Local rings of embedding codepth at most 3 have only trivial semidualizing complexes

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Semidualizing modules

Assumption

$(R, \mathfrak{m}, k)$ is a commutative noetherian local ring.

Definition (Foxby 1972, Grothendieck 1961)

(a) A finitely generated R-module C is semidualizing if $R \cong \operatorname{Hom}_R(C, C)$ and $\operatorname{Ext}_R^{\geqslant 1}(C, C) = 0$.

(b) A semidualizing module of finite inj. dimension is dualizing.

(c) $\mathfrak{S}_0(R)$ is the set of isomorphism classes of SDMs.

Fact

(a) *R is a semidualizing R-module.*

(b) *R is a dualizing R-module if and only if R is Gorenstein.*

(c) *R has a dualizing module if and only if it is Cohen-Macaulay and a homomorphic image of a Gorenstein ring.*

Applications and examples of semidualizing modules

Remark

Semidualizing modules are useful.

(a) Bass series of local ring homomorphisms.

(b) Composition of ring homomorphisms of finite G-dimension.

(c) Growth of Bass numbers of R.

(d) Structure of quasi-deformations.

Example

$R_1 = k[\![X, Y]\!]/(X, Y)^2$ with dualizing module D_1.

$R_2 = k[\![Z, W]\!]/(Z, W)^2$ with dualizing module D_2.

$$R = R_1 \otimes_k R_2 \cong \frac{k[\![X, Y, Z, W]\!]}{(X, Y)^2 + (Z, W)^2}$$

SDMs: $R \qquad D_1 \otimes_k R_2 \qquad R_1 \otimes_k D_2 \qquad D = D_1 \otimes_k D_2$

Smaller rings

Fact

Assume that R is artinian with $e = \operatorname{edim}(R)$.

(a) *If $e \leq 1$, then R is CI, so the only SDM is R.*

(b) *If $e = 2$, then R is Golod, so the only SDMs are R and D.*

(c) *The previous example has $e = 4$ with four distinct SDMs.*

Sketch of proof of (b).

Let C be a semidualizing R-module.

Gerko: $\operatorname{Hom}_R(C, D)$ is SDM and Tor-independent with C.

D. Jorgensen: $\operatorname{Hom}_R(C, D)$ or C has finite projective dimension.

Therefore, $C \cong D$ or $C \cong R$. □

Question

What if $e = 3$?

Embedding codepth 3

Theorem (Nasseh-SW)

Assume that $e = \operatorname{edim}(R) - \operatorname{depth}(R) \leq 3$. *Then* R *has at most two SDMs, namely* R *and a dualizing module if* R *has one.*

Sketch of proof

The completion $\widehat{R}$ is a homomorphic image of a regular local ring Q with $\operatorname{edim}(Q) = \operatorname{edim}(R)$.

$$R \longrightarrow \widehat{R} \longrightarrow K^{\widehat{R}} \cong \widehat{R} \otimes_Q K^Q \simeq F \otimes_Q K^Q \simeq F \otimes_Q k$$

$$\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}_0(\widehat{R}) \hookrightarrow \mathfrak{S}(K^{\widehat{R}}) \xrightarrow{\quad\cong\quad} \mathfrak{S}(F \otimes_Q k)$$

$K^{\widehat{R}}$ has the structure of a DG $\widehat{R}$-algebra.

Auslander-Buchsbaum: The minimal free resolution F of $\widehat{R}$ over Q has length ≤ 3.

Buchsbaum-Eisenbud: F is a DG Q-algebra. □

DG algebras and DG modules

Example (Koszul complex as DG algebra)

Let $\mathbf{x} = x_1, \ldots, x_n \in R$, and set $K = K^R(\mathbf{x})$. The exterior algebra $\wedge R^n \cong \oplus_{i=0}^n K_n$ is a graded commutative R-algebra. With this multiplication, K satisfies the Leibniz rule:

$$\partial^K(ab) = \partial^K(a)b + (-1)^{|a|}a\partial^K(b).$$

Definition

A DG R-algebra is a non-negatively graded R-complex A such that $A^\natural = \oplus_i A_i$ is a graded commutative R-algebra and with this multiplication A satisfies the Leibniz rule.

A DG A-module is an R-complex M such that $\oplus_i M_i$ is a graded $A^\natural$-module and M satisfies the Leibniz rule.

Example

R is a DG R-algebra. DG R-modules are just R-complexes.

Semidualizing DG modules

Definition (Christensen-SW 2009)

Let A be a DG R-algebral A homologically finite DG A-module C is semidualizing if $A \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_A(C, C)$.

Example

"Semidualizing DG R-module" = "semidualizing R-complex".

Theorem (Nasseh-SW)

Let B be a finite-dimensional DG k-algebra, and let $W \neq 0$ be a positively graded finite dimensional k-vector space. Consider the trivial extension $A = B \ltimes W$. Given two homologically finite DG A-modules M and N, if $\mathrm{Tor}^A_{\gg 0}(M, N) = 0$, then either M or N has finite projective dimension over A.

Conclusion of proof

$|\mathfrak{S}(F \otimes_Q k)| \leq 2$.

Set $A = F \otimes_Q k$.

If R is Gorenstein, then so is A, hence $|\mathfrak{S}(A)| = 1$.

If $A \cong B \ltimes W$ for a positively graded vector space $W \neq 0$, argue as in the Golod case to conclude that $|\mathfrak{S}(A)| \leq 2$.

Weyman and Avramov-Kustin-Miller: the only case that remains is when there is a positively graded finite dimensional k-vector space $V \neq 0$ such that

$$A \cong (k \ltimes V) \otimes_k (k \ltimes \Sigma k) \cong (k \ltimes V) \otimes_k K^k(0) \cong K^{k \ltimes V}(0).$$

$$|\mathfrak{S}(A)| = |\mathfrak{S}(K^{k \ltimes V}(0))| = |\mathfrak{S}(k \ltimes V)| \leq 2.$$

By a DG version of Auslander-Ding-Solberg's lifting theorem.

By the previous theorem. □