Ext^1: What is it good for?

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Introduction

Goals

1. Give some indication of how various research areas think about modules.
2. Give an example of how algebraic information is encoded geometrically and how some geometric information is encoded algebraically.

Cast of Characters

1. Rings and Modules
2. Homological Algebra
3. Linear Algebra
4. Representation Theory
5. Group Theory
6. Algebraic Geometry
7. Algebraic Topology $*$

Rings and Modules

Assumption

Let R be a non-zero commutative ring with identity.

Example

- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}/p\mathbb{Z}$
- Polynomial rings $A[x_1, \ldots, x_n]$ and quotients $A[x_1, \ldots, x_n]/I$

Slogan

To study a ring, study its modules.

Example

Vector space and abelian groups

Fact

Every R-module has a basis if and only if R is a field.

Homological Algebra, First View

Definition

A sequence of R-module homomorphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact if $\mathrm{Ker}(\beta) = \mathrm{Im}(\alpha)$.

Fact

Given an R-module L and a short exact sequence

$$0 \to N' \to N \to N'' \to 0$$

the induced sequence is exact:

$$0 \to \mathrm{Hom}(L, N') \to \mathrm{Hom}(L, N) \to \mathrm{Hom}(L, N'') \to \mathrm{Ext}^1(L, N') \cdots$$

Slogan

Ext measures the "exactness defect" on the right.

Linear Algebra

Assumption

R is a finite dimensional algebra over a field $F = \overline{F}$.

Facts

1. Every R-module has a "canonical" F-vector space structure by "restriction of scalars".
2. Every non-zero F-vector space has many distinct R-module structures.

Example

Let $R = F[x]/(x(x-1))$ and $M = R/xR$ and $N = R/(x-1)R$. Then M and N are isomorphic over F, but not over R.

Homological Algebra, Second View

Definition

A short exact sequence $0 \to N' \xrightarrow{f'} N \xrightarrow{f} N'' \to 0$ of R-module homomorphisms splits if there is an R-module homomorphism $g\colon N \to N'$ such that $g \circ f' = 1_{N'}$.

Facts

1. Every short exact sequence $0 \to N' \to N \to N'' \to 0$ of R-modules splits over F, but not necessarily over R.
2. If $\operatorname{Ext}^1_R(N'', N') = 0$, then every short exact sequence $0 \to N' \to N \to N'' \to 0$ splits over R.
3. $\operatorname{Ext}^1_R(N'', N')$ is (isomorphic to) the set of equivalence classes of short exact sequences $0 \to N' \to N \to N'' \to 0$.

Slogan

Ext parameterizes “extensions”.

Representation Theory

Slogan

To study R-modules, fix $V = F^n$ and study all the ways to make V into an R-module.

Recall

An R-module structure on V is a bilinear map $m\colon R \times V \to V$ satisfying certain axioms (associative and unital).

Note

$R \times V$ and V are F-vector spaces, but m is not a linear transformation. This is one reason why we need tensor products.

More Linear Algebra

Construction

Let $b_1, \ldots, b_d \in R$ and $e_1, \ldots, e_n \in V$ be bases over F. The tensor product $R \otimes_F V$ is the vector space over F with basis $b_1 \otimes e_1, \ldots, b_d \otimes e_n$.

Facts

1. The bilinear maps $R \times V \to V$ are in bijection with the linear maps $R \otimes_F V \to V$, i.e., the $n \times dn$ matrices over F.
2. An R-module structure on V is a matrix in $\mathrm{M}_{n \times dn}(F)$ satisfying certain axioms (associative and unital).
3. Given variables x_{ij} to represent the entries of a matrix in $\mathrm{M}_{n \times dn}(F)$, the R-module axioms are characterized by polynomial equations in the x_{ij}.

Notation

$\mathrm{Mod}_R(V) \subseteq \mathrm{M}_{n \times dn}(F)$ is the set of R-module structures on V. $*$

Group Theory

Question

When are two module structures in $\mathrm{Mod}_R(V)$ isomorphic?

Notation

$\mathrm{GL}_n(F)$ is the set of invertible $n \times n$ matrices over F.

Facts

1. $\mathrm{GL}_n(F)$ acts on $\mathrm{Mod}_R(V)$ by conjugation: Given $\phi \in \mathrm{GL}_n(F)$ and $\mu \in \mathrm{Mod}_R(V)$, set
$$\phi \cdot \mu = \phi \circ \mu \circ (R \otimes_F \phi^{-1}).$$
2. Two module structures $\mu, \lambda \in \mathrm{Mod}_R(V)$ are isomorphic over R if and only if $\lambda = \phi \cdot \mu$ for some $\phi \in \mathrm{GL}_n(F)$.
3. The isomorphism classes in $\mathrm{Mod}_R(V)$ are precisely the orbits under the action of $\mathrm{GL}_n(F)$. $*$

Facts

1. The R-module axioms are defined by polynomial equations so $\mathrm{Mod}_R(V)$ is a closed subset of $\mathrm{M}_{n\times dn}(F) \cong F^{dn^2}$.
2. This is the definition of “closed” in the Zariski topology.
3. If $F = \mathbb{C}$, then it is closed in the euclidean topology.
4. $\mathrm{GL}_n(F)$ is an open subset of $\mathrm{M}_{n\times n}(F) \cong F^{n^2}$, and isomorphic to a closed subset of F^{n^2+1}.
5. The group operations in $\mathrm{GL}_n(F)$ and the action of $\mathrm{GL}_n(F)$ on $\mathrm{Mod}_R(V)$ are defined by polynomial functions.
6. Each orbit in $\mathrm{Mod}_R(V)$ is locally closed.
7. For $M \in \mathrm{Mod}_R(V)$, there is an inclusion of tangent spaces

$$\mathrm{T}_M^{\mathrm{GL}_n(F)\cdot M} \subseteq \mathrm{T}_M^{\mathrm{Mod}_R(V)}.*$$

Homological Algebra, Third View

Theorem

Given $M \in \mathrm{Mod}_R(V)$, there is an isomorphism

$$\mathrm{T}_M^{\mathrm{Mod}_R(V)} \,/\, \mathrm{T}_M^{\mathrm{GL}_n(F)\cdot M} \cong \mathrm{Ext}_R^1(M,M).*$$

Corollary

Given $M \in \mathrm{Mod}_R(V)$, the orbit $\mathrm{GL}_n(F) \cdot M$ is open in $\mathrm{Mod}_R(V)$ if and only if $\mathrm{Ext}_R^1(M,M) = 0$.

Corollary

The set of isomorphism classes of R-modules M such that $\mathrm{Hom}_R(M,M) \cong R$ and $\mathrm{Ext}_R^1(M,M) = 0$ is finite.

Question

How to prove the second corollary for rings that are not finite dimensional algebras over a field?

Answer

When R is local, replace R with an appropriate finite dimensional differential graded (DG) F-algebra U:

1. U is a a graded commutative F-algebra $U = \oplus_{i=0}^{e} U_i$,
2. U has a differential, i.e., a sequence of R-linear maps $\partial_i^U \colon U_i \to U_{i-1}$ such that $\partial_i^U \partial_{i+1}^U = 0$ for all i, and
3. ∂^U satisfies the Leibniz Rule: for all $a_i \in U_i$ and $a_j \in U_j$

$$\partial_{i+j}^U(a_i a_j) = \partial_i^U(a_i)a_j + (-1)^i a_i \partial_j^U(a_j).$$

Note

The starting point for this replacement is to take the Koszul complex on a minimal generating sequence for the maximal ideal $\mathfrak{m} \subset R$.

Algebraic Topology, cont.

Solution

1. One needs to work with DG U-modules: U-modules with extra data (a differential that satisfies the Leibniz Rule), and one has to encode the extra data into the geometric object $\mathrm{DGMod}_U(V)$.
2. One has to consider a product of GL's for the group action.
3. The quotient of tangent spaces is still isomorphic to an Ext-module, but it is the wrong Ext-module.
4. There are two distinct kinds of Ext over U!
   DG-Ext corresponds to $\mathrm{Ext}^1_R(M, M)$ under passage to U.
   Yoneda-Ext parametrizes extensions.
   They are not generally the same.
5. With a little work one can adjust things so that DG Ext-vanishing implies Yoneda Ext-vanishing, and the rest of the proof goes through. $*$

Conclusion

Remarks

1. Algebra does not exist in an algebraic vacuum.
2. Linear algebra is important.
3. Group actions are not only useful for the Algebra prelim.
4. Geometry can encode algebraic information.
5. Algebra can encode geometric information.
6. Sometime to prove a theorem about rings, you have to be flexible about your definition of “ring”.