Reflexivity and connectedness

Sean Sather-Wagstaff

North Dakota State University

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Semidualizing modules and Bass classes

Assumption

R is a commutative noetherian ring with identity.

Definition (Foxby '72, Vasconcelos '74)

A finitely generated R-module C is semidualizing if

$$R \xrightarrow{\cong} \operatorname{Hom}_R(C, C) \qquad \text{and} \qquad \operatorname{Ext}_R^{\geqslant 1}(C, C) = 0.$$

Fact

1. *R is semidualizing.*
2. *D is dualizing iff it is semidualizing and* $\operatorname{id}_R(D) < \infty$.
3. *The semidualizing property is local: TFAE:*
    1. *U is semidualizing over R.*
    2. *$U_{\mathfrak{p}}$ is semidualizing over $R_{\mathfrak{p}}$ for all prime $\mathfrak{p}$.*
    3. *$U_{\mathfrak{m}}$ is semidualizing over $R_{\mathfrak{m}}$ for all maximal $\mathfrak{m}$.*

Bass classes

Definition (Foxby '94, Christensen '01)

Let U be a finitely generated R-module. The Bass class with respect to U is the class $\mathcal{B}_U(R)$ of all R-modules M such that

$$U \otimes_R \operatorname{Hom}_R(U, M) \xrightarrow{\cong} M \qquad \text{and}$$

$$\operatorname{Ext}_R^{\geqslant 1}(U, M) = 0 = \operatorname{Tor}^R_{\geqslant 1}(U, \operatorname{Hom}_R(U, M)).$$

Fact

Let U be a finitely generated R-module. TFAE.

1. *U is semidualizing.*
2. *Every R-module N with $\operatorname{id}_R(N) < \infty$ is in $\mathcal{B}_U(R)$.*
3. *$\mathcal{B}_U(R)$ contains a faithfully injective R-module.*

Bass classes, cont.

Fact

Let U be a finitely generated R-module.

1. $0 \in \mathcal{B}_U(R)$.
2. *If C is semidualizing, then $C \in \mathcal{B}_C(R)$.*
3. *Membership in $\mathcal{B}_U(R)$ is a local property: TFAE:*
    1. $N \in \mathcal{B}_U(R)$.
    2. $N_{\mathfrak{p}} \in \mathcal{B}_{U_{\mathfrak{p}}}(R_{\mathfrak{p}})$ *for all prime* $\mathfrak{p}$.
    3. $N_{\mathfrak{m}} \in \mathcal{B}_{U_{\mathfrak{m}}}(R_{\mathfrak{m}})$ *for all maximal* $\mathfrak{m}$.

Example

If U is a finitely generated R-module such that $0 \neq U \in \mathcal{B}_U(R)$, then U may not be semidualizing.
Let $R = k \times k$ and $U = k \times 0$. Then $0 \neq U \in \mathcal{B}_U(R)$, but U is not semidualizing.

Bass class vs. semidualizing - the local case

Proposition (SSW)

If R is local and U is a finitely generated R-module such that $0 \neq U \in \mathcal{B}_U(R)$, then U must be semidualizing.

Sketch of proof

$U \in \mathcal{B}_U(R)$ implies that $\operatorname{Ext}_R^{\geqslant 1}(U, U) = 0$.

It remains to show that $\chi\colon R \xrightarrow{\cong} \operatorname{Hom}_R(U, U)$.

$$\begin{array}{ccc} U \otimes_R R & & \\ {\scriptstyle U\otimes_R\chi}\downarrow & \searrow^{\cong} & \\ U \otimes_R \operatorname{Hom}_R(U, U) & \xrightarrow[\cong]{} & U \end{array}$$

$U \otimes_R \operatorname{Coker}(\chi) = 0$, so Nakayama's Lemma implies χ is onto.

$0 \to \operatorname{Ker}(\chi) \to R \to \operatorname{Hom}_R(U, U) \to 0$ exact.

Long exact sequence in $\operatorname{Tor}^R(U, -)$ implies $\operatorname{Tor}^R(U, \operatorname{Ker}(\chi)) = 0$.

So χ is 1-1. □

Bass class vs. semidualizing - the general case

Theorem (SSW)

Let U be a finitely generated R-module s.t. $0 \neq U \in \mathcal{B}_U(R)$, but U is not semidualizing.

1. *There are non-zero R_1, R_2 such that $R \cong R_1 \times R_2$.*
2. *There is a semidualizing R_1-module C_1 s.t. $U \cong C_1 \times 0$.*

Sketch of proof

It suffices to show that $\operatorname{Supp}_R(U)$ is Zariski open in $\operatorname{Spec}(R)$. Since the Bass class condition is local, the Proposition implies

$$\operatorname{Supp}_R(U) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid U_\mathfrak{p} \text{ is semidualizing for } R_\mathfrak{p}\}.$$

The exact sequence

$$0 \to \operatorname{Ker}(\chi) \to R \to \operatorname{Hom}_R(U, U) \to \operatorname{Coker}(\chi) \to 0$$

implies that

$$\operatorname{Supp}_R(U) = \operatorname{Spec}(R) \smallsetminus (\operatorname{Supp}_R(\operatorname{Ker}(\chi)) \cup \operatorname{Supp}_R(\operatorname{Coker}(\chi)))$$

which is open. □

Total reflexivity

Definition

Let G and U be f.g. R-modules. Then G is totally U-reflexive if

$$G \xrightarrow{\cong} \mathrm{Hom}_R(\mathrm{Hom}_R(G, U), U) \qquad \text{and}$$

$$\mathrm{Ext}_R^{\geqslant 1}(G, U) = 0 = \mathrm{Ext}_R^{\geqslant 1}(\mathrm{Hom}_R(G, U), U).$$

Fact

Let U be a finitely generated R-module.

1. 0 *is totally U-reflexive.*
2. *If C is semidualizing, then C is totally C-reflexive.*
3. *Being totally U-reflexive is a local property.*

Example

Let $R = k \times k$ and $U = k \times 0$. Then $U \neq 0$ is totally U-reflexive, but U is not semidualizing.

Total reflexivity vs. semidualizing

Proposition (SSW)

If R is local and $U \neq 0$ is a finitely generated R-module that is totally U-reflexive, then U must be semidualizing.

Theorem (SSW)

Let $U \neq 0$ be a finitely generated R-module that is totally U-reflexive but not semidualizing.

1. *There are non-zero R_1, R_2 such that $R \cong R_1 \times R_2$.*
2. *There is a semidualizing R_1-module C_1 s.t. $U \cong C_1 \times 0$.*

A derived version

Theorem (SSW)

Let U be a homologically finite R-complex. TFAE:

1. $0 \not\simeq U \in \mathcal{B}_U(R)$ *and U is not semidualizing.*
2. *U is derived U-reflexive and not semidualizing and* $U \not\simeq 0$.
3. *There are non-zero R_1, R_2 such that $R \cong R_1 \times R_2$, and there is a semidualizing R_1-complex C_1 s.t.* $U \simeq C_1 \times 0$.