Some Questions About Ext

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Large and Small Support

Assumption

R is a commutative noetherian ring, and M, N are R-modules.

Definition

Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and set $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

(a) $\mathfrak{p} \in \operatorname{Supp}_R(M)$ if $M_{\mathfrak{p}} \neq 0$, i.e., $R_{\mathfrak{p}} \otimes_R M \neq 0$, i.e., $\operatorname{Tor}^R(R_{\mathfrak{p}}, M) \neq 0$

(b) (Foxby '79) $\mathfrak{p} \in \operatorname{supp}_R(M)$ if $\operatorname{Tor}^R(\kappa(\mathfrak{p}), M) \neq 0$

Fact

- $\operatorname{supp}_R(M) \subseteq \operatorname{Supp}_R(M)$
- $M \neq 0$ iff $\operatorname{supp}_R(M) \neq \emptyset$ iff $\operatorname{Supp}_R(M) \neq \emptyset$
- $\operatorname{Tor}^R(M, N) \neq 0$ if and only if $\operatorname{supp}_R(M) \cap \operatorname{supp}_R(N) \neq \emptyset$
- If M is finitely generated, then $\operatorname{supp}_R(M) = \operatorname{Supp}_R(M)$.

Large and Small Co-Support

Definition

(a) $\mathfrak{p} \in \text{Co-supp}_R(M)$ if $\text{Ext}_R(R_\mathfrak{p}, M) \neq 0$

(b) (Benson-Iyengar-Krause '12) $\mathfrak{p} \in \text{co-supp}_R(M)$ if $\text{Ext}_R(\kappa(\mathfrak{p}), M) \neq 0$

Fact

- $\text{co-supp}_R(M) \subseteq \text{Co-supp}_R(M)$
- $M \neq 0$ iff $\text{co-supp}_R(M) \neq \emptyset$ iff $\text{Co-supp}_R(M) \neq \emptyset$
- $\text{Ext}_R(M, N) \neq 0$ if and only if $\text{supp}_R(M) \cap \text{co-supp}_R(N) \neq \emptyset$
- If M is finitely generated, then $\text{co-supp}_R(M) \subseteq \text{supp}_R(M)$.

Example

Assume that $(R, \mathfrak{m}, k)$ is local and not artinian. For $E = E_R(k)$:

$$\text{co-supp}_R(E) = \text{Spec}(R) \not\subseteq \{\mathfrak{m}\} = \text{supp}_R(E).$$

The First Question

Question

- If M is finitely generated, what is $\text{co-supp}_R(M)$?
- What is $\text{co-supp}_R(R)$?

Fact (BIK '12)

Assume that M is finitely generated. Let $\mathfrak{a} \subset R$ be an ideal.

- $\text{co-supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$ if and only if M is $\mathfrak{a}$-adically complete.
- If $(R, \mathfrak{m})$ is local, then $\text{co-supp}_R(R) = \{\mathfrak{m}\}$ if and only if R is complete.
- If R is a 1-dimensional local domain, then

$$\text{co-supp}_R(R) = \begin{cases} \{\mathfrak{m}\} & \text{if } R \text{ is complete} \\ \mathrm{Spec}(R) & \text{otherwise.} \end{cases}$$

- $\text{co-supp}_R(M)$ and $\text{supp}_R(M)$ have the same maximal elements with respect to containment.

Results

Theorem (SW-Wicklein '14)

If R is a 1-dimensional domain with a dualizing complex, then

$$\text{co-supp}_R(R) = \begin{cases} \text{m-Spec}(R) & \textit{if } R \textit{ is local and complete} \\ \text{Spec}(R) & \textit{otherwise.} \end{cases}$$

Proof.

If R is local, we are done.
Assume that R is not local, and let D be dualizing.
We know that $\text{m-Spec}(R) \subseteq \text{supp}_R(R)$.
Maximum condition implies that $\text{m-Spec}(R) \subseteq \text{co-supp}_R(R)$.
Need to show that $0 \in \text{co-supp}_R(R)$.

$D : 0 \to Q \to \oplus_{\mathfrak{m}} E(R/\mathfrak{m}) \to 0$

$$\text{Hom}_R(Q, D) : 0 \to \underbrace{\text{Hom}_R(Q, Q)}_{\cong Q} \to \underbrace{\text{Hom}_R(Q, \oplus_{\mathfrak{m}} E(R/\mathfrak{m}))}_{\text{rank}_Q \geq 2} \to 0$$

So, $\text{Ext}^1_R(Q, D) = 0$ and $0 \in \text{co-supp}_R(D) = \text{co-supp}_R(R)$. □

Results, cont.

Theorem (SW-Wicklein '14)

Let $\mathfrak{p} \in \mathrm{Spec}(R)$ *such that* $\dim(R/\mathfrak{p}) = 1$. *If* R *has a dualizing complex, then* $\mathfrak{p} \in \text{co-supp}_R(R)$ *if and only if* $R/\mathfrak{p}$ *is not local or not complete.*

Example

Let k be a field and let R be a localization of the polynomial $k[X_1, \ldots, X_d]$. Given $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\dim(R/\mathfrak{p}) = 1$, we have $\mathfrak{p} \in \text{co-supp}_R(R)$ if and only if $R/\mathfrak{p}$ is not local.

Question

Let k be a field and set $R = k[X, Y]$ or $R = k[X, Y]_{(X,Y)}$. Do we have $0 \in \text{co-supp}_R(R)$? i.e., do we have $\mathrm{Ext}_R(Q(R), R) \neq 0$?

The Second Question

Assumption

$(R, \mathfrak{m})$ is local.

Fact (Frankild, SW, R. Wiegand; '08)

TFAE:

(i) *R is complete.*

(ii) $\mathrm{Ext}^i_R(\widehat{R}, R) = 0$ *for all $i \geq 1$.*

(iii) $\mathrm{Ext}^i_R(\widehat{R}, R)$ *is f.g. over $\widehat{R}$ for $i = 1, \ldots, \dim_R(R)$.*

Note

If R is a DVR, Anderson-SW explicitly describe $\mathrm{Ext}^1_R(\widehat{R}, R)$.

Question

For which i do we have $\mathrm{Ext}^i_R(\widehat{R}, R) \neq 0$?

A Result

Fact (Frankild, SW, R. Wiegand '08)

$\operatorname{Hom}_R(\widehat{R}, R) \neq 0$ *if and only if* R *has a non-zero complete ideal.*

Theorem (SW)

Assume that R *is an incomplete domain and that* $f, g \in \mathfrak{m}$.

(a) *If* R/fR *is complete, then* $\operatorname{Ext}^1_R(\widehat{R}, R) \neq 0$.

(b) *If* R/gR *has no non-zero complete ideals (e.g., if* R/gR *is an incomplete domain), then* $\operatorname{Ext}^1_R(\widehat{R}, R)$ *has no* g*-torsion.*

Example

$R = k[X]_{(X)}[\![\underline{Y}]\!]$ or $k[\![\underline{Y}]\!][X]_{(X,\underline{Y})}$ with $f = X$ and $g = g(\underline{Y})$.

Question

Let $R = k[X, Y]_{(X,Y)}$, so $\widehat{R} = k[\![X, Y]\!]$.

For which $i = 1, 2$ do we have $\operatorname{Ext}^i_R(\widehat{R}, R) \neq 0$?

Conclusion

Question

(a) For which $\mathfrak{p}$ do we have $\operatorname{Ext}_R(\kappa(\mathfrak{p}), R) \neq 0$?

(b) For which i do we have $\operatorname{Ext}^i_R(\widehat{R}, R) \neq 0$?

Intuition

Each question has to do with how far R is from being complete:

(a) the smallest $\mathrm{V}(\mathfrak{a})$ such that R is $\mathfrak{a}$-adically complete,

(b) the lengths of sequences $\underline{f} = f_1, \ldots, f_n$ such that $R/(\underline{f})$ is complete.

Question

Let k be a field and set $R = k[X, Y]_{(X,Y)}$.

(a) Do we have $\operatorname{Ext}_R(Q(R), R) \neq 0$?

(b) For which i do we have $\operatorname{Ext}^i_R(\widehat{R}, R) \neq 0$?