Derived Categories in Commutative Algebra

Olgur Celikbas[1] Sean Sather-Wagstaff[2]

[1]University of Connecticut

[2]North Dakota State University

20 September 2014
Directions in Commutative Algebra: Past, Present, Future
(dedicated to the memory of H.-B. Foxby)
AMS Central Section Meeting
University of Wisconsin-Eau Claire

Outline

G-dimension

Assumption

$(R, \mathfrak{m}, k)$ is a commutative noetherian local ring.

Definition (Auslander-Bridger '67)

(a) A finitely generated R-module G is totally reflexive if $G \cong \operatorname{Hom}_R(\operatorname{Hom}_R(G, R), R)$ and $\operatorname{Ext}_R^{\geqslant 1}(G, R) = 0 = \operatorname{Ext}_R^{\geqslant 1}(\operatorname{Hom}_R(G, R), R)$.

(b) A finitely generated R-module has finite G-dimension if there is an exact sequence $0 \to G_n \to \cdots \to G_0 \to M \to 0$ such that each G_i is totally reflexive.

Fact (Auslander-Bridger '67)

If R is Gorenstein then every finitely generated R-module has finite G-dimension.

If $\text{G-dim}_R(\mathfrak{m}) < \infty$, then R is Gorenstein.

Integrally closed $\mathfrak{m}$-primary ideals

Assumption

I is an integrally closed $\mathfrak{m}$-primary ideal of R.

Fact (Burch '68)

If $\mathrm{pd}(I) < \infty$, then R is regular.

Fact (Goto-Hayasaka '02)

Assume I contains a non-zerodivisor of R, or R satisfies (S_1).
If $\mathrm{G\text{-}dim}_R(I) < \infty$, then R is Gorenstein.

Theorem (O. Celikbas-SSW)

If $\mathrm{G\text{-}dim}_R(I) < \infty$*, then R is Gorenstein.*

Test Modules

Assumption

Let H-dim be pd or G-dim.

Definition (O. Celikbas-Dao-Takahashi ’14, O. Celikbas-SSW)

A finitely generated R-module M is H-dim-test if for all finitely generated R-modules N one has $\operatorname{Tor}^R_{\gg 0}(M, N) = 0$ if and only if $\text{H-dim}_R(N) < \infty$.

Example

k is a pd-test module.

Fact

1. If $\widehat{M}$ is H-dim-test over $\widehat{R}$, then M is H-dim-test over R.
2. Assume that M is H-dim-test over R. If R is Gorenstein, then $\text{G-dim}_R(M) < \infty$.

Test Modules, cont

H-dim-test: $\operatorname{Tor}^R_{\gg 0}(M, N) = 0$ if and only if $\text{H-dim}_R(N) < \infty$.

Theorem (O. Celikbas-SSW)

1. *If M is* H-dim-*test over R, then $\widehat{M}$ is* H-dim-*test over $\widehat{R}$.*
2. *Assume that M is* H-dim-*test over R. If* $\text{G-dim}_R(M) < \infty$, *then R is Gorenstein.*

Note

- Our first theorem is an application of the second one: Corso, Huneke, Katz, Vasconcelos: I is pd-test.
- The proof of the second one uses the derived category.
- Special cases of part (2) by O. Celikbas-Dao-Takahashi and O. Celikbas-Gheibi-Sadeghi-Zargar.
- Jon Totushek will present another application.

Chain complexes: objects of $\mathcal{C}(R)$

Definition

- R-complexes are indexed homologically:

  $X = \cdots \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \xrightarrow{\partial^X_{i-1}} \cdots$ such that $\partial^X_{i-1}\partial^X_i = 0$.
- $\mathrm{H}_i(X) := \mathrm{Ker}(\partial^X_i)/\,\mathrm{Im}(\partial^X_{i+1})$.

Example

- R-modules M and N, or a Koszul complex K
- truncated projective resolution P or injective resolution I
- $\mathrm{Hom}_R(P, N)$ and $\mathrm{Hom}_R(M, I)$ and $\mathrm{Hom}_R(P, I)$
- $P \otimes_R N$ and $M \otimes_R Q$ and $P \otimes_R Q$

Problem

The operation $(M, N) \mapsto \mathrm{Hom}_R(P, N)$ is not well-defined.

Chain maps: morphisms in $\mathcal{C}(R)$

Definition

- chain map: $f \colon X \to Y$ makes the ladder diagram commute.
- induced map: $\mathrm{H}_i(f) \colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$.
- quasiisomorphism: $f \colon X \xrightarrow{\simeq} Y$ if $\mathrm{H}_i(f) \colon \mathrm{H}_i(X) \xrightarrow{\cong} \mathrm{H}_i(Y)$.

Example

- chain maps are the elements of $\mathrm{Ker}(\partial_0^{\mathrm{Hom}(X,Y)})$
- exact $\cdots \to P_1 \to P_0 \to M \to 0$ gives rise to $P \xrightarrow{\simeq} M$
- $\mathrm{Hom}_R(P,N) \xrightarrow{\simeq} \mathrm{Hom}_R(P,I) \xleftarrow{\simeq} \mathrm{Hom}_R(M,I)$
- $P \otimes_R N \xleftarrow{\simeq} P \otimes_R Q \xrightarrow{\simeq} M \otimes_R Q$

Problem

"There is a quasiisomorphism $X \xrightarrow{\simeq} Y$" is not symmetric.

Homotopic chain maps

Definition

- a chain map $f\colon X \to Y$ is null-homotopic ($f \sim 0$) if

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_{i+1} & \longrightarrow & X_i & \longrightarrow & X_{i-1} & \longrightarrow & \cdots \\
 & & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & & \\
\cdots & \longrightarrow & Y_{i+1} & \longrightarrow & Y_i & \longrightarrow & Y_{i-1} & \longrightarrow & \cdots
\end{array}$$

  i.e., $f \in \operatorname{Im}(\partial_1^{\operatorname{Hom}(X,Y)})$
- chain maps $f, g\colon X \to Y$ are homotopic ($f \sim g$) if $f - g \sim 0$

Fact (Lifting Lemma)

Let M and N be R-modules with projective resolutions $P \xrightarrow{\simeq} M$ and $Q \xrightarrow{\simeq} N$. Given a homomorphism $M \to N$ there is a lift $P \to Q$, and any two such lifts are homotopic.

The homotopy category $\mathcal{K}(R)$

Definition

- Objects of $\mathcal{K}(R)$: chain complexes
- Morphisms of $\mathcal{K}(R)$: homotopy classes of chain maps, i.e., $\mathrm{Mor}_{\mathcal{K}(R)}(X,Y) := \mathrm{Mor}_{\mathcal{C}(R)}(X,Y)/\sim\, = \mathrm{H}_0(\mathrm{Hom}_R(X,Y))$

Fact

- The functor $\mathcal{M}(R) \to \mathcal{K}(R)$ with $M \mapsto P$ is well-defined.
- The bifunctors $\mathcal{M}(R) \times \mathcal{M}(R) \to \mathcal{K}(R)$ with $(M,N) \mapsto \mathrm{Hom}_R(P,N)$ and $P \otimes_R N$ are well-defined.
- Similarly for injective versions, $(M,N) \mapsto \mathrm{Hom}_R(P,I)$, etc.

Problem

The complexes $\mathrm{Hom}_R(P,N)$ and $\mathrm{Hom}_R(P,I)$ and $\mathrm{Hom}_R(M,I)$ need not be isomorphic in $\mathcal{K}(R)$.

The derived category $\mathcal{D}(R)$

Definition (Grothendieck-Verdier '63)

- Objects of $\mathcal{D}(R)$: chain complexes
- Morphisms of $\mathcal{D}(R)$: localize $\mathrm{Mor}_{\mathcal{K}(R)}(X, Y)$ by formally inverting the quasiisomorphisms

Fact

- Quasiisomorphisms in $\mathcal{C}(R)$ give isomorphisms in $\mathcal{D}(R)$
- We have bifunctors $\mathcal{M}(R) \times \mathcal{M}(R) \to \mathcal{D}(R)$ with

$$\mathbf{R}\mathrm{Hom}_R(M, N) = \mathrm{Hom}_R(P, N) \simeq \mathrm{Hom}_R(P, I) \simeq \mathrm{Hom}_R(M, I)$$

$$M \otimes^{\mathbf{L}}_R N = P \otimes_R N \simeq P \otimes_R Q \simeq M \otimes_R Q$$

- More generally these bifunctors are defined $\mathcal{D}(R) \times \mathcal{D}(R) \to \mathcal{D}(R)$

Useful results

Fact

Given $X, Y, Z \in \mathcal{D}(R)$, there are isomorphisms:

- $R \otimes^{\mathbf{L}}_R X \simeq X$ and $\mathbf{R}\mathrm{Hom}_R(R, X) \simeq X$
- $X \otimes^{\mathbf{L}}_R Y \simeq Y \otimes^{\mathbf{L}}_R X$ and $(X \otimes^{\mathbf{L}}_R Y) \otimes^{\mathbf{L}}_R Z \simeq X \otimes^{\mathbf{L}}_R (Y \otimes^{\mathbf{L}}_R Z)$
- $\mathbf{R}\mathrm{Hom}_R(X \otimes^{\mathbf{L}}_R Y, Z) \simeq \mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\mathrm{Hom}_R(Y, Z))$

Fact (Foxby-Yassemi '95)

A finitely generated R-module has finite G-dimension if and only if $\mathrm{Ext}_R^{\geqslant 0}(M, R) = 0$ and $M \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R)$.

Fact (Avramov-Iyengar-Lipman '10)

A finitely generated R-module has finite G-dimension if and only if $M \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R)$.

Homological dimensions

Definition

Let $X \in \mathcal{D}(R)$. Then $\mathrm{pd}_R(X) < \infty$ if $X \simeq P$ in $\mathcal{D}(R)$ where P is a bounded complex of projective R-modules.

Fact (Foxby-Yassemi '95, Avramov-Iyengar-Lipman '10)

Let $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. TFAE

- $X \simeq G$ in $\mathcal{D}(R)$ where G is a bounded complex of totally reflexive R-modules
- $\mathbf{R}\mathrm{Hom}_R(X, R) \in \mathcal{D}_{\mathrm{b}}(R)$ and $X \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R)$
- $X \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, R), R)$

Complexes satisfying these conditions have $\text{G-dim}_R(X) < \infty$.

Test objects

Definition (O. Celikbas-Dao-Takahashi ’14, O. Celikbas-SSW)

Let H-dim be pd or G-dim. A finitely generated R-module M is H-dim-test if for all finitely generated R-modules N one has $\operatorname{Tor}^R_{\gg 0}(M, N) = 0$ if and only if $\text{H-dim}_R(N) < \infty$.

Definition (O. Celikbas-SSW)

Let H-dim be pd or G-dim. An R-complex $M \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ is H-dim-test if for all $N \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ one has $M \otimes^{\mathbf{L}}_R N \in \mathcal{D}_{\mathrm{b}}(R)$ if and only if $\text{H-dim}_R(N) < \infty$.

Fact (O. Celikbas-SSW)

A finitely generated R-module is an H-dim-test module if and only if it is an H-dim-test complex.

Sketches of proofs

Theorem (O. Celikbas-SSW)

If M is H-dim*-test over R, then $\widehat{M}$ is* H-dim*-test over $\widehat{R}$.*

Proof.

Let N be a finitely generated $\widehat{R}$-module where $\operatorname{Tor}^{\widehat{R}}_{\gg 0}(\widehat{M}, N) = 0$.
That is, $M \otimes^{\mathbf{L}}_R N \in \mathcal{D}_{\mathrm{b}}(R)$.
Let K be the Koszul complex over R on a generating set for $\mathfrak{m}$.
Then $K \otimes^{\mathbf{L}}_R N \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and

$$M \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R N) \simeq K \otimes^{\mathbf{L}}_R (M \otimes^{\mathbf{L}}_R N) \in \mathcal{D}_{\mathrm{b}}(R).$$

M is H-dim-test over R, so $\text{H-dim}_R(K \otimes^{\mathbf{L}}_R N) < \infty$.
Hence $\text{H-dim}_{\widehat{R}}(K \otimes^{\mathbf{L}}_R N) < \infty$.
Hence $\text{H-dim}_{\widehat{R}}(N) < \infty$. □

Sketches of proofs

Theorem (O. Celikbas-SSW)

Assume that M is H-dim*-test over R. If* $\operatorname{G-dim}_R(M) < \infty$*, then R is Gorenstein.*

Proof.

Previous result implies that $\widehat{M}$ is H-dim-test over $\widehat{R}$.
Also $\operatorname{G-dim}_{\widehat{R}}(\widehat{M}) < \infty$.
O. Celikbas-Dao-Takahashi imply that $\widehat{R}$ is Gorenstein.
So R is Gorenstein. □

Theorem (O. Celikbas-SSW)

Let M be a G-dim*-test complex. Let C be a semidualizing R-complex such that that* $\mathbf{R}\mathrm{Hom}_R(M, C) \in \mathcal{D}^f_b(R)$*. Then C is dualizing for R.*