Detecting completeness from Ext-vanishing

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Let R be a commutative Noetherian ring.

Theorem. (Jensen '70) *If B is a flat R-module, then* $\operatorname{pd}_R(B) \leq \dim(R)$.

Corollary. *If B is a flat R-module, then* $\operatorname{Ext}_R^{>\dim(R)}(B, M) = 0$ *for all R-modules M.*

Theorem. (Jensen '70) *The following are equivalent.*

(i) $R \cong \prod_{i=1}^{n} R_i$ *with each R_i local and complete.*

(ii) $\operatorname{Ext}_R^{\geq 1}(B, M) = 0$ *for all R-modules B, M with B flat and M finitely generated.*

(iii) $\operatorname{Ext}_R^1(B, R) = 0$ *for all flat R-modules B.*

Theorem. (Buchweitz-Flenner '06) *Let $\mathfrak{m} \subset R$ be a maximal ideal and let B, M be R-modules. If M is $\mathfrak{m}$-adically complete, then* $\mathrm{Ext}_R^{\geq 1}(B, M) = 0$.

Example. Let $(R, \mathfrak{m})$ be a local domain with $\dim(R) > 0$ and field of fractions $M = E_R(R)$. Since M is divisible, $\mathfrak{m}^n M = M$ for each $n \geq 1$. Hence, $\widehat{M} = 0$ and M is not $\mathfrak{m}$-adically complete. However, M is injective, so $\mathrm{Ext}_R^{\geq 1}(B, M) = 0$.

Theorem A. (AJF-SSW '06) *Let $\mathfrak{a} \subseteq \mathrm{Jac}(R)$. If M is a finitely generated R-module such that $\mathrm{Ext}_R^{\geq 1}(\widehat{R}^{\mathfrak{a}}, M) = 0$, then M is $\mathfrak{a}$-adically complete.*

Corollary B. *Let $(R, \mathfrak{m})$ be local and M finitely generated. TFAE.*

(i) *M is $\mathfrak{m}$-adically complete.*

(ii) $\mathrm{Ext}_R^{\geq 1}(B, M) = 0$ *for all flat R-modules B.*

(iii) $\mathrm{Ext}_R^{\geq 1}(\widehat{R}, M) = 0$.

Sketch of proof of Theorem A.

Assume that M is finitely generated and $\operatorname{Ext}^{\geq 1}_R(\widehat{R}^{\mathfrak{a}}, M) = 0$.

Want to show that M is $\mathfrak{a}$-adically complete.

Assume that M is indecomposable.

$\operatorname{Ext}^{\geq 1}_R(\widehat{R}^{\mathfrak{a}}, M) = 0$ implies $\operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, M) \neq 0$.

Hence, M contains a nonzero $\mathfrak{a}$-adically complete submodule.

M has a unique maximal $\mathfrak{a}$-adically complete submodule $C^{\mathfrak{a}}_M \subseteq M$.

Need to show $C^{\mathfrak{a}}_M = M$, i.e., the inclusion $C^{\mathfrak{a}}_M \xrightarrow{h_M} M$ is surjective.

The map h_M decomposes as the following composition

$$C^{\mathfrak{a}}_M \xrightarrow{\cong} \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, C^{\mathfrak{a}}_M) \xrightarrow[\operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, h_M)]{\cong} \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, M) \xrightarrow{\epsilon_M} M$$

$$c \longmapsto (r \mapsto rc) \qquad\qquad \phi \longmapsto \phi(1)$$

It remains to show that ϵ_M is surjective.

It suffices to show that $\widehat{\epsilon_M}^{\mathfrak{a}} \colon \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)^{\widehat{\ }\mathfrak{a}} \to \widehat{M}^{\mathfrak{a}}$ is surjective.

Local homology. To see where the Ext-vanishing is used.

Let $\mathfrak{a} = (a_1, \ldots, a_m)R$ and consider the Cech complex

$$C(\mathfrak{a}) = 0 \to \underbrace{R}_{\text{deg } 0} \to \underbrace{\oplus_i R_{a_i}}_{\text{deg } -1} \to \underbrace{\oplus_{i,j} R_{a_i a_j}}_{\text{deg } -2} \to \cdots \to \underbrace{R_{a_1 \cdots a_m}}_{\text{deg } -m} \to 0$$

To see what the maps are, realize $C(\mathfrak{a})$ as a tensor product of short Cech complexes

$$C(a_i) = 0 \to \underbrace{R}_{\text{deg } 0} \to \underbrace{R_{a_i}}_{\text{deg } -1} \to 0 \qquad C(\mathfrak{a}) = C(a_1) \otimes_R \cdots \otimes_R C(a_m).$$

Local cohomology modules are computed via tensor product

$$\mathrm{H}^i_{\mathfrak{a}}(N) = \mathrm{H}_{-i}(C(\mathfrak{a}) \otimes_R N)$$

so it makes sense that the modules in the Cech complex are flat. For local *homology*, we use $\mathrm{Hom}_R(-, N)$ instead of $- \otimes_R N$, so we need a *projective* resolution of $C(\mathfrak{a})$.

Start with a projective resolution of the short Cech complex $C(a_i)$:

$$R_{a_i} \cong R[X]/(a_i X - 1)$$

$$0 \longrightarrow R[X] \xrightarrow{a_i X - 1} R[X] \xrightarrow{\alpha_i} R_{a_i} \longrightarrow 0$$

$$X \longmapsto 1/a_i$$

$$\begin{array}{ccccccccc}
L(a_i) & = & 0 & \longrightarrow & R[X] \oplus R & \xrightarrow{(a_i X - 1 \;\; i)} & R[X] & \longrightarrow & 0 \\
\simeq \big\downarrow & & & & \big\downarrow (0 \;\; 1) & & \big\downarrow \alpha_i & & \\
C(a_i) & = & 0 & \longrightarrow & R & \longrightarrow & R_{a_i} & \longrightarrow & 0
\end{array}$$

$$L(\mathfrak{a}) = L(a_1) \otimes_R \cdots \otimes_R L(a_m).$$

The *ith local homology module* of N at $\mathfrak{a}$ is

$$\mathrm{H}_i^{\mathfrak{a}}(N) = \mathrm{H}_i(\mathrm{Hom}_R(L(\mathfrak{a}), N)).$$

Fact 1. $\mathrm{H}_i^{\mathfrak{a}}(-)$ is a well-defined additive covariant functor.

Fact 2. If $X \to Y$ is a quasiisomorphism, then so is the induced morphism $\mathrm{Hom}_R(L(\mathfrak{a}), X) \to \mathrm{Hom}_R(L(\mathfrak{a}), Y)$.

Fact 3. The morphism $L(\mathfrak{a}) \cong L(\mathfrak{a}) \otimes_R R \to L(\mathfrak{a}) \otimes_R \widehat{R}^{\mathfrak{a}}$ is a quasiisomorphism.

Fact 4. If N is finitely-generated, then $\mathrm{H}_i^{\mathfrak{a}}(N) \cong \begin{cases} \widehat{N}^{\mathfrak{a}} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$

The proof of Theorem A will be complete once we prove:

Lemma C. (AJF-SSW '06) *If M is finitely generated and $\mathrm{Ext}_R^{\geq 1}(\widehat{R}^a, M) = 0$, then $\widehat{\epsilon_M}^{\mathfrak{a}} \colon \mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)^{\widehat{}\mathfrak{a}} \to \widehat{M}^{\mathfrak{a}}$ is bijective.*

Proof. Since M and $\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)$ are finitely generated, we need to show that $\mathrm{H}_0^{\mathfrak{a}}(\epsilon_M) \colon \mathrm{H}_0^{\mathfrak{a}}(\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)) \to \mathrm{H}_0^{\mathfrak{a}}(M)$ is bijective. We will show that $\mathrm{Hom}_R(L(\mathfrak{a}), \epsilon_M)$ is a quasiisomorphism.

Let $\iota\colon M \xrightarrow{\simeq} J$ be an injective resolution.
$\operatorname{Ext}_R^{\geq 1}(\widehat{R}^a, M) = 0$ implies that $\operatorname{Hom}_R(\widehat{R}, \iota)$ is a quasiisomorphism.

$$
\begin{array}{ccc}
\operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, M) & \xrightarrow{\epsilon_M} & M \\
{\scriptstyle \operatorname{Hom}_R(\widehat{R},\iota)}\downarrow\simeq & \circlearrowleft & \simeq\downarrow{\scriptstyle \iota} \\
\operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, J) & \xrightarrow{\epsilon_J} & J
\end{array}
$$

Applying $\operatorname{Hom}_R(L(\mathfrak{a}), -)$ yields the top half of the next diagram.

$$
\begin{array}{ccc}
\operatorname{Hom}_R(L(\mathfrak{a}), \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, M)) & \xrightarrow{\operatorname{Hom}_R(L(\mathfrak{a}),\epsilon_M)} & \operatorname{Hom}_R(L(\mathfrak{a}), M) \\
{\scriptstyle \text{Fact 1}}\downarrow\simeq & \circlearrowleft & \simeq\downarrow{\scriptstyle \text{Fact 1}} \\
\operatorname{Hom}_R(L(\mathfrak{a}), \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, J)) & \longrightarrow & \operatorname{Hom}_R(L(\mathfrak{a}), J) \\
{\scriptstyle \text{adjointness}}\downarrow\cong & \circlearrowleft & \cong\downarrow \\
\operatorname{Hom}_R(L(\mathfrak{a}) \otimes_R \widehat{R}^{\mathfrak{a}}, J) & \underset{\simeq}{\xrightarrow{\text{Fact 2}}} & \operatorname{Hom}_R(L(\mathfrak{a}) \otimes_R R, J)
\end{array}
$$

$\square$

Question. How many Ext-vanishings need to be checked?

Answer 1. Jensen's result implies $\operatorname{Ext}_R^{>\dim(R)}(\widehat{R}^{\mathfrak{a}}, M) = 0$, so one need only check $\operatorname{Ext}_R^i(\widehat{R}^{\mathfrak{a}}, M) = 0$ for $i = 1, \ldots, \dim(R)$.

We can do slightly better.

Theorem D. *If M is an R-module, then* $\operatorname{Ext}_R^{>\dim(M)}(\widehat{R}^{\mathfrak{a}}, M) = 0$.