Associated Primes of Derived Local Cohomology

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Set-up

Assumption (R is a commutative noetherian ring with identity)

$\mathfrak{a} \subsetneq R$ is an ideal
M is an R-module with injective resolution I

Definition (Grothendieck 1967)

$\mathfrak{a}$-torsion submodule:

$$\Gamma_{\mathfrak{a}}(M) := \{m \in M \mid \mathfrak{a}^i m = 0 \text{ for } i \gg 0\}$$

local cohomology with support in $\mathfrak{a}$:

$$\mathrm{H}^i_{\mathfrak{a}}(M) := \mathrm{H}^i(\Gamma_{\mathfrak{a}}(I))$$

A Motivating Example

Example (Let $(R, \mathfrak{m})$ be a d-dimensional local Gorenstein ring)

The minimal injective resolution of R over itself has the form

$$I = 0 \to \bigoplus_{\mathrm{ht}(\mathfrak{p})=0} E_R(R/\mathfrak{p}) \to \bigoplus_{\mathrm{ht}(\mathfrak{p})=1} E_R(R/\mathfrak{p}) \to \cdots \to E_R(R/\mathfrak{m}) \to 0$$

$$\Gamma_{\mathfrak{m}}(I) = 0 \to 0 \to 0 \to \cdots \to 0 \to E_R(R/\mathfrak{m}) \to 0$$

$$\mathrm{H}^i_{\mathfrak{m}}(R) = \begin{cases} 0 & \text{if } i \neq d \\ E_R(R/\mathfrak{m}) & \text{if } i = d \end{cases}$$

In particular, if $d \geq 1$, then $\mathrm{H}^d_{\mathfrak{m}}(R)$ is not finitely generated.

Question

If M is finitely generated, does $\mathrm{H}^i_{\mathfrak{a}}(M)$ have any reasonable finiteness properties?

Some Finiteness Properties

Facts (Assume M is finitely generated)

1. If $\mathfrak{m} \subset R$ is maximal, then $\mathrm{H}^i_{\mathfrak{m}}(M)$ is artinian.
2. (Lyubeznik 1993) If $R = \mathbb{C}[x_1, \ldots, x_d]$, then $H^i_{\mathfrak{a}}(M)$ is a "holonomic D-module" where D is a Weyl algebra.
3. (Huneke-Sharp, Lyubeznik 1993) If R is an unramified regular local ring, then $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))$ is a finite set.

$$\mathrm{Ass}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} = \mathrm{Ann}_R(x) \text{ for some } x \in M\}.$$

Questions (Huneke 1992)

Assume M is finitely generated.

1. Is $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))$ finite?
2. Is $\mathrm{min}(\mathrm{Supp}_\mathrm{R}(\mathrm{H}^\mathrm{d}_{\mathfrak{a}}(\mathrm{M})))$ finite?
3. Are the Bass numbers $\mu^j_R(\mathfrak{p}, \mathrm{H}^i_{\mathfrak{a}}(M))$ all finite?

Infinitely Many Associated Primes

Example (Singh 2000)

If $\mathfrak{a} = (x, y, z)R$ where

$$R = \frac{\mathbb{Z}[u, v, w, x, y, z]}{(ux + vy + wz)}$$

then $\mathrm{Ass}_R(\mathrm{H}^3_{\mathfrak{a}}(R))$ is infinite.

Example (Katzman 2002)

If k is a field and $\mathfrak{a} = (x, y)R$ where

$$R = \frac{k[s, t, u, v, x, y]}{(su^2x^2 - (s + t)uxvy + tv^2y^2)}$$

then $\mathrm{Ass}_R(\mathrm{H}^2_{\mathfrak{a}}(R))$ is infinite.

Singh and Swanson (2004) improve upon this.

The Derived Category Perspective

Maxim: Treat a complex as a single algebraic object instead of breaking it into its separate (co)homology modules.

Fact (Derived Local Cohomology Complex)

1. *The complex* $\mathbf{R}\Gamma_{\mathfrak{a}}(M) := \Gamma_{\mathfrak{a}}(I)$ *is a well defined object up to isomorphism in the derived category* $\mathcal{D}(R)$.
2. *For all* $\mathfrak{p} \in \operatorname{Spec}(R)$ *we have*
$$\mu_R^i(\mathfrak{p}, \mathbf{R}\Gamma_{\mathfrak{a}}(M)) = \begin{cases} \mu_R^i(\mathfrak{p}, M) & \text{if } \mathfrak{a} \subseteq \mathfrak{p} \\ 0 & \text{otherwise.} \end{cases}$$
In particular, each Bass number $\mu_R^i(\mathfrak{p}, \mathbf{R}\Gamma_{\mathfrak{a}}(M))$ *is finite.*
3. $\operatorname{Supp}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(M)) = \operatorname{Supp}_R(M) \cap V(\mathfrak{a}) = \operatorname{Supp}_R(M/\mathfrak{a}M)$.
In particular, $\min(\operatorname{Supp}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$ *is finite.*

Good news: answers to related questions.
Bad news: not helpful for answering the original questions.

Associated Primes of Complexes

Definition ($(R, \mathfrak{m})$ local)

$\operatorname{depth}_R(M) = \inf\{i \geq 0 \mid \operatorname{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\}$
In particular, $\operatorname{depth}_R(0) = \infty$.

Fact

$\operatorname{Ass}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 0\}.$

Definition (Christensen 2001)

For any $Y \in \mathcal{D}_\mathrm{b}(R)$, set

$$\operatorname{Ass}_R(Y) = \{\mathfrak{p} \in \operatorname{Supp}_R(Y) \mid \operatorname{depth}_{R_\mathfrak{p}}(Y_\mathfrak{p}) = -\sup(Y_\mathfrak{p})\}.$$

$$\sup(Y_\mathfrak{p}) := \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(Y_\mathfrak{p}) \neq 0\} \geq -\operatorname{depth}_{R_\mathfrak{p}}(Y_\mathfrak{p})$$

Finitely Many Associated Primes

Theorem (SSW)

$\mathrm{Ass}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$ *is a finite set.*

Sketch of proof

Set $K = K^R(a_1, \ldots, a_n)$ where $\mathfrak{a} = (a_1, \ldots, a_n)R$.

$$\begin{aligned} K \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) &\simeq K \otimes^{\mathbf{L}}_R (\mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_R M) \\ &\simeq \mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R M) \\ &\simeq K \otimes^{\mathbf{L}}_R M \\ \mathrm{Ass}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(M)) &\subseteq \mathrm{Ass}_R(K \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M)) \\ &= \mathrm{Ass}_R(K \otimes^{\mathbf{L}}_R M) \end{aligned}$$

$\mathrm{H}(K \otimes^{\mathbf{L}}_R M)$ is finitely generated, so $\mathrm{Ass}_R(K \otimes^{\mathbf{L}}_R M)$ is finite. □