Semidualizing modules

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Throughout, $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring with canonical module $\omega = \omega_R$.

Definition. A finitely generated R-module C is *semidualizing* if

(a) The natural homothety homomorphism

$$\chi_C^R \colon R \to \mathrm{Hom}_R(C, C)$$

is an isomorphism, and

(b) $\mathrm{Ext}_R^i(C, C) = 0$ for each $i \neq 0$.

Notation. $\mathfrak{S}_0(R)$ is the set of isomorphism classes of semidualizing R-modules.

Example. R and ω are semidualizing.

Example. R is Gorenstein if and only if $\mathfrak{S}_0(R) = \{[R]\}$.

Example. If R has minimal multiplicity, e.g., if $\mathfrak{m}^2 = 0$, then $\mathfrak{S}(R) = \{[R], [\omega]\}$.

Example. Let $\varphi\colon Q \to R$ be a finite local homomorphism of finite projective dimension and ω_Q a canonical module for Q. E.g.,

$$R \cong Q \ltimes Q^r \cong Q[X_1, \ldots, X_r]/(X_1, \ldots, X_r)^2$$

If $d = \operatorname{depth} Q - \operatorname{depth} R$, then $\mathfrak{S}_0(R)$ contains the following:

$$\omega_R \cong \operatorname{Ext}^d_Q(R, \omega_Q) \qquad\qquad \omega_Q \otimes_Q R$$

$$\omega_\varphi := \operatorname{Ext}^d_Q(R, Q) \qquad\qquad R \cong Q \otimes_Q R$$

Problem. Give a complete description of $\mathfrak{S}_0(R)$.

Question. Is $\mathfrak{S}(R)$ finite?

Theorem. $\mathfrak{S}_0(R)$ *has the structure of a nontrivial metric space.*

Theorem. *The following conditions are equivalent.*
(i) R *is Gorenstein;*
(ii) *there exists a semidualizing* R*-module* C *with* $C \cong \operatorname{Hom}_R(C, \omega)$*;*
(iii) $\mathfrak{S}_0(R)$ *is a finite set with odd cardinality.*

Assume for the remainder that R is a normal domain.

Proposition. *Each semidualizing R-module is reflexive with rank 1. Thus, there is a natural inclusion $\mathfrak{S}_0(R) \subseteq \mathrm{Cl}(R)$.*

Corollary. *If $\mathrm{Cl}(R)$ is finite, then so is $\mathfrak{S}_0(R)$.*

Example. Let k be a field and X a symmetric $n \times n$ matrix of variables. With $0 < r < n$, set $R = k[X]_{(X)}/I_{r+1}(X)$. Then R is a Cohen-Macaulay normal local domain with $\mathrm{Cl}(R) \cong \mathbb{Z}/(2)$. Furthermore, R is non-Gorenstein if and only if $n \equiv r \pmod 2$. That is, $\mathfrak{S}_0(R) = \{[R], [\omega]\}$ and

(a) $\#\mathfrak{S}_0(R) = 2$ iff $n \equiv r \pmod 2$;

(b) $\#\mathfrak{S}_0(R) = 1$ iff $n \not\equiv r \pmod 2$.

Theorem. *Let X be an $m \times n$ matrix of variables*