Semidualizing modules give a defective Gorenstein defect

Sean Sather-Wagstaff

Clemson University

23 Sep 2017 (the end of the world?)
AMS Southeastern Section Meeting, Central Florida University
Special Session on Commutative Algebra: Interactions with Algebraic Geometry and Algebraic Topology

joint with Saeed Nasseh, Georgia Southern University
Ryo Takahashi, Nagoya University
Keller VandeBogert, University of South Carolina

Big Picture

Goals

1. Explain why you might hope that the set of isomorphism classes of semidualizing modules over a Cohen-Macaulay local ring does not grow under localization.
2. Show how to find a Cohen-Macaulay local ring where the set of isomorphism classes of semidualizing modules does grow under localization.

Assumption

R is a (commutative noetherian) ring (with identity).

Two Examples

Fact (Assume R is local and CM with canonical module D.)

1. *D is finitely generated over R,*
2. $\operatorname{Ext}^i_R(D, D) = 0$ *for all* $i \neq 0$*, and*
3. $R \xrightarrow{\chi} \operatorname{Hom}_R(D, D)$ *is an isomorphism.*

Fact (The module R has similar properties.)

1. *R is finitely generated over R,*
2. $\operatorname{Ext}^i_R(R, R) = 0$ *for all* $i \neq 0$*, and*
3. $R \xrightarrow{\chi} \operatorname{Hom}_R(R, R)$ *is an isomorphism.*

Remark

Each of these examples is used for an important duality.

Semidualizing Modules

Definition (Foxby '72)

An R-module C is semidualizing if

1. C is finitely generated over R,
2. $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \neq 0$, and
3. $R \xrightarrow{\chi} \mathrm{Hom}_R(C, C)$ is an isomorphism.

$\mathfrak{S}_0(R) = \{$isomorphism classes of semidualizing R-modules$\}$

Applications

1. Understanding compositions of local ring homomorphisms of finite G-dimension.
2. Understanding growth of Bass numbers of local rings.

Properties of Semidualizing Modules

Fact (Assume R is local)

1. $\mathfrak{S}_0(R)$ *is a finite set. (Christensen-SSW, Nasseh-SSW)*
2. *If R is Gorenstein, then $\mathfrak{S}_0(R) = \{R\}$; the converse holds when R is Cohen-Macaulay with a canonical module.*
3. $|\mathfrak{S}_0(R)|$ *is a candidate for the Gorenstein defect of R, if R is Cohen-Macaulay with a canonical module.*

Fact (Let $C \in \mathfrak{S}_0(R)$ and $U \subseteq R$ multiplicatively closed.)

1. *Then $U^{-1}C$ is semidualizing over $U^{-1}R$.*
2. *This gives a well-defined map $\mathfrak{S}_0(R) \to \mathfrak{S}_0(U^{-1}R)$.*
3. *If R is a normal domain, then*

$$\begin{array}{ccc} \mathfrak{S}_0(R) & \xrightarrow{\text{surj?}} & \mathfrak{S}_0(U^{-1}R) \\ \downarrow & & \downarrow \\ \mathrm{Cl}(R) & \twoheadrightarrow & \mathrm{Cl}(U^{-1}R). \end{array}$$

The Question

Question

If R is Cohen-Macaulay local with a canonical module, and $\mathfrak{p} \in \mathrm{Spec}(R)$, then $|\mathfrak{S}_0(R_{\mathfrak{p}})| \leq |\mathfrak{S}_0(R)|$?

Problem

The class of local rings where we can completely describe $\mathfrak{S}_0(R)$ is small:

1. Gorenstein: $\mathfrak{S}_0(R) = \{R\}$
2. $\mathfrak{m}^2 = 0$ (min. multiplicity) e.g., $k \ltimes k^e$: $\mathfrak{S}_0(R) = \{R, D\}$
   Veronese subrings of $k[\![X_1, \ldots, X_n]\!]$ (Celikbas-Dao)
   some other rings of invariants (Sanders)
3. determinantal extensions, e.g., $A[X]_{(X)}/I_r(X)$ where A is a normal domain (SSW): $\mathfrak{S}_0(R) \equiv \mathfrak{S}_0(A) \times \{0, 1\}$
4. (SSW) $e_i \geq 2 \implies |\mathfrak{S}_0((k \ltimes k^{e_1}) \otimes_k \cdots \otimes_k (k \ltimes k^{e_n}))| = 2^n$

Fiber Products: $(R, \mathfrak{m}, k)$ and $(S, \mathfrak{n}, k)$ are local rings

Definition ($R \xrightarrow{\tau_R} k \xleftarrow{\tau_S} S$ are the canonical projections)

$$\{(r,s) \in R \times S \mid \tau_R(r) = \tau_S(s) \in k\} = R \times_k S \to R$$

$$\begin{array}{ccc} R \times_k S & \to & R \\ \downarrow & & \downarrow \tau_R \\ S & \xrightarrow{\tau_S} & k \end{array}$$

Fact ($R \times_k S$ is a local ring with $\mathfrak{m}_{R \times_k S} = \mathfrak{m} \oplus \mathfrak{n}$)

1. $\dim(R \times_k S) = \max\{\dim(R), \dim(S)\}$
2. $\operatorname{depth}(R \times_k S) = \min\{\operatorname{depth}(R), \operatorname{depth}(S), 1\}$
3. *$R \times_k S$ is Cohen-Macaulay iff R, S are Cohen-Macaulay with* $\dim(R) = \dim(S) \leq 1$.
4. $|\mathfrak{S}_0(R \times_k S)| \leq 2$. *(Nasseh-SSW)*

Example

$$k[\![X]\!] \times_k k[\![Y]\!] \cong \frac{k[\![X, Y]\!]}{(XY)} \qquad \frac{k[\![\underline{X}]\!]}{I} \times_k k[\![T]\!] \cong \frac{k[\![\underline{X}, T]\!]}{(I, \underline{X}T)}$$

The Example

$$S = (k \ltimes k^2)^{\otimes_k^n} \cong \frac{k[\![X_1, Y_1, \ldots, X_n, Y_n]\!]}{(X_1, Y_1)^2 + \cdots + (X_n, Y_n)^2}$$

$$R = S[\![Z]\!] \times_k k[\![T]\!] \qquad \text{CM dim 1}$$

$$\cong \frac{k[\![X_1, Y_1, \ldots, X_n, Y_n, Z]\!]}{(X_1, Y_1)^2 + \cdots + (X_n, Y_n)^2} \times_k k[\![T]\!]$$

$$\cong \frac{k[\![X_1, Y_1, \ldots, X_n, Y_n, Z, T]\!]}{(X_1, Y_1)^2 + \cdots + (X_n, Y_n)^2 + (X_1T, Y_1T, \ldots, X_nT, Y_nT, ZT)}$$

Then $\mathfrak{S}(R) = \{R, D^R\}$. Set $\mathfrak{p} = (X_1, Y_1, \ldots, X_n, Y_n, T)R$.

$$R_{\mathfrak{p}} \cong \frac{k[\![X_1, Y_1, \ldots, X_n, Y_n, Z, T]\!]_{(X_1, Y_1, \ldots, X_n, Y_n, T)}}{(X_1, Y_1)^2 + \cdots + (X_n, Y_n)^2 + (X_1T, Y_1T, \ldots, X_nT, Y_nT, ZT)}$$

$$\cong \frac{k[\![X_1, Y_1, \ldots, X_n, Y_n, Z]\!]_{(X_1, Y_1, \ldots, X_n, Y_n)}}{(X_1, Y_1)^2 + \cdots + (X_n, Y_n)^2}$$

This is a flat local S-algebra, so $|\mathfrak{S}_0(R_{\mathfrak{p}})| \geq |\mathfrak{S}_0(S)| = 2^n$.

Questions

1. (Dao) Does there exist a 2-dimensional complete local normal domain R such that $|\mathfrak{S}_0(R)| > 2$?
2. Does there exists a complete local CM normal domain R with a prime ideal $\mathfrak{p}$ such that $|\mathfrak{S}_0(R_\mathfrak{p})| > |\mathfrak{S}_0(R)|$?
3. If R is CM local, then $|\mathfrak{S}_0(R)| = 2^n$ for some $n \in \mathbb{N}$?