Multiplicities and Localization

(preliminary report)

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Abstract: Given a Noetherian local ring $(R, \mathfrak{m})$ with R-regular sequence $\underline{x} = x_1, \ldots, x_k \in \mathfrak{m}$, define the *multiplicity* of $\underline{x}$ as $m_R(\underline{x}) = e(R/(\underline{x})R)$ where $e(A)$ is the Samuel multiplicity of the local ring A. If $\mathfrak{p}$ is a prime ideal of R such that $\underline{x} \in \mathfrak{p}$, we consider the inequality $m_{R_\mathfrak{p}}(\underline{x}) \leq m_R(\underline{x})$ and a question of P. C. Roberts involving Serre's intersection multiplicity.

I. Background and Introduction

Assume that $(R, \mathfrak{m})$ is a regular local ring with ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$. Under these assumptions, Serre defined the *intersection multiplicity* of $R/\mathfrak{p}$ and $R/\mathfrak{q}$ as

$$\chi(R/\mathfrak{p}, R/\mathfrak{q}) = \sum_{i=0}^{\dim(R)} (-1)^i \operatorname{len}(\operatorname{Tor}_i^R(R/\mathfrak{p}, R/\mathfrak{q}))$$

and proved that

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R).$$

Furthermore, he conjectured:

(Vanishing) If $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) < \dim(R)$, then $\chi(R/\mathfrak{p}, R/\mathfrak{q}) = 0$.

(Nonnegativity) $\chi(R/\mathfrak{p}, R/\mathfrak{q}) \geq 0$.

(Positivity) If $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$, then $\chi(R/\mathfrak{p}, R/\mathfrak{q}) > 0$.

Recently, K. Kurano and P. Roberts proved the following theorem.

Theorem 1. *Assume that* $(R, \mathfrak{m})$ *is a regular local ring which either contains a field or is ramified. Also, assume that* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are prime ideals in* R *such that* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ *and* $\operatorname{ht}\mathfrak{p} + \operatorname{ht}\mathfrak{q} = \dim R$. *If* $\chi(R/\mathfrak{p}, R/\mathfrak{q}) > 0$ *then*

$$\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1} \text{ for all } n \geq 0 \qquad (1)$$

As a result, they conjectured that (1) should hold in all regular local rings. More specifically,

Conjecture 2. *(Kurano-Roberts) Assume that* $(R, \mathfrak{m})$ *is a regular local ring and that* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are prime ideals in* R *such that* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ *and* $\operatorname{ht}\mathfrak{p} + \operatorname{ht}\mathfrak{q} = \dim R$. *Then* $\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$, *for all* $n \geq 0$.

As positivity is known when R contains a field, this conjecture holds for equicharacteristic regular local rings. Kurano-Roberts asked whether there was a proof in this case which does not use positivity.

Definition 3. Assume that $(R, \mathfrak{m})$ is a Noetherian local ring. Given a finitely generated R-module $M \neq 0$ and a proper ideal $\mathfrak{a}$ of R such that $M/\mathfrak{a}M$ has finite length, the *Hilbert polynomial* of $\mathfrak{a}$ on M, denoted $H[\mathfrak{a}, M](n)$, is the polynomial in n of degree $r = \dim(M)$ with rational coefficients such that for $n \gg 0$

$$H[\mathfrak{a}, M](n) = \mathrm{len}(M/\mathfrak{a}^{n+1}M).$$

If e_r is the leading coefficient of $H[\mathfrak{a}, M](n)$, then the *multiplicity* of $\mathfrak{a}$ on M is $e(\mathfrak{a}, M) = r!e_r$. We denote $e(\mathfrak{m}, M)$ by $e(M)$. Given an R-regular sequence $x_1, \ldots, x_k \in \mathfrak{m}$, define the *multiplicity* of $\mathbf{x} = x_1, \ldots, x_k$ as

$$m_R(\mathbf{x}) = e(R/(\mathbf{x})R).$$

If R is regular and $k = 1$, then $m_R(x)$ is exactly the degree of x with respect to $\mathfrak{m}$. If $x \in \mathfrak{p}$ for some prime ideal then a classical result implies that $\mathfrak{p}^{(n)} \subseteq \mathfrak{m}^n$ so that $m_{R_\mathfrak{p}}(x) \le m_R(x)$. This motivates the following.

Conjecture 4. *(Roberts) Assume that $(R, \mathfrak{m})$ is a regular local ring with prime ideal $\mathfrak{p}$ and R-regular sequence $\mathbf{x} = x_1, \ldots, x_k \in \mathfrak{p}$. Then*

$$m_{R_\mathfrak{p}}(\mathbf{x}) \le m_R(\mathbf{x}).$$

Example 5. Let

$$R = k[\![X, Y, Z, W]\!]/(XY - ZW)$$

and $\mathfrak{p} = (X, Z)R$. Then

$$R/ZR = k[\![X, Y, W]\!]/(XY)$$

so that $m_{R_\mathfrak{p}}(Z) = 1$ and $m_R(Z) = 2$. In particular, the assumption that R is regular is crucial.

Using multiplicities, Conjecture 2 can be generalized to regular sequences of length k.

Conjecture 6. *(Roberts) Assume that* $(R, \mathfrak{m})$ *is a regular local ring with prime ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$ *such that* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$. *Assume that* $x_1, \ldots, x_k \in \mathfrak{p} \cap \mathfrak{q}$ *is an* R*-regular sequence such that* $m_{R_\mathfrak{p}}(\mathbf{x}) = m_R(\mathbf{x})$. *Then*

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R) - k.$$

This is the key reinterpretation which allows us to apply other tools to Conjecture 2.

Example 7. Let $R = k[\![X, Y, Z, W]\!]$, with $\mathfrak{p} = (X, Z)$ and $\mathfrak{q} = (Y, W)$. Then $m_R(XY - ZW) = 2$, $m_{R_\mathfrak{p}}(XY - ZW) = 1$ and

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 4 > \dim(R) - 1.$$

Thus, the requirement that $m_{R_\mathfrak{p}}(\mathbf{x}) = m_R(\mathbf{x})$ is necessary.

Note that regularity may not be necessary, and we may be able to replace the regular sequence $\mathbf{x}$ with an arbitrary ideal $\mathfrak{a}$, as will be shown below.

II. Conjecture 4 for Excellent Rings

Definition 8. A Noetherian local ring $(A, \mathfrak{m})$ is *equidimensional* if $\dim(R/\mathfrak{p}) = \dim(R)$ for every minimal prime $\mathfrak{p}$ of A. A is *quasi-unmixed* if its completion is equidimensional.

Theorem 9. *Assume that $(A, \mathfrak{m})$ is an equidimensional, excellent local ring with prime ideal $\mathfrak{p}$. Then $e(R_{\mathfrak{p}}) \leq e(R)$.*

Note. The result remains true if we replace "excellent" by "pseudo-geometric".

Corollary 10. *Assume that* $(R, \mathfrak{m})$ *is a pseudo-geometric, Cohen-Macaulay local ring with prime ideal* $\mathfrak{p}$ *and regular sequence* $\mathbf{x} \in \mathfrak{p}$*. Then* $m_{R_\mathfrak{p}}(\mathbf{x}) \leq m_R(\mathbf{x})$*.*

Example 11. Let $R = k[\![X]\!] \times_k k[\![Z]\!]/(Z^n)$ where $n > 1$. Then R is local with maximal ideal $\mathfrak{m} = ((X, 0), (0, Z))$. If $\mathfrak{p} = ((0, Z))$, then $e(R_\mathfrak{p}) = n$ and $e(R) = 1$. Thus, the hypothesis that A is equidimensional is crucial for Theorem 9.

III. Conjecture 6 when $R/\mathfrak{p}$ is Regular

Notation. For an ideal I in a local ring A, let $s(I)$ denote the analytic spread of I. If A has infinite residue field, then $s(I)$ is the smallest number i such that I has a reduction ideal generated by i elements. Recall that $\operatorname{ht}(I) \leq s(I) \leq \dim(A)$.

Lemma 12. *Let R be a quasi-unmixed local ring and let $\mathfrak{p}$ be a prime ideal of R for which $R/\mathfrak{p}$ is regular. Then the following conditions are equivalent.*
(i) $e(R) = e(R_\mathfrak{p})$ *(ii)* $\operatorname{ht}(\mathfrak{p}) = s(\mathfrak{p})$

Example 13. Let $R = k[\![X_1, \ldots, X_6]\!]$ and $\mathfrak{p} = I_2 \begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{pmatrix}$. Since R is regular, $e(R) = 1 = e(R_\mathfrak{p})$. However, $\operatorname{ht}(\mathfrak{p}) = 2$ and $s(\mathfrak{p}) = 3$. Thus, the requirement that $R/\mathfrak{p}$ be regular is necessary.

Theorem 14. *Assume that $(A, \mathfrak{m})$ is an equidimensional, catenary Noetherian local ring of dimension d with infinite residue field and prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $A/\mathfrak{p}$ is regular. Furthermore, assume that $\mathfrak{p} \cap \mathfrak{q}$ contains an unmixed ideal $\mathfrak{a}$ of height k such that $A/\mathfrak{a}$ is quasi-unmixed and $e(A_\mathfrak{p}/\mathfrak{a}_\mathfrak{p}) = e(A/\mathfrak{a})$. Then $\dim(A/\mathfrak{p}) + \dim(A/\mathfrak{q}) \leq d - k$.*

Sketch of proof. By replacing A by the quotient $A/\mathfrak{a}$, we may assume that $\mathfrak{a} = (0)$, A is unmixed and quasi-unmixed, and $e(A_{\mathfrak{p}}) = e(A)$; thus, we need to show that $\dim(A/\mathfrak{p}) + \dim(A/\mathfrak{q}) \leq d$. By Lemma 12, $\mathfrak{p}$ contains a sequence $y_1, \ldots, y_i$ which generate a minimal reduction of $\mathfrak{p}$ and $i = \mathrm{ht}\,(\mathfrak{p})$. Since $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$, we see that $\mathfrak{q}$ is an ideal of definition for $A/\mathfrak{p}$ and therefore $\mathfrak{q}$ contains a system of parameters $z_1, \ldots, z_j$ for $A/\mathfrak{p}$. In particular $j = \dim(A/\mathfrak{p})$. It is easily verified that $y_1, \ldots, y_i, z_1, \ldots, z_j$ is a system of parameters for A. Furthermore, since $\sqrt{(\mathbf{y})A} = \mathfrak{p}$, in the ring $A/\mathfrak{q}$ the images of $\mathbf{y}$ generate an ideal which is primary to $\mathfrak{m}/\mathfrak{q}$. Thus, $i \geq \dim(A/\mathfrak{q})$, so that

$$\dim(A) = i + j \geq \dim(A/\mathfrak{q}) + \dim(A/\mathfrak{p}).$$

QED.

Corollary 15. *Assume that* R *is a regular local ring of dimension* d *with infinite residue field and prime ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$ *such that* $A/\mathfrak{p}$ *is regular. Furthermore, assume that* $\mathfrak{p} \cap \mathfrak{q}$ *contains a regular sequence* $\mathbf{x}$ *of length* k *such that* $m_{R_\mathfrak{p}}(\mathbf{x}) = m_R(\mathbf{x})$*. Then*

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R) - k.$$

Proof. The ring $A = R/(\mathbf{x})R$ satisfies the hypotheses of Theorem 14. QED.