Gorenstein Presentations and Semidualizing Modules

Sean Sather-Wagstaff

Department of Mathematics
North Dakota State University

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Dualizing Modules

Throughout this talk $(R, \mathfrak{m}, k)$ is a local Cohen-Macaulay ring.

Definition. Let C be an R-module. The *homothety morphism* associated to C is the map $\chi_C^R \colon R \to \mathrm{Hom}_R(C, C)$ given by $\chi_C^R(r)(c) = rc$.

Definition. (Grothendieck '67) An R-module D is *dualizing* if:

(1) D is finitely generated,
(2) $\chi_D^R \colon R \to \mathrm{Hom}_R(D, D)$ is an isomorphism;
(3) $\mathrm{Ext}_R^{\geqslant 1}(D, D) = 0$, and
(4) D has finite injective dimension.

Example. If R is artinian, then $E_R(k)$ is a dualizing R-module.

Fact. R is Gorenstein if and only if R is a dualizing R-module.

Fact. (Ferrand-Raynaud '70) In general, R may not have a dualizing module.

Existence of Dualizing Modules

Theorem. (Sharp '71) *Assume that there exist a Gorenstein local ring Q and an ideal $I \subset Q$ such that $R \cong Q/I$. Then R has a dualizing module.*

Sketch of proof. Set $g = \dim(Q) - \dim(R)$. Then $\mathrm{Ext}_Q^g(R, Q)$ is a dualizing R-module and $\mathrm{Ext}_Q^{i \neq g}(R, Q) = 0$. □

Example.

(a) If R is artinian (more generally, if it is complete) then there is a regular local ring Q and an ideal I such that $R \cong Q/I$ by Cohen's Structure Theorem; so R has a dualizing module.

(b) If R is essentially of finite type over a complete local ring or over a Gorenstein local ring, then R has a dualizing module.

Question. Is this the only way R can have a dualizing module?

Dualizing Modules Yield Gorenstein Presentations

Theorem. (Foxby '72, Reiten '72) *If R has a dualizing module, then there exists a Gorenstein local ring Q and an ideal $I \subset Q$ such that $R \cong Q/I$ and $\dim(Q) = \dim(R)$.*

Sketch of proof. Let D be a dualizing module for R, and let $Q = R \ltimes D$ denote Nagata's "idealization" of D:

(1) as an additive abelian group, we have $R \ltimes D = R \oplus D$;
(2) multiplication on $R \ltimes D$ is $(r, d)(r'd') = (rr', rd' + r'd)$.

With this structure $R \ltimes D$ is a commutative local ring with maximal ideal $\mathfrak{m} \oplus D$, and the natural surjection $\tau \colon R \ltimes D \to R$ given by $(r, d) \mapsto r$ is a ring epimorphism.

The kernel of τ is $0 \oplus D$ which satisfies $(0 \oplus D)^2 = 0$, and so $\dim(R \ltimes D) = \dim(R)$.

Because D is dualizing for R, the ring $R \ltimes D$ is Gorenstein. □

Philosophy and Goal

Philosophy. The existence of a dualizing R-module imposes structural conditions on R, namely, that R is of the form Q/I where Q is a Gorenstein local ring.

Also, there are other seemingly different methods for constructing dualizing modules, but they are all equivalent to Sharp's construction.

Goal. Explain how the existence of other R-modules impose further conditions on the presentation $R \cong Q/I$, specifically, further conditions on the structure of I.

Semidualizing Modules

Definition. (Foxby '72, Golod '84, Vasconcelos '74, Wakamatsu '88) An R-module C is *semidualizing* if:

(1) C is finitely generated,
(2) $\chi^R_C \colon R \to \operatorname{Hom}_R(C, C)$ is an isomorphism; and
(3) $\operatorname{Ext}^{\geqslant 1}_R(C, C) = 0$.

Example. The R-module R is semidualizing.

Example. An R-module D is dualizing if and only if it is semidualizing and has finite injective dimension.

Fact. If R is Gorenstein, then the only semidualizing R-module is R. The converse holds when R has a dualizing module.

Fact. Assume that R has a dualizing module D. If C is a semidualizing R-module, then $\operatorname{Ext}^{\geqslant 1}_R(C, D) = 0$ and $C^\dagger = \operatorname{Hom}_R(C, D)$ is a semidualizing R-module.

Existence of Nontrivial Semidualizing Modules

Example. (Foxby '87) Let A be a local Cohen-Macaulay ring that is complete and not Gorenstein. The ring

$$R = A \ltimes A^2 \cong A[\![X, Y]\!]/(X, Y)^2$$

is complete, local, Cohen-Macaulay and non-Gorenstein with $\dim(R) = \dim(A)$. Thus, R has a dualizing module $D \ncong R$.

The R-module $C = \mathrm{Hom}_A(R, A)$ is semidualizing and $C \ncong R$ and $C \ncong D$. Also, the module $C^\dagger = \mathrm{Hom}_R(C, D)$ satisfies the same properties.

Note that R has a nontrivial decomposition as a tensor product. Indeed, since A is complete, there is a Gorenstein local ring Q and an ideal J such that $A \cong Q/J$. Then

$$R \cong \frac{Q[\![X, Y, Z, W]\!]}{(J, Z, W)} \otimes_{Q[\![X,Y,Z,W]\!]} \frac{Q[\![X, Y, Z, W]\!]}{(X, Y)^2}.$$

Construction of Nontrivial Semidualizing Modules

Fact. Assume that there exist a Gorenstein local ring Q and ideals $I_1, I_2 \subset Q$ such that $R \cong Q/(I_1 + I_2)$. Then

$$R \cong Q/(I_1 + I_2) \cong Q/I_1 \otimes_Q Q/I_2.$$

Set $g_i = \dim(Q/I_i) - \dim(R)$, and assume the following:

(1) each Q/I_i is Cohen-Macaulay and not Gorenstein;
(2) $\operatorname{Tor}^Q_{\geqslant 1}(Q/I_1, Q/I_2) = 0$; and (3) $\operatorname{pd}_Q(Q/I_i) < \infty$ for $i = 1, 2$.

Then $C_i = \operatorname{Ext}^{g_i}_{Q/I_i}(R, Q/I_i)$ is semidualizing and $D \ncong C_i \ncong R$.

Note that (2) implies that $I_1 \cap I_2 = I_1 I_2$ because
$0 = \operatorname{Tor}^Q_1(Q/I_1, Q/I_2) \cong (I_1 \cap I_2)/(I_1 I_2)$.

In (3) we only need $\text{G-dim}_{Q/I_i}(R) < \infty$ for $i = 1, 2$.

Question. Is this the only way R can have a nontrivial semidualizing module?

Semidualizing Modules Yield Decompositions of I

Theorem. (DJ-GL-SSW '08) *Assume that R has a dualizing module D and a semidualizing module C such that $D \ncong C \ncong R$. Then there exist a Gorenstein local ring Q and ideals $I_1, I_2 \subset Q$ such that $R \cong Q/(I_1 + I_2) \cong Q/I_1 \otimes_Q Q/I_2$ and*

(1) each Q/I_i is Cohen-Macaulay and not Gorenstein;
(2) $I_1 \cap I_2 = I_1 I_2$; and (3) $\text{G-dim}_{Q/I_i}(R) < \infty$ for $i = 1, 2$.

Sketch of Proof. Set $C^\dagger = \text{Hom}_R(C, D)$.

Then $R_1 = R \ltimes C$ is Cohen-Macaulay and local with dualizing module $D_1 = C^\dagger \oplus D$; the R_1-module structure is given by

$$(r, c)(\phi, d) = (r\phi, \phi(c) + rd).$$

It follows that the ring

$$Q = R_1 \ltimes D_1 \cong R \oplus C \oplus C^\dagger \oplus D$$

is local and Gorenstein.

Sketch of Proof, cont.

The ring structure on $Q \cong R \oplus C \oplus C^\dagger \oplus D$ is given by

$$(r,c,\phi,d)(r',c',\phi',d') = (rr', rc' + r'c, r\phi' + r'\phi, \phi'(c) + \phi(c') + rd' + r'd).$$

The subsets

$$I_1 = 0 \oplus C \oplus 0 \oplus D \qquad \text{and} \qquad I_2 = 0 \oplus 0 \oplus C^\dagger \oplus D$$

are ideals of Q such that

$$I_1 \cap I_2 = 0 \oplus 0 \oplus 0 \oplus D = I_1 I_2 \qquad \text{and} \qquad R_1 = R \ltimes C \cong Q/I_2$$
$$I_1 + I_2 = 0 \oplus C \oplus C^\dagger \oplus D \qquad \text{and} \qquad R \cong Q/(I_1 + I_2).$$

The rest of the conclusions require more work to show. □