Finite Generation of Ext and ascent of module structures

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Motivation: A Result of Buchweitz and Flenner

Assumption. $(R, \mathfrak{m}, k)$ is a (commutative noetherian) local ring.

Theorem. (Buchweitz, Flenner 2006) *If M is a complete R-module, then $\operatorname{Ext}^i_R(F, M) = 0$ for all flat R-modules F and all $i \geqslant 1$, that is, M is cotorsion.*

Question. What about the converse?

Note. Converse fails without extra assumptions.

Example. If I is an injective R-module, then $\operatorname{Ext}^i_R(F, M) = 0$ for all R-modules F and all $i \geqslant 1$, but I is almost never complete.

Question. What about the converse when M is finitely generated?

Previous Results

Theorem. (Frankild, Sather-Wagstaff 2008) *Let $\mathfrak{a}$ be a proper ideal of R, and let M be a finitely generated R-module. TFAE:*

1. *M is $\mathfrak{a}$-adically complete;*
2. $\mathrm{Ext}^i_R(\widehat{R}^{\mathfrak{a}}, M) = 0$ *for all* $i \geqslant 1$*; and*
3. *M has an $\widehat{R}^{\mathfrak{a}}$-module structure compatible with its R-module structure via the natural map $R \to \widehat{R}^{\mathfrak{a}}$.*

Note. Uses big guns: derived local (co)homology.

Note. Originally motivated by questions about G-dimensions.

Question. What about other R-algebras, like R^h?

Previous Results

Let M be a finitely generated R-module, and let $\varphi\colon R \to S$ be a flat local ring homomorphism such that $R/\mathfrak{m} \xrightarrow{\cong} S/\mathfrak{m}S$.

Theorem. (Frankild, Sather-Wagstaff, R. Wiegand 2008) *TFAE:*

1. *M has an S-module structure compatible with its R-module structure via φ;*
2. $\operatorname{Ext}^i_R(S, M) = 0$ *for all* $i \geqslant 1$*;*
3. $\operatorname{Ext}^i_R(S, M)$ *is finitely generated over R for all* $i \geqslant 1$*;*
4. *The map* $\operatorname{Hom}_R(S, M) \to M$ *taking* ψ *to* $\psi(1)$ *is bijective;*
5. *The map* $M \to S \otimes_R M$ *taking* m *to* $1 \otimes m$ *is bijective; and*
6. *$S \otimes_R M$ is finitely generated as an R-module.*

Note. Proof is easier, using Koszul complex, though it uses the Amplitude Inequality of Foxby and Iyengar (and Iversen) which is a consequence of the New Intersection Theorem.

Previous Results

Let M be a finitely generated R-module, and let $\varphi\colon R \to S$ be a flat local ring homomorphism such that $R/\mathfrak{m} \xrightarrow{\cong} S/\mathfrak{m}S$.

Lemma. (Frankild, Sather-Wagstaff, R. Wiegand 2008) *The map* $\operatorname{Hom}_R(S, M) \to M$ *taking* ψ *to* $\psi(1)$ *is injective.*

Theorem. (Christensen, Sather-Wagstaff 2010) *If R is Gorenstein and* $\operatorname{Ext}^i_R(S, M)$ *is finitely generated over S for all* $i \geqslant 1$*, then* $\operatorname{Ext}^i_R(S, M) = 0$ *for all* $i \geqslant 1$ *and M has an S-module structure compatible with its R-module structure via* φ*.*

Note. This is a corollary of a result about G-dimensions.

Question. Is the Gorenstein assumption necessary?

New Results

Let M be a finitely generated R-module, and let $\varphi\colon R \to S$ be a flat local ring homomorphism such that $R/\mathfrak{m} \xrightarrow{\cong} S/\mathfrak{m}S$.

Theorem. (Anderson, Coykendall, Sather-Wagstaff 2010) *If* $\operatorname{Ext}^i_R(S, M)$ *satisfies Nakayama's Lemma (e.g., if* $\operatorname{Ext}^i_R(S, M)$ *is finitely generated over S) for all* $i \geqslant 1$*, then* $\operatorname{Ext}^i_R(S, M) = 0$ *for all* $i \geqslant 1$ *and* M *has an* S*-module structure compatible with its* R*-module structure via* φ*.*

Definition. An R-module N *satisfies Nakayama's Lemma* if either $N = 0$ or $N/\mathfrak{m}N \neq 0$.

Note. Proof is easier, using only basic properties of the Koszul complex.

New Results

Let M be an R-module, and let $\varphi\colon R \to S$ be a flat local ring homomorphism such that $R/\mathfrak{m} \xrightarrow{\cong} S/\mathfrak{m}S$.

Lemma. (Anderson, Coykendall, Sather-Wagstaff 2010) *If M satisfies Krull's Intersection Theorem, then the map* $\operatorname{Hom}_R(S, M) \to M$ *taking ψ to $\psi(1)$ is injective.*

Definition. An R-module N *satisfies Krull's Intersection Theorem* if $\cap_{i=1}^{\infty} \mathfrak{m}^i N = 0$.

Example. Let k be a field, and consider the rings $R = k[X]_{(X)}$ and $S = \widehat{R} \cong k[\![X]\!]$. Then $\operatorname{Ext}^i_R(S, R) = 0$ for all $i \neq 1$, but $\operatorname{Ext}^1_R(S, R) \cong \operatorname{Hom}_R(\widehat{R}, E) \cong E \oplus k(\!(X)\!)^{(\mu)}$ for some $\mu \neq 0$.