if and only if the sequence $\operatorname{Hom}_R(S, E)$ is exact. This is equivalent to the following condition: for each R-module M, one has $M = 0$ if and only if $\operatorname{Hom}_R(M, E) = 0$. The term *faithfully projective* is defined dually.

Example A.2.3. For instance, the R-module $E = \coprod_{\mathfrak{m}} E_R(R/\mathfrak{m})$ is faithfully injective where the coproduct is taken over the set of maximal ideals $\mathfrak{m} \subset R$. [ref] ichikawa? Every non-zero free R-module is faithfully projective.

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Bibliography

1. M. Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, Séminaire d'Algèbre Commutative dirigé par Pierre Samuel, vol. 1966/67, Secrétariat mathématique, Paris, 1967. MR 37 #1435
2. M. Auslander and M. Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, R.I., 1969. MR 42 #4580
3. M. Auslander and D. A. Buchsbaum, *Homological dimension in local rings*, Trans. Amer. Math. Soc. **85** (1957), 390–405. MR 19,249d
4. L. L. Avramov, *Infinite free resolutions*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 1–118. MR 99m:13022
5. L. L. Avramov and H.-B. Foxby, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 241–270. MR 98d:13014
6. L. L. Avramov, V. N. Gasharov, and I. V. Peeva, *Complete intersection dimension*, Inst. Hautes Études Sci. Publ. Math. (1997), no. 86, 67–114 (1998). MR 1608565 (99c:13033)
7. L. L. Avramov, S. B. Iyengar, and J. Lipman, *Reflexivity and rigidity for complexes I. Commutative rings*, Algebra and Number Theory, to appear, `arXiv:math/0904.4695v1`.
8. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, revised ed., Studies in Advanced Mathematics, vol. 39, University Press, Cambridge, 1998. MR 1251956 (95h:13020)
9. L. W. Christensen, *Semi-dualizing complexes and their Auslander categories*, Trans. Amer. Math. Soc. **353** (2001), no. 5, 1839–1883. MR 2002a:13017
10. I. S. Cohen, *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc. **59** (1946), 54–106. MR 0016094 (7,509h)
11. A. Frankild and S. Sather-Wagstaff, *The set of semidualizing complexes is a nontrivial metric space*, J. Algebra **308** (2007), no. 1, 124–143. MR 2290914
12. A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 167. MR 0217085 (36 #177c)
13. ______, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361.
14. D. Hilbert, *über die theorie der algebraischen formen*, Math. Ann. **36** (1890), 473–534.
15. H. Holm and P. Jørgensen, *Semi-dualizing modules and related Gorenstein homological dimensions*, J. Pure Appl. Algebra **205** (2006), no. 2, 423–445. MR 2203625
16. H. Matsumura, *Commutative ring theory*, second ed., Studies in Advanced Mathematics, vol. 8, University Press, Cambridge, 1989. MR 90i:13001
17. J. J. Rotman, *An introduction to homological algebra*, Pure and Applied Mathematics, vol. 85, Academic Press Inc., New York, 1979. MR 538169 (80k:18001)
18. S. Sather-Wagstaff, *Semidualizing modules and the divisor class group*, Illinois J. Math. **51** (2007), no. 1, 255–285. MR 2346197
19. S. Sather-Wagstaff and S. Spiroff, *Maps on divisor class groups induced by ring homomorphisms of finite flat dimension*, J. Commut. Algebra **1** (2009), no. 3, 567–590. MR 2524867
20. J.-P. Serre, *Sur la dimension homologique des anneaux et des modules noethériens*, Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955 (Tokyo), Science Council of Japan, 1956, pp. 175–189. MR 19,119a
21. D. White, *Gorenstein projective dimension with respect to a semidualizing module*, J. Commut. Algebra, to appear, `arXiv:math.AC/0611711v2`.