# Weakly Spherical Modules

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28 October 2012
AMS Western Section Meeting
University of Arizona

# Spherical Objects

## Definition (Seidel and Thomas, 2000)

Let $X$ be a smooth complex projective variety, and let $\mathcal{D}^b(X)$ denote the bounded derived category of coherent sheaves on $X$. An object $\mathcal{E} \in \mathcal{D}^b(X)$ is spherical if $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$ and

$$\operatorname{Hom}^r_{\mathcal{D}^b(X)}(\mathcal{E}, \mathcal{E}) \cong \begin{cases} \mathbb{C} & \text{if } r = 0, \dim(X) \\ 0 & \text{if } r \neq 0, \dim(X). \end{cases}$$

## Motivation

Given certain collections of spherical objects, Seidel and Thomas construct braid group actions on $\mathcal{D}^b(X)$ related to homological mirror symmetry, the McKay correspondence, etc.

## Motivation

I love Ext.

# Weakly Spherical Modules

## Assumption

$(R, \mathfrak{m}, k)$ is a commutative local noetherian ring and $n$ is a positive integer.

## Definition

A finitely generated $R$-module $M$ is weakly $n$-spherical if

$$\operatorname{Ext}_R^r(M, M) \cong \begin{cases} k & \text{if } r = 0, n \\ 0 & \text{if } r \neq 0, n. \end{cases}$$

## Example

If $R$ is a DVR, then $k$ is weakly 1-spherical over $R$.

## Question

Are there any other examples?

# No

**Theorem**

*Assume that $R$ admits a weakly $n$-spherical module $M$. Then $R$ is a DVR and $M \cong k$.*

**Proof.**

Assumption: $\mathrm{Ext}^r_R(M, M) \cong \begin{cases} k & \text{if } r = 0, n \\ 0 & \text{if } r \neq 0, n. \end{cases}$

We have $0 \neq R/\operatorname{Ann}_R(M) \hookrightarrow \mathrm{Hom}_R(M, M) \cong k$.

So $M \cong k^\beta$ for some $\beta \neq 0$.

It follows that $k \cong \mathrm{Hom}_R(k^\beta, k^\beta) \cong k^{\beta^2}$, so $\beta = 1$ and $M = k$.

Hence $0 = \mathrm{Ext}^r_R(M, M) = \mathrm{Ext}^r_R(k, k)$ for all $r \gg 0$.

We conclude that $R$ is regular. Let $d = \dim(R)$.

Then $\mathrm{Ext}^r_R(k, k) = k^{\binom{d}{r}}$ for all $r$, so $\mathrm{Ext}^r_R(k, k) \neq 0$ for $0 \leq r \leq d$.

The Assumption implies that $d = 1$, so $R$ is a DVR. □

# Weakly Spherical Complexes

## Definition

A homologically finite $R$-complex $X$ is weakly $n$-spherical if

$$\operatorname{Ext}^r_R(X,X) \cong \begin{cases} k & \text{if } r = 0, n \\ 0 & \text{if } r \neq 0, n. \end{cases}$$

Here $\operatorname{Ext}^i_R(X,X) = \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(X,X))$.

## Example

If $R$ is a DVR, then $\Sigma^i k$ is weakly 1-spherical over $R$ for all $i$.

## Question

Are there any other examples?

# No

**Theorem**

*Assume that $R$ admits a weakly $n$-spherical complex $X$. Then $R$ is a DVR and $M \simeq \Sigma^j k$ for some $j \in \mathbb{Z}$.*

**Sketch of proof.**

The fact that $X$ is homologically finite such that $\mathrm{Ext}^r_R(X, X)$ has finite length for all $r$ implies that $\mathrm{H}_i(X)$ has finite length for all $i$.
From this, one deduces that there is an integer $j$ such that $\mathrm{H}_i(X) = 0$ for all $i \neq j$.
So there is a finitely generated $R$-module $M$ such that $X \simeq \Sigma^j M$.
The previous theorem implies that $M \cong k$. □

**Assumption**

Let $\mathbf{x} = x_1, \ldots, x_n \in \mathfrak{m}$ and set $K = K^R(\mathbf{x})$, the Koszul complex.
We view $K$ as a DG $R$-algebra via the wedge product.

# Weakly Spherical DG Modules

## Definition

A homologically finite DG $K$-module $X$ is weakly $n$-spherical if

$$\operatorname{Ext}^r_K(X,X) \cong \begin{cases} k & \text{if } r = 0, n \\ 0 & \text{if } r \neq 0, n. \end{cases}$$

## Example

Let $R$ be a regular local ring with regular system of parameters $x_1, \ldots, x_n, t$. Then the DG $K$-module $K^R(\mathbf{x}) \otimes_R K^R(t) = K^R(x_1, \ldots, x_n, t)$ is a weakly 1-spherical.

## Question

Are there any other examples?

# Not Many, If Any

## Theorem

*Assume that $K$ admits a weakly $n$-spherical DG module $X$.*

(a) *Then $n = 1$ or $n = 2$.*

(b) *If $n = 1$, then $R$ is a regular local ring with rsop $x_1, \ldots, x_n, t$ and $X \simeq \Sigma^i K^R(x_1, \ldots, x_n, t)$ for some $i \in \mathbb{Z}$.*

## Hint of proof.

Assume that $R$ is complete and that $n \neq 2$.
It follows that $\mathrm{Ext}^2_R(X, X) = 0$ so a lifting result of Nasseh and Sather-Wagstaff (á la Auslander, Ding, and Solberg) provides a homologically finite $R$-complex $M$ such that $X \cong K \otimes^{\mathbf{L}}_R M$.
Prove that $n = 1$ and $M \simeq \Sigma^i K^R(t)$, as described above. □

## Note

The case $n = 2$ is still open.