Complete Intersection Dimension for Complexes

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I. CI-dimension for modules.

Let R be a local ring. A *quasi-deformation* of R is a diagram of local ring homomorphisms $R \xrightarrow{f} R' \xleftarrow{g} Q$ such that f is flat and g is surjective with kernel generated by a Q-regular sequence. In this situation, if M is an R-module, let $M' = M \otimes_R R'$.

If R is a local ring and $M \neq 0$ a finite module over R, the *CI-dimension* of M is defined to be

$$\mathrm{CI}-\mathrm{dim}_R(M) = \inf\{\mathrm{pdim}_Q M' - \mathrm{pdim}_Q R'\}$$

with the inf taken over all quasi-deformations $R \to R' \leftarrow Q$. Set $CI - dim_R(0) = 0$.

If R is any ring and M is an R-module then

$$\mathrm{CI}-\mathrm{dim}_R(M) = \sup\{\mathrm{CI}-\mathrm{dim}_{R_\mathfrak{m}}(M_\mathfrak{m})\}$$

with the sup taken over all $\mathfrak{m} \in \mathrm{m}-\mathrm{Spec}(R)$.

Theorem. If $(R, \mathfrak{m}, k)$ is a local ring, TFAE.
1. R is a complete intersection.
2. Every finite R-module has finite $\mathrm{CI-dim}$.
3. $\mathrm{CI-dim}_R(k) < \infty$.

For a finite R-module M, we say $\mathrm{G-dim}_R(M) = 0$ if M is reflexive and $\mathrm{Ext}^i_R(M, R) = 0 = \mathrm{Ext}^i_R(M^*, R)$ for $i \neq 0$. In general, $\mathrm{G-dim}_R(M) = n$ if there exists a resolution

$$0 \to G_n \to \cdots \to G_0 \to M \to 0$$

such that each $\mathrm{G-dim}_R(G_p) = 0$ and no shorter resolution of this form exists.

Theorem. If M is a finite R-module, then

$$\mathrm{G-dim}_R(M) \leq \mathrm{CI-dim}_R(M) \leq \mathrm{pdim}_R(M)$$

with equality holding to the left of any finite quantity.

Theorem. Let M be a finite R-module.
1. For $\mathfrak{p} \in \mathrm{Spec}(R)$,
$\mathrm{CI-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \mathrm{CI-dim}_R(M)$.
2. Let $\mathbf{x} = x_1, \ldots, x_n$ be an R-regular sequence and $\overline{R} = R/(\mathbf{x})R$. If $(\mathbf{x})M = 0$, then

$$\mathrm{CI-dim}_R(M) \leq \mathrm{CI-dim}_{\overline{R}}(M) + n$$

with equality holding when $\mathbf{x} \in \mathrm{Jac}(R)$.
3. If $R \to S$ is faithfully flat, then

$$\mathrm{CI-dim}_R(M) \leq \mathrm{CI-dim}_S(M \otimes_R S)$$

and equality holds if $\mathrm{CI-dim}_S(M \otimes_R S) < \infty$.
4. If $I \subsetneq R$ is an ideal and R^* the I-adic completion of R, then

$$\mathrm{CI-dim}_{R^*}(M \otimes_R R^*) \leq \mathrm{CI-dim}_R(M)$$

with equality holding if $I \subseteq \mathrm{Jac}(R)$.
5. If R is local and $\mathrm{CI-dim}_R(M) < \infty$ then

$$\mathrm{CI-dim}_R(M) = \mathrm{depth}(R) - \mathrm{depth}_R(M).$$

Furthermore, the sequence of Betti numbers $b_n^R(M)$ is bounded above by a polynomial in n.

II. Background on complexes.

We write complexes $X_\bullet$ of R-modules with decreasing indices: $\cdots \to X_p \stackrel{d_p^X}{\to} X_{p-1} \to \cdots$.

A complex $X_\bullet$ is *homologically bounded* if $\mathsf{H}_p(X_\bullet) = 0$ for all but finitely many p.
It is *bounded* if $X_p = 0$ for all but finitely many p.
It is *homologically finite* if it is homologically bounded and every $\mathsf{H}_p(X_\bullet)$ is a finite R-module.
The *supremum* of $X_\bullet$ is defined as

$$\sup(X_\bullet) = \sup\{p | \mathsf{H}_p(X_\bullet) \neq 0\}.$$

If $X_\bullet \stackrel{f_\bullet}{\to} Y_\bullet$ is a morphism of complexes such that every induced map $\mathsf{H}_p(X_\bullet) \stackrel{\mathsf{H}_p(f_\bullet)}{\to} \mathsf{H}_p(Y_\bullet)$ is an isomorphism, we say that $f_\bullet$ is a *quasi-isomorphism*.

In general, $X_\bullet$ and $Y_\bullet$ are *equivalent*, denoted $X_\bullet \simeq Y_\bullet$ if there exists a sequence

$$X_\bullet \xrightarrow{\cong} X(1)_\bullet \xleftarrow{\cong} X(2)_\bullet \xrightarrow{\cong} \cdots \xrightarrow{\cong} Y_\bullet$$

of quasi-isomorphisms.

A *projective resolution* of $X_\bullet$ is a complex $P_\bullet \simeq X_\bullet$ such that every P_p is projective. If $X_\bullet$ is homologically bounded, then a projective resolution of $X_\bullet$ exists. If $X_\bullet$ is homologically finite, then $X_\bullet$ has a projective resolution $F_\bullet \xrightarrow{\cong} X_\bullet$ such that each F_p is a finite, free R-module.

If $(R, \mathfrak{m}, k)$ is local, the pth Betti number of $X_\bullet$ is

$$b_p^R(X_\bullet) = \mathsf{rank}_k(\mathsf{H}_p(P_\bullet \otimes_R k))$$

where $P_\bullet \simeq X_\bullet$ is a projective resolution. The Betti numbers are independent of the choice of projective resolution.

If $X_\bullet$ has a projective resolution $P_\bullet \xrightarrow{\simeq} X_\bullet$ where $P_\bullet$ is bounded then $X_\bullet$ has *finte projective dimension*. If $X_\bullet$ has finite projective dimension, we write $\mathrm{pdim}_R(X_\bullet) \leq n$ if there exists $P_\bullet \xrightarrow{\simeq} X_\bullet$ such that $P_p = 0$ for all $p > n$. We write $\mathrm{pdim}_R(X_\bullet) = n$ if $\mathrm{pdim}_R(X_\bullet) \leq n$ and $\mathrm{pdim}_R(X_\bullet) \not\leq n - 1$.

If $X_\bullet$ is a complex over a local ring $(R, \mathfrak{m})$, let $K_\bullet$ denote the Koszul complex on a set of n generators for $\mathfrak{m}$. The *depth* of $X_\bullet$ is defined to be

$$\mathrm{depth}_R(X_\bullet) = n - \sup(X_\bullet \otimes_R K_\bullet).$$

If $X_\bullet$ is homologically bounded, then $0 \leq \mathrm{depth}_R(X_\bullet) \leq n$. If $\mathrm{pdim}_R(X_\bullet) < \infty$, the Auslander-Buchsbaum formula holds:

$$\mathrm{pdim}_R(X_\bullet) = \mathrm{depth}(R) - \mathrm{depth}_R(X_\bullet).$$

III. CI-dim for complexes

Given a quasi-deformation $R \to R' \leftarrow Q$ of a local ring R and a complex $X_\bullet$ over R, let $X'_\bullet = X_\bullet \otimes_R R'$.

If $X_\bullet \not\simeq 0$, define the *CI-dimension* of $X_\bullet$ to be

$$\mathrm{CI}-\mathrm{dim}_R(X_\bullet) = \inf\{\mathrm{pdim}_Q X'_\bullet - \mathrm{pdim}_Q R'\}$$

taken over all quasi-deformations $R \to R' \leftarrow Q$. If $X_\bullet \simeq 0$, define $CI - dim_R(X_\bullet) = 0$.

If R is a Noetherian ring and $X_\bullet$ is a complex over R then

$$\mathrm{CI}-\mathrm{dim}_R(M) = \sup\{\mathrm{CI}-\mathrm{dim}_{R_\mathfrak{m}}(M_\mathfrak{m})\}$$

taken over all maximal ideals $\mathfrak{m} \subset R$.

Theorem. If $(R, \mathfrak{m}, k)$ is a local ring, TFAE.
1. R is a complete intersection.

2. Every homologically finite complex over R has finite $\mathsf{CI}-\mathsf{dim}$.
3. Every finite R-module has finite $\mathsf{CI}-\mathsf{dim}$.
4. $\mathsf{CI}-\mathsf{dim}_R(k) < \infty$.

For a complex $X_\bullet$ over R, we say $\mathsf{G}-\mathsf{dim}_R(X_\bullet) < \infty$ if there exists a resolution $G_\bullet \simeq X_\bullet$ such that $G_\bullet$ is bounded and $\mathsf{G}-\mathsf{dim}(G_p) = 0$ for all p. If $X_\bullet$ has finite $\mathsf{G}-\mathsf{dim}$, we write $\mathsf{G}-\mathsf{dim}_R(X_\bullet) \leq n$ if there exists $G_\bullet \simeq X_\bullet$ such that $G_p = 0$ for all $p > n$. We write $\mathsf{pdim}_R(X_\bullet) = n$ if $\mathsf{pdim}_R(X_\bullet) \leq n$ and $\mathsf{pdim}_R(X_\bullet) \not\leq n-1$.

Theorem. If $X_\bullet$ a homologically finite complex over R, then

$$\mathsf{G}-\mathsf{dim}_R(X_\bullet) \leq \mathsf{CI}-\mathsf{dim}_R(X_\bullet) \leq \mathsf{pdim}_R(X_\bullet)$$

with equality holding to the left of any finite quantity.

Theorem. Let $X_\bullet$ be a homologically finite complex over R.
1. For $\mathfrak{p} \in \mathrm{Spec}(R)$,
$\mathrm{CI}-\dim_{R_\mathfrak{p}}((X_\bullet)_\mathfrak{p}) \leq \mathrm{CI}-\dim_R(M)$.
2. Let $\mathbf{x} = x_1, \ldots, x_n$ be an R-regular sequence and $\overline{R} = R/(\mathbf{x})R$. If $(\mathbf{x})M = 0$, then

$$\mathrm{CI}-\dim_R(M) \leq \mathrm{CI}-\dim_{\overline{R}}(M) + n$$

with equality holding when $\mathbf{x} \in \mathrm{Jac}(R)$.
3. If $R \to S$ is faithfully flat, then

$$\mathrm{CI}-\dim_R(X_\bullet) \leq \mathrm{CI}-\dim_S(X_\bullet \otimes_R S)$$

and equality holds if $\mathrm{CI}-\dim_S(X_\bullet \otimes_R S) < \infty$.
4. If $I \subsetneq R$ is an ideal and R^* the I-adic completion of R, respectively, then

$$\mathrm{CI}-\dim_{R^*}(X_\bullet \otimes_R R^*) \leq \mathrm{CI}-\dim_R(X_\bullet)$$

with equality holding if $I \subseteq \mathrm{Jac}(R)$.
5. If R is local and $\mathrm{CI}-\dim_R(X_\bullet) < \infty$ then

$$\mathrm{CI}-\dim_R(X_\bullet) = \mathrm{depth}(R) - \mathrm{depth}_R(X_\bullet).$$

Furthermore, the sequence of Betti numbers $b_n^R(X_\bullet)$ is bounded above by a polynomial in n.