Descent of semidualizing complexes

Sean Sather-Wagstaff

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Joint with Lars W. Christensen

Assumption. $f\colon R \to S$ is a local ring homomorphism of finite flat dimension.

Example. a. The completion map $f_R\colon R \to \widehat{R}$.

b. The surjection $\pi_{\mathbf{x}}\colon R \to R/(\mathbf{x})$ where $\mathbf{x} \in R$ is an R-sequence.

Definition. A *semidualizing R-complex* is a homologically finite complex of finitely generated free R-modules

$$P = \cdots \xrightarrow{\partial_P^{i+2}} P_{i+1} \xrightarrow{\partial_P^{i+1}} P_i \to 0$$

such that the homothety morphism is a quasiisomorphism.

$$R \xrightarrow{\simeq} \operatorname{Hom}_R(P,P)$$

Notation. $\mathfrak{S}(R)$ is the set of shift-quasiisomorphism classes of semidualizing R-complexes.

Application. (Avramov-Foxby '97) There is a semidualizing $\widehat{S}$-complex $D^{f_S f}$ such that $I_S^S(t) = I_R^R(t) P^S_{D^{f_S f}}(t)$.

Example. a. R is R-semidualizing.

b. (Hartshorne '66) P is R-dualizing if and only if it is R-semidualizing and $\mathrm{id}_R(P) < \infty$.

c. Let P be a projective resolution of a finite R-module C. Then P is a semidualizing R-complex if and only if C is a semidualizing R-module, i.e., $R \xrightarrow{\cong} \mathrm{Hom}_R(C, C)$ and $\mathrm{Ext}_R^{>0}(C, C) = 0$.

Fact. a. (LWC '01) If R is CM and P is R-semidualizing, then P is a shift of a projective resolution of a semidualizing R-module.

b. (SSW '04) If R is a CM normal domain, then $\mathfrak{S}(R) \subseteq \mathrm{Cl}(R)$.

c. (LWC '01, Frankild-SSW '04) The assignment $[P] \mapsto [P \otimes_R S]$ describes an injective map $\mathfrak{S}(f)\colon \mathfrak{S}(R) \to \mathfrak{S}(S)$.

d. (Frankild-SSW '04) If P is a homologically finite complex of free R-modules with $P_i = 0$ for $i \ll 0$ and $P \otimes_R S$ is S-semidualizing, then P is R-semidualizing.

Question. When is $\mathfrak{S}(f)$ surjective?

Assumption. S admits a dualizing complex D^S.

Fact. a. (Frankild-SSW '04) If R is complete, then $\mathfrak{S}(\pi_{\mathbf{x}})$ is surjective.

b. (Avramov-Foxby '94) $[D^S]$ is in the image of $\mathfrak{S}(f)$ if and only if R admits a dualizing complex D^R and $D^S \sim D^R \otimes_R S$.

c. Even if R admits a dualizing complex the map $\mathfrak{S}(f)$ will not usually be surjective.

Example. If R is a field and S a non-Gorenstein local R-algebra, then the structure map $f\colon R \to S$ is flat and R is R-dualizing, but $D^R \otimes_R S \cong R \otimes_R S \cong S \not\simeq D^S$.

Question. When is $\mathfrak{S}(f_R)$ surjective?

Fact. (Ogoma '82) R may not admit a dualizing complex.

Question. How badly can surjectivity of $\mathfrak{S}(f_R)$ fail?

Example. Fix an integer $n \geq 1$. There exists a complete CM normal domain S containing $\mathbb{Q}$ such that $\dim(S) \geq 2$ and

$$\operatorname{card}(\mathfrak{S}(S)) = 2^n.$$

There exists a local UFD R such that $\widehat{R} \cong S$. In particular,

$$\mathfrak{S}(R) \subseteq \operatorname{Cl}(R) = \{[R]\} \quad \Longrightarrow \quad \operatorname{card}(\mathfrak{S}(R)) = 1$$

so the image of $\mathfrak{S}(f_R)$ is trivial.

Fact. (Khinich '93, Rotthaus '96) If R has the approximation property, then R admits a dualizing complex.

Theorem. (LWC-SSW '06) *If R has the approximation property, then $\mathfrak{S}(f_R)$ is surjective.*

Proof. Let P be a semidualizing $\widehat{R}$-complex.

Case 1. R is CM of dimension d. We will exhibit a semidualizing R-complex $\widetilde{F}$ such that $\widetilde{F} \otimes_R \widehat{R} \simeq P$.

Let $\mathbf{x}$ be a system of parameters for R. It suffices to construct a homologically finite complex of free R-modules $\widetilde{F}$ such that $\widetilde{F}_i = 0$ for $i \ll 0$ and there exists a $\widehat{R}/(\mathbf{x})$-quasiisomorphism $\widetilde{F} \otimes_R \widehat{R}/(\mathbf{x}) \xrightarrow{\simeq} P \otimes_{\widehat{R}} \widehat{R}/(\mathbf{x})$.

$$
\begin{array}{ccccccc}
\widetilde{F} & & R & \xrightarrow{f_R} & \widehat{R} & \qquad \widetilde{F} \otimes_R \widehat{R} & \qquad P \\
 & & \Big\downarrow{\scriptstyle \pi_{\mathbf{x}}} & & \Big\downarrow{\scriptstyle \widehat{\pi_{\mathbf{x}}}} & & \\
 & & R/(\mathbf{x}) & \xrightarrow{\cong} & \widehat{R}/(\mathbf{x}) & \qquad \widetilde{F} \otimes_R \widehat{R}/(\mathbf{x}) & \xrightarrow{\simeq} P \otimes_{\widehat{R}} \widehat{R}/(\mathbf{x})
\end{array}
$$

For $c \in \mathbb{Z}$ consider the hard truncation $P_{\leq c}$. The approximation property guarantees the existence of a bounded complex of free R-modules F and an isomorphism $F \otimes_R \widehat{R}/(\mathbf{x}) \xrightarrow{\cong} P_{\leq c} \otimes_{\widehat{R}} \widehat{R}/(\mathbf{x})$.

$$
\begin{array}{ccccccc}
F & & R & \xrightarrow{f_R} & \widehat{R} & \qquad F \otimes_R \widehat{R} & \qquad P_{\leq c} \\
& & \downarrow{\scriptstyle \pi_{\mathbf{x}}} & & \downarrow{\scriptstyle \widehat{\pi_{\mathbf{x}}}} & & \\
& & R/(\mathbf{x}) & \xrightarrow{\cong} & \widehat{R}/(\mathbf{x}) & \qquad F \otimes_R \widehat{R}/(\mathbf{x}) \xrightarrow{\cong} & P_{\leq c} \otimes_{\widehat{R}} \widehat{R}/(\mathbf{x})
\end{array}
$$

Let $\widetilde{F}$ be the complex of free R-modules obtained by splicing F with a R-free resolution of $\mathrm{Ker}(\partial_c^F)$.

$$
\begin{array}{ccccccccc}
 & & & & 0 & \longrightarrow & F_c & \xrightarrow{\partial_c^F} & F_{c-1} \xrightarrow{\partial_{c-1}^F} \cdots \\
 & & & & & \nearrow & \uparrow & & \\
\cdots \xrightarrow{\partial_{c+3}^{\widetilde{F}}} & \widetilde{F}_{c+2} & \xrightarrow{\partial_{c+2}^{\widetilde{F}}} & \widetilde{F}_{c+1} & \longrightarrow & & \mathrm{Ker}(\partial_c^F) & \longrightarrow & 0
\end{array}
$$

Choosing $c \gg 0$ we have

$$F \otimes_R \widehat{R} \to \widetilde{F} \otimes_R \widehat{R}$$

$$F \to \widetilde{F}$$

$$\begin{array}{ccc} R & \xrightarrow{f_R} & \widehat{R} \\ \downarrow{\scriptstyle \pi_{\mathbf{x}}} & & \downarrow{\scriptstyle \widehat{\pi_{\mathbf{x}}}} \\ R/(\mathbf{x}) & \xrightarrow{\cong} & \widehat{R}/(\mathbf{x}) \end{array}$$

$$P_{\leq c} \to P$$

$$\begin{array}{ccc} F \otimes_R \widehat{R}/(\mathbf{x}) & \longrightarrow & \widetilde{F} \otimes_R \widehat{R}/(\mathbf{x}) \\ \downarrow{\scriptstyle \simeq} & & \vdots{\scriptstyle \simeq} \\ P_{\leq c} \otimes_{\widehat{R}} \widehat{R}/(\mathbf{x}) & \longrightarrow & P \otimes_{\widehat{R}} \widehat{R}/(\mathbf{x}) \end{array}$$

Case 2. General case. In place of $\widehat{R}/(\mathbf{x})$, use a Koszul complex with the DGA structure given by the wedge product.

Let $\mathbf{x}$ be a generating sequence for $\mathfrak{m}_R$. It suffices to construct a homologically finite complex of free R-modules $\widetilde{F}$ such that $\widetilde{F}_i = 0$ for $i \ll 0$ and and there exists a $K^{\widehat{R}}(\mathbf{x})$-quasiisomorphism $\widetilde{F} \otimes_R K^{\widehat{R}}(\mathbf{x}) \xrightarrow{\simeq} P \otimes_{\widehat{R}} K^{\widehat{R}}(\mathbf{x})$.

$$\widetilde{F} \qquad \begin{array}{ccc} R & \xrightarrow{f_R} & \widehat{R} \\ \downarrow & & \downarrow \\ K^R(\mathbf{x}) & \xrightarrow{\simeq} & K^{\widehat{R}}(\mathbf{x}) \end{array} \qquad \begin{array}{ccc} \widetilde{F} \otimes_R \widehat{R} & & P \\ & & \\ \widetilde{F} \otimes_R K^{\widehat{R}}(\mathbf{x}) & \xrightarrow{\simeq} & P \otimes_{\widehat{R}} K^{\widehat{R}}(\mathbf{x}) \end{array}$$

For $c \in \mathbb{Z}$ the approximation property guarantees the existence of a bounded complex of free R-modules F and a $K^{\widehat{R}}(\mathbf{x})$-isomorphism $F \otimes_R K^{\widehat{R}}(\mathbf{x}) \xrightarrow{\cong} P_{\leq c} \otimes_{\widehat{R}} K^{\widehat{R}}(\mathbf{x})$.

$$\begin{array}{ccccccc}
F & & R & \xrightarrow{f_R} & \widehat{R} & & F \otimes_R \widehat{R} \qquad P_{\leq c} \\
 & & \downarrow & & \downarrow & & \\
 & & K^R(\mathbf{x}) & \xrightarrow{\simeq} & K^{\widehat{R}}(\mathbf{x}) & & F \otimes_R K^{\widehat{R}}(\mathbf{x}) \xrightarrow{\cong} P_{\leq c} \otimes_{\widehat{R}} K^{\widehat{R}}(\mathbf{x})
\end{array}$$

Let $\widetilde{F}$ be the complex of free R-modules obtained by splicing F with a R-free resolution of $\operatorname{Ker}(\partial_c^F)$.

$$\begin{array}{ccccccccc}
 & & & & 0 & \longrightarrow & F_c & \xrightarrow{\partial_c^F} & F_{c-1} \xrightarrow{\partial_{c-1}^F} \cdots \\
 & & & & & \nearrow & \uparrow & & \\
\cdots & \xrightarrow{\partial_{c+3}^{\widetilde{F}}} & \widetilde{F}_{c+2} & \xrightarrow{\partial_{c+2}^{\widetilde{F}}} & \widetilde{F}_{c+1} & \longrightarrow & \operatorname{Ker}(\partial_c^F) & \longrightarrow & 0
\end{array}$$

Choosing $c \gg 0$ we have

$$F \otimes_R \widehat{R} \to \widetilde{F} \otimes_R \widehat{R}$$

$$F \to \widetilde{F}$$

$$\begin{array}{ccc} R & \xrightarrow{f_R} & \widehat{R} \\ \downarrow & & \downarrow \\ K^R(\mathbf{x}) & \xrightarrow{\simeq} & K^{\widehat{R}}(\mathbf{x}) \end{array}$$

$$P_{\leq c} \to P$$

$$\begin{array}{ccc} F \otimes_R K^{\widehat{R}}(\mathbf{x}) & \longrightarrow & \widetilde{F} \otimes_R K^{\widehat{R}}(\mathbf{x}) \\ \downarrow \simeq & & \simeq \vdots \\ P_{\leq c} \otimes_{\widehat{R}} K^{\widehat{R}}(\mathbf{x}) & \longrightarrow & P \otimes_{\widehat{R}} K^{\widehat{R}}(\mathbf{x}) \end{array}$$

$\square$

Corollary. *Let R have the approximation property.*

(a) *For an ideal $I \subseteq \mathfrak{m}_R$ let $\widehat{R}^I$ be the I-adic completion of R with $f_R^I \colon R \to \widehat{R}^I$. Then $\mathfrak{S}(f_R^I) \colon \mathfrak{S}(R) \to \mathfrak{S}(\widehat{R}^I)$ is surjective.*

(b) *For a R-sequence $\mathbf{x} \in \mathfrak{m}_R$ the map $\mathfrak{S}(\pi_{\mathbf{x}}) \colon \mathfrak{S}(R) \to \mathfrak{S}(R/(\mathbf{x}))$ is surjective.*

Proof. Diagram chase: each map is injective.

$$
\begin{array}{ccc}
\mathfrak{S}(R) & \longrightarrow & \mathfrak{S}(\widehat{R}^I) \\
& {\scriptstyle\cong}\searrow & \downarrow \\
& & \mathfrak{S}(\widehat{R})
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathfrak{S}(R) & \xrightarrow{\ \cong\ } & \mathfrak{S}(\widehat{R}) \\
\downarrow & & \downarrow{\scriptstyle\cong} \\
\mathfrak{S}(R/(\mathbf{x})) & \longrightarrow & \mathfrak{S}(\widehat{R}/(\mathbf{x}))
\end{array}
$$

$\square$

Corollary. *Let R be an excellent $\mathbb{Q}$-algebra.*

(a) *The map $\mathfrak{S}(R^h) \to \mathfrak{S}(\widehat{R})$ is surjective.*

(b) *If $\mathfrak{S}(\widehat{R})$ is finite, then there exists a pointed étale neighborhood $R \to R'$ such that the map $\mathfrak{S}(R') \to \mathfrak{S}(\widehat{R})$ is surjective.*

Proof. (a) R^h has the approximation property.

(b) If $\mathfrak{S}(\widehat{R})$ is finite, then only finitely many equations need to be solved to descend every element of $\mathfrak{S}(\widehat{R})$ to $\mathfrak{S}(R^h)$. The solutions to these equations live in a pointed étale neighborhood R' of R, so each element of $\mathfrak{S}(\widehat{R})$ descends to $\mathfrak{S}(R')$. □