Incomparability of semidualizing modules

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Assumption. R is a Cohen-Macaulay local ring.

Much research in commutative algebra is devoted to duality.

Examples. Investigate $\mathrm{Hom}_R(-, R)$.

(Grothendieck-Hartshorne) Investigate $\mathrm{Hom}_R(-, D)$ when D is dualizing (canonical) module for R.

(Foxby-Golod) Investigate $\mathrm{Hom}_R(-, C)$ when C is semidualizing.

Definition. A finitely generated R-module C is *semidualizing* if

(a) The natural homothety homomorphism

$$\chi^R_C\colon R \to \mathrm{Hom}_R(C, C)$$

is an isomorphism, and

(b) $\mathrm{Ext}^i_R(C, C) = 0$ for each $i \neq 0$.

Notation. The set of isomorphism classes of semidualizing R-modules is denoted $\mathfrak{S}_0(R)$ with $[C] \in \mathfrak{S}_0(R)$.

Example. R is semidualizing.

Fact. D is a dualizing module if and only if it is semidualizing and $\mathrm{id}_R(D)$ is finite.

Fact. When R admits a dualizing module, R is Gorenstein if and only if $\mathfrak{S}_0(R) = \{[R]\}$.

Example. (Foxby) Let $\varphi\colon Q \to R$ be a finite local homomorphism with $\mathrm{pd}_Q(R) < \infty$ and D_Q a dualizing module for Q. The following R-modules are semidualizing where $d = \operatorname{depth} Q - \operatorname{depth} R$.

$$D_R \cong \mathrm{Ext}^d_Q(R, D_Q) \qquad\qquad D_Q \otimes_Q R$$

$$D_\varphi := \mathrm{Ext}^d_Q(R, Q) \qquad\qquad R \cong Q \otimes_Q R$$

These modules can be pairwise nonisomorphic.

General motivation. Increase the understanding of $\mathfrak{S}_0(R)$.

Definition. If C, M are finitely generated R-modules with C semidualizing, then M is *totally C-reflexive* if

(a) The natural biduality homomorphism

$$\delta_M^C \colon M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)$$

is an isomorphism, and

(b) $\operatorname{Ext}_R^i(M, C) = 0 = \operatorname{Ext}_R^i(\operatorname{Hom}_R(M, C), C)$ for each $i \neq 0$.

Example. R is totally C-reflexive.

Example. When D is dualizing, the nonzero totally D-reflexive modules are the maximal Cohen-Macaulay modules.

Example. The nonzero totally R-reflexive modules are the modules of G-dimension 0.

Notation. For $[C], [C'] \in \mathfrak{S}(R)$ write $[C] \trianglelefteq [C']$ whenever C' is totally C-reflexive.

Facts. Fix $[C], [C'] \in \mathfrak{S}_0(R)$.

(a) $[C] = [C']$ if and only if $[C] \trianglelefteq [C']$ and $[C'] \trianglelefteq [C]$.

(c) $[C] \trianglelefteq [R]$.

(d) If $[C] \trianglelefteq [C']$, then $\mathrm{Hom}_R(C', C)$ is semidualizing and $[C] \trianglelefteq [\mathrm{Hom}_R(C', C)]$.

Assuming that D is dualizing:

(e) $[D] \trianglelefteq [C]$. In particular, $\mathrm{Hom}_R(\mathrm{Hom}_R(C, D), D) \cong C$, and $\mathrm{Hom}_R(C, D)$ is semidualizing, e.g., $\mathrm{Hom}_R(R, D) \cong D$ and $\mathrm{Hom}_R(D, D) \cong R$.

(g) $[C] \trianglelefteq [C']$ if and only if $[\mathrm{Hom}_R(C', D)] \trianglelefteq [\mathrm{Hom}_R(C, D)]$.

Question. Is the ordering transitive?

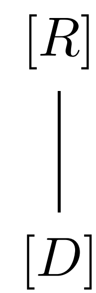
Construction. Let $\Gamma(R)$ be the graph with

Vertex set: $\mathfrak{S}_0(R)$

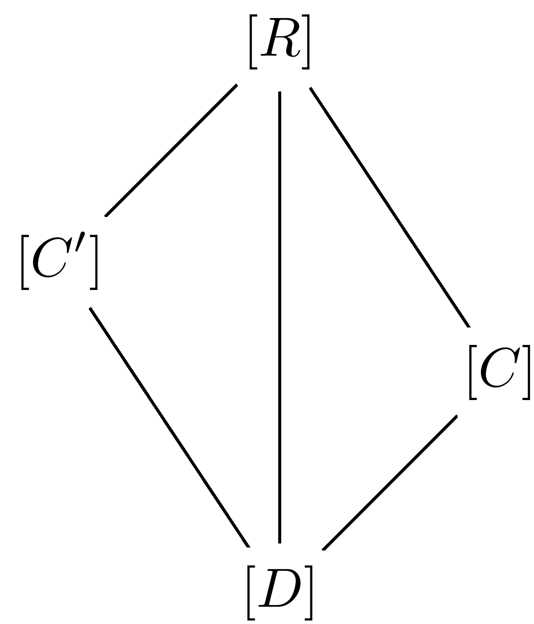
Edge set: $[C] - [C']$ whenever $[C] \trianglelefteq [C']$ or $[C'] \trianglelefteq [C]$.

Examples. $\operatorname{card}(\mathfrak{S}_0(R)) = 1$ $\quad [R]$

$\operatorname{card}(\mathfrak{S}_0(R)) = 2$



$\operatorname{card}(\mathfrak{S}_0(R)) = 4$



Note. $\mathsf{\Gamma}(R)$ is connected since there is a path $[C] - [R] - [C']$.

Question. What about the case $\mathrm{card}(\mathfrak{S}_0(R)) = 3$?

Question. Does $\mathsf{\Gamma}(R)$ always have a nontrivial symmetry?

Question. When is $\mathsf{\Gamma}(R)$ complete?

Fact. If D is dualizing for R, then the functor $\mathrm{Hom}_R(-, D)$ induces an involution of $\mathsf{\Gamma}(R)$; hence the symmetry in this case.

Construction, cont. When $[C] \trianglelefteq [C']$ assign the corresponding edge of $\mathsf{\Gamma}(R)$ the weight (or length)

$$\delta([C], [C']) = \mathrm{curv}_R(\mathrm{Hom}_R(C', C)) = \limsup_{n\to\infty} \sqrt[n]{\beta_n^R(\mathrm{Hom}_R(C', C))}.$$

The distance $\mathrm{dist}_R([C], [C'])$ between general $[C], [C'] \in \mathfrak{S}_0(R)$ is the infimum of the lengths of the paths from $[C]$ to $[C']$.

Theorem. $\mathfrak{S}_0(R)$ *is a metric space.*

Note. The metric space topology on $\mathfrak{S}_0(R)$ is trivial.

Question. When does $\mathfrak{S}_0(R)$ admit nontrivial open balls? That is, when does there exists $[C] \in \mathfrak{S}(R)$ and $\delta > 0$ such that the open ball $B([C], \delta)$ satisfies $\{[C]\} \subsetneq B([C], \delta) \subsetneq \mathfrak{S}(R)$?

Fact. If D is dualizing for R, then $\operatorname{Hom}_R(-, D)$ induces an isometric involution of $\mathfrak{S}_0(R)$.

Question. What happens if this involution fixes a point of $\mathfrak{S}_0(R)$?

Theorem. *If D is dualizing for R, then TFAE.*
(i) There exists $[C] \in \mathfrak{S}_0(R)$ such that $[C] = [\operatorname{Hom}_R(C, D)]$;
(ii) R is Gorenstein;
(iii) $\mathfrak{S}_0(R)$ is finite with odd cardinality.

Note. (ii) $\iff$ (iii) extends an earlier fact: When R admits a dualizing module, R is Gorenstein if and only if $\mathfrak{S}_0(R) = \{[R]\}$.

Question. Must $\operatorname{card}(\mathfrak{S}_0(R))$ be finite?

Question. Must $\operatorname{card}(\mathfrak{S}_0(R))$ be even? a power of 2?

Theorem. *The following conditions on a ring R are equivalent:*
(i) *There exist elements of $\mathfrak{S}_0(R)$ that are not comparable;*
(ii) $\mathfrak{S}_0(R)$ *has cardinality at least 3;*
(iii) $\mathfrak{S}_0(R)$ *admits a nontrivial open ball.*

Theorem. *If $[C] \lhd [C'] \lhd [R]$, then $[C']$ and $[\mathrm{Hom}_R(C', C)]$ are not comparable in the ordering on $\mathfrak{S}_0(R)$.*

Theorem. *Let $X_1, \ldots, X_m$ homologically bounded below and degreewise finite complexes with $m \geq 1$. If the complex $X_1 \otimes^{\mathbf{L}}_R X_1 \otimes^{\mathbf{L}}_R X_2 \otimes^{\mathbf{L}}_R \cdots \otimes^{\mathbf{L}}_R X_m$ is semidualizing, then X_1 is shift-isomorphic to R in the derived category $\mathsf{D}(R)$.*

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