Homotopy-theory techniques in commutative algebra

Sean Sather-Wagstaff

Department of Mathematics
North Dakota State University

03 April 2008
Texas Tech University
Mathematics and Statistics Colloquium
Joint with Lars W. Christensen
`arXiv: math.AC/0612311`

Rings and Modules

Throughout this talk, let R be a commutative ring with identity. Examples include:

- The rings of integers $\mathbb{Z}$ and p-adic integers $\mathbb{Z}_p$
- A field k like $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}/p\mathbb{Z}$
- Polynomial rings $A[x_1, \ldots, x_n]$ with coefficients in a commutative ring A with identity
- Quotient rings $A[x_1, \ldots, x_n]/I$

R-modules are objects that can be "acted upon" by the ring R.

- If k is a field, then "k-module"="k-vector space".
- "$\mathbb{Z}$-module"="abelian group".
- If I is an ideal of R, then I and R/I are R-modules.

Free Modules

An R-module is free if it has a basis. If F is free and has a finite basis $e_1, \ldots, e_r$ then $F \cong R^r$.

Most R-modules are not free.

Example. If $n \geq 2$, then the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ does not have a basis since $nb = 0$ for all $b \in \mathbb{Z}/n\mathbb{Z}$: linear independence fails.

Theorem. *R is a field if and only if every R-module is free.*

Moral. Ring theoretic properties for R are mirrored in structure results for R-modules.

Theorem. *If R is a PID, then every finitely generated R-module is a direct sum of cyclic R-modules.*

Presentations

Assume: R is Noetherian, M is a finitely generated R-module.
For example, $R = A[x_1, \ldots, x_n]/I$ where $A = k$ or $A = \mathbb{Z}$.

- There exists a surjection $\tau_0 \colon R^m \twoheadrightarrow M$.
- R is Noetherian, so $\mathrm{Ker}(\tau_0) \subseteq R^m$ is finitely generated.
- There exists a surjection $\tau_1 \colon R^n \twoheadrightarrow \mathrm{Ker}(\tau_0)$.
- The composition $R^n \to \mathrm{Ker}(\tau_0) \to R^m$ is given by an $m \times n$ matrix B with entries in R, and $M \cong R^m/C$ where C is the submodule of R^m generated by the columns of B.
- The exact sequence $R^n \xrightarrow{B} R^m \xrightarrow{\tau_0} M \to 0$ is a finite free presentation of M.

More generally M has a free resolution:

$$\cdots \to R^{r_3} \to R^{r_2} \to R^{r_1} \to R^{r_0} \to M \to 0.$$

Applications of Presentations

Moral. Presentations: Use linear algebra to study modules.

Application. Build examples of R-modules: Let $B \in \mathcal{M}_{m\times n}(R)$. The submodule $C \subseteq R^m$ generated by the columns of B, and the quotient R^m/C, are finitely generated R-modules.

Application. Describe tensor products: Let M be an R-module with finite free presentation $R^n \xrightarrow{B} R^m \to M \to 0$.
For each R-module N, the tensor product $N \otimes_R M$ has a presentation $N^n \xrightarrow{B} N^m \to N \otimes_R M \to 0$.
If $\varphi\colon R \to S$ is a ring homomorphism, then $S \otimes_R M$ has a finite free presentation $S^n \xrightarrow{\varphi(B)} S^m \to S \otimes_R M \to 0$ as an S-module.

Question. Does describe every finitely generated S-module?

Answer. Not in general.

Descent Problem for S-modules

Problem. Given a ring homomorphism $\varphi\colon R \to S$, provide sufficient (or necessary) conditions for an S-module N to be of the form $S \otimes_R M$ for some finitely generated R-module M.

Example. If N is free, then $N \cong S^n \cong S \otimes_R R^n$.

Example. If $\varphi = \mathrm{id}_R\colon R \to S = R$, then $N \cong R \otimes_R N = S \otimes_R N$.

Naive idea. Let k be a field.

$$\begin{array}{ccccc}
\exists ? M & \longrightarrow & S \otimes_R M & \stackrel{?}{\cong} & N \\
\downarrow & & \downarrow & & \downarrow \\
& & k \otimes_R M \cong & & \\
k \otimes_R M \cong L & \longrightarrow & k \otimes_k L & \cong & k \otimes_S N
\end{array}$$

$$\begin{array}{ccc}
R = k[x,y] & \longrightarrow & k[\![x,y]\!] = S \\
\downarrow & & \downarrow \\
k & \xrightarrow{\cong} & k
\end{array}$$

This is naive because $k \otimes_S N$ does not retain enough data.

Building the Koszul Complex I: Exterior Powers

Fix $n \geq 1$ and let $e_1, \ldots, e_n \in R^n$ be a basis for R^n.

For each integer d the dth exterior power of R^n is the free R-module $\wedge^d R^n \cong R^{\binom{n}{d}}$ with basis

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d} \mid 1 \leq i_1 < i_2 < \cdots < i_d \leq n\}.$$

Example. $n = 3$:

	$d=4$	$d=3$	$d=2$	$d=1$	$d=0$	$d=-1$
$\wedge^d R^n$	0	R^1	R^3	R^3	R^1	0
basis		$e_1 \wedge e_2 \wedge e_3$	$e_1 \wedge e_2$	e_1	1	
			$e_1 \wedge e_3$	e_2		
			$e_2 \wedge e_3$	e_3		

Building the Koszul Complex II: The Differential

Let $\mathbf{x} = x_1, \ldots, x_n \in R$, and build the Koszul complex $K^R(\mathbf{x})$:

$$0 \xrightarrow{\partial^K_{n+1}} \underset{d=n}{\wedge^n R^n} \xrightarrow{\partial^K_n} \underset{d=n-1}{\wedge^{n-1} R^n} \xrightarrow{\partial^K_{n-1}} \cdots \xrightarrow{\partial^K_3} \underset{d=2}{\wedge^2 R^n} \xrightarrow{\partial^K_2} \underset{d=1}{R^n} \xrightarrow{\partial^K_1} \underset{d=0}{R} \xrightarrow{\partial^K_0} 0$$

Define homomorphisms $\partial^K_d \colon \wedge^d R^n \to \wedge^{d-1} R^n$ for each d.

$$\partial^K_1(e_i) = x_i$$

$$\partial^K_2(e_{i_1} \wedge e_{i_2}) = x_{i_1} e_{i_2} - x_{i_2} e_{i_1}$$

$$\partial^K_d(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d}) = \sum_{j=1}^{d} (-1)^{j+1} x_{i_j} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_d}$$

- $\partial^K_{d-1} \circ \partial^K_d = 0$ for each d, so this is a chain complex.
- When $R = A[x_1, \ldots, x_n]$, this is a free resolution of $R/(\mathbf{x})$.

Examples of Koszul Complexes

$n = 0$:

$$0 \to R \to 0$$

$n = 1$:

$$0 \to R^1 \xrightarrow{[\,x_1\,]} R^1 \to 0$$

$n = 2$:

$$0 \to R^1 \xrightarrow{\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}} R^2 \xrightarrow{[\,x_1 \;\; x_2\,]} R^1 \to 0$$

$n = 3$:

$$0 \to R^1 \xrightarrow{\begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \\ 0 & -x_1 & -x_2 \end{bmatrix}} R^3 \xrightarrow{[\,x_1 \;\; x_2 \;\; x_3\,]} R^1 \to 0$$

Algebra Structure on the Koszul Complex

The wedge product provides a product on the Koszul complex:
For $v \in \wedge^d R^n$ and $w \in \wedge^{d'} R^n$, define $vw := v \wedge w \in \wedge^{d+d'} R^n$.

This gives rise to elements of the form

$$(e_{i_1} \wedge \cdots \wedge e_{i_d})(e_{j_1} \wedge \cdots \wedge e_{j_{d'}}) = e_{i_1} \wedge \cdots \wedge e_{i_d} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{d'}}$$

which are not necessarily basis elements.

To make sense of such elements, use the following relations:

$$e_{i_1} \wedge \cdots \wedge e_{i_j} \wedge e_{i_{j+1}} \wedge \cdots \wedge e_{i_{d+d'}} = -e_{i_1} \wedge \cdots \wedge e_{i_{j+1}} \wedge e_{i_j} \wedge \cdots \wedge e_{i_{d+d'}}$$
$$e_{i_1} \wedge \cdots \wedge e_{i_j} \wedge e_{i_j} \wedge \cdots \wedge e_{i_{d+d'}} = 0$$

Example. $(e_1 \wedge e_2) \wedge e_1 = e_1 \wedge e_2 \wedge e_1 = -e_1 \wedge e_1 \wedge e_2 = 0.$

DG Algebra Structure on the Koszul Complex

Basic properties. For $v \in \wedge^d R^n$ and $w \in \wedge^{d'} R^n$

- $wv = (-1)^{dd'} vw$
- $v^2 = 0$ when d is odd
- Leibniz Rule: $\partial^K_{d+d'}(vw) = \partial^K_d(v)w - (-1)^d v \partial^K_{d'}(w)$

This gives the Koszul complex the structure of a differential graded commutative algebra or DG algebra for short.

- The ring R is a DG algebra concentrated in degree 0.
- The natural map $R \to K^R$ is a DG algebra homomorphism.

$$\begin{array}{c} R \\ \downarrow{\scriptstyle \iota_R} \\ K^R(\mathbf{x}) \end{array} \qquad \begin{array}{ccccccccccc} & & & & & & 0 & \longrightarrow & R & \longrightarrow & 0 \\ & & & & & & \downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow \\ 0 & \longrightarrow & R^{\binom{n}{n}} & \longrightarrow & \cdots & \longrightarrow & R^{\binom{n}{1}} & \longrightarrow & R^{\binom{n}{0}} & \longrightarrow & 0 \end{array}$$

Compatibility with Ring Homomorphisms

A ring homomorphism $\varphi\colon R \to S$ induces a DG algebra homomorphism $K^R(\mathbf{x}) \to K^S(\varphi(\mathbf{x}))$

$$
\begin{array}{ccccccccccccc}
K^R(\mathbf{x}) & & 0 & \longrightarrow & R^{\binom{n}{n}} & \longrightarrow & \cdots & \longrightarrow & R^{\binom{n}{1}} & \longrightarrow & R^{\binom{n}{0}} & \longrightarrow & 0 \\
\downarrow{\scriptstyle K^{\varphi}} & & & & \downarrow{\scriptstyle \varphi^{\binom{n}{n}}} & & & & \downarrow{\scriptstyle \varphi^{\binom{n}{1}}} & & \downarrow{\scriptstyle \varphi^{\binom{n}{0}}} & & \\
K^S(\varphi(\mathbf{x})) & & 0 & \longrightarrow & S^{\binom{n}{n}} & \longrightarrow & \cdots & \longrightarrow & S^{\binom{n}{1}} & \longrightarrow & S^{\binom{n}{0}} & \longrightarrow & 0
\end{array}
$$

These maps make the following diagram commute.

$$
\begin{array}{ccc}
R & \xrightarrow{\varphi} & S \\
{\scriptstyle \iota_R}\downarrow & & \downarrow{\scriptstyle \iota_S} \\
K^R(\mathbf{x}) & \xrightarrow{K^{\varphi}} & K^S(\varphi(\mathbf{x}))
\end{array}
$$

The Completion of a Local Ring

R is a local noetherian ring with maximal ideal $\mathfrak{m}$. For $r, s \in R$

$$\operatorname{ord}(r) = \sup\{n \geq 0 \mid r \in \mathfrak{m}^n\} \qquad \operatorname{dist}(r,s) = 2^{-\operatorname{ord}(r-s)}$$

The function $\operatorname{dist}(-,-)$ is a metric on R. The topological completion of R is denoted $\widehat{R}$.

$\widehat{R}$ is a noetherian local ring equipped with a canonical ring homomorphism $\varphi_R \colon R \to \widehat{R}$ and maximal ideal $\varphi_R(\mathfrak{m})\widehat{R}$.

Example. If $R = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}/(f_1, \ldots, f_m)$, then $\widehat{R} \cong k[\![x_1, \ldots, x_n]\!]/(f_1, \ldots, f_m)$.

If $\mathfrak{m} = (x_1, \ldots, x_n)R$, then the induced map on Koszul complexes $K^R \to K^{\widehat{R}}$ is a homology isomorphism.

A Descent Theorem

Return to naive idea.

$$\begin{array}{ccccc}
\exists ? M & \longrightarrow & \widehat{R}\otimes_R M & \stackrel{?}{\cong} & N \\
\downarrow & \begin{array}{ccc} R & \longrightarrow & \widehat{R} \\ \downarrow & & \downarrow \\ K^R & \xrightarrow{\simeq} & K^{\widehat{R}} \end{array} & \downarrow & & \downarrow \\
& & K^{\widehat{R}}\otimes_R M \simeq & & \\
K^R \otimes_R M \simeq L & \longrightarrow & K^{\widehat{R}}\otimes_{K^R} L & \simeq & K^{\widehat{R}}\otimes_{\widehat{R}} N
\end{array}$$

R has the approximation property when, for every finite system of polynomial equations $P = \{f_i(X_1, \ldots, X_N) = 0\}_{i=1}^t$ over R, if P has a solution in $\widehat{R}$, then it has a solution in R.

Theorem. (L.W.Christensen-SSW) *Assume that R has the approximation property. For each finitely generated $\widehat{R}$-module N, there exists an R-module M such that $K^{\widehat{R}} \otimes_R M \simeq K^{\widehat{R}} \otimes_{\widehat{R}} N$.*

Further Descent Results

Theorem. (L.W.Christensen-SSW) *Assume that R has the approximation property. For each semidualizing $\widehat{R}$-module C, there exists a semidualizing R-module B such that $\widehat{R} \otimes_R B \cong C$.*

Corollary. (V.Hinich) *If R has the approximation property, then R admits a dualizing module.*

Moral. The DG algebra structure on the Koszul complex allows us to solve certain problems about commutative rings by leaving the realm of commutative rings.