Semidualizing Modules and Bass Numbers

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Themes

Throughout this talk $(R, \mathfrak{m}, k)$ is a commutative local artinian ring with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

Theme. Module theory is representation theory for rings. Ring-theoretic information about R determines and is determined by information about the R-modules.

Example. The ring R is a field if and only if every R-module is free.

Theme. Patterns in homological invariants (numbers associated to R) yield information about R.

Theme. Existence of nontrivial R-modules can force patterns to occur in homological invariants.

Gorenstein Rings

Fact. The R-module R is always free, hence projective.

Fact. The R-module R is frequently not injective.

Definition. The ring R is Gorenstein if it is self-injective, that is, if it is injective as an R-module.

Example. Let k be a field.

(a) The ring $k[\![X]\!]/(X^m)$ is Gorenstein, as are the rings $k[\![X_1,\ldots,X_n]\!](X_1^{m_1},\ldots,X_n^{m_n})$.

(b) The ring $k[X,Y]/(X^2,XY,Y^2)$ is not Gorenstein.

Fact. If R is Gorenstein, then $\mathrm{Ext}_R^i(M,R)=0$ for every R-module M and every integer $i\geq 1$.

If $\mathrm{Ext}_R^i(k,R)=0$ for some $i\geq 1$, then R is Gorenstein.

Bass Numbers

Definition. The ith Bass number of R is the integer $\mu^i = \mu^i_R(R) = \dim_k(\mathrm{Ext}^i_R(k, R))$.

Fact. If $\mu^i = 0$ for some $i \geq 1$, then R is Gorenstein and hence $\mu^i = 0$ for all $i \geq 1$.

Theme. Patterns in homological invariants (Bass numbers) yield information about R.

Questions. (Huneke)

(a) If the sequence $\{\mu^i\}_i$ is bounded, must R be Gorenstein?

(b) If the sequence $\{\mu^i\}_i$ is bounded by a polynomial in i, must R be Gorenstein?

Recent progress by Jorgensen and Leuschke; Christensen, Striuli and Veliche; Lorestani, Sather-Wagstaff and Yassemi.

Fact. Since R is artinian and local, there exists a unique indecomposable injective R-module $E = E_R(k)$.

We have $\mathrm{Hom}_R(E, E) \cong R$ and $\mathrm{Ext}^i_R(E, E) = 0$ for all $i \geq 1$.

There exists an exact sequence

$$0 \to R \to E^{\mu^0} \to E^{\mu^1} \to E^{\mu^2} \to \cdots$$

called a minimal injective resolution of R.

Apply $\mathrm{Hom}_R(-, E)$ to find an exact sequence

$$\cdots \to R^{\mu^2} \to R^{\mu^1} \to R^{\mu^0} \to E \to 0$$

which is a minimal free resolution of E.

So, the Bass numbers of R are the Betti numbers of E.

Semidualizing Modules

In this setting E is a dualizing module for R, so-named because of its good duality properties: For every finitely generated R-module M, we have $M \cong \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)$.

Definition. Let C be an R-module. The homothety morphism associated to C is the map

$$\chi_C^R \colon R \to \mathrm{Hom}_R(C, C)$$

given by $\chi_C^R(r)(c) = rc$.

Definition. (Foxby '72, Golod '84, Vasconcelos '74, Wakamatsu '88) An R-module C is semidualizing if:

(1) C is finitely generated,
(2) $\chi_C^R \colon R \to \mathrm{Hom}_R(C, C)$ is an isomorphism; and
(3) $\mathrm{Ext}_R^i(C, C) = 0$ for all $i \geq 1$.

Example. The R-modules R and E are semidualizing.

Semidualizing Modules, Properties

Definition. An R-module C is semidualizing if:

(1) C is finitely generated,
(2) $\chi^R_C\colon R \to \operatorname{Hom}_R(C, C)$ is an isomorphism; and
(3) $\operatorname{Ext}^i_R(C, C) = 0$ for all $i \geq 1$.

Example. One has $C \cong E$ if and only if C is semidualizing and injective (equivalently, has finite injective dimension).

One has $C \cong R$ if and only if C is semidualizing and projective (equivalently, has finite projective dimension).

In other words, the only tilting R-module is R, and the only cotilting R-module is E.

Fact. The following conditions are equivalent:

(i) R is Gorenstein;
(ii) $R \cong E$; and
(iii) R has only one semidualizing module, up to isomorphism.

Semidualizing Modules, Examples

Theme. Module theory is representation theory for rings. Ring-theoretic information about R determines and is determined by information about the R-modules.

Example. Let k be a field.

The ring $R = k[\![X_1, \ldots, X_n]\!](X_1^{m_1}, \ldots, X_n^{m_n})$ has only one semidualizing module, up to isomorphism: R.

The ring $R = k[X, Y]/(X^2, XY, Y^2)$ has exactly two non-isomorphic semidualizing modules: R and E.

The ring $S = k[X, Y, Z, W]/(X^2, XY, Y^2, Z^2, ZW, W^2)$ has exactly four non-isomorphic semidualizing modules.

Problem. Characterize the local artinian rings with exactly two non-isomorphic semidualizing modules.

Theorem. (SSW 2008) *If R has a semidualizing module C such that $C \not\cong R$ and $C \not\cong E$, then the sequence $\{\mu^i\}_i$ is unbounded.*

Theme. Existence of nontrivial R-modules can force patterns to occur in homological invariants.

Proof. Consider minimal free resolutions

$$\cdots \to R^{a_2} \to R^{a_1} \to R^{a_0} \to C \to 0$$

$$\cdots \to R^{b_2} \to R^{b_1} \to R^{b_0} \to \operatorname{Hom}_R(C,E) \to 0.$$

Since $C \not\cong R$ and $C \not\cong E$, we have $a_i, b_i \geq 1$ for all $i \geq 0$.

Since C is semidualizing, we have $E \cong C \otimes_R \operatorname{Hom}_R(C,E)$ and $\operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C,E)) = 0$ for $i \geq 1$.

Hence, the minimal free resolution of E is the tensor product of the minimal free resolutions of C and $\operatorname{Hom}_R(C,E)$.

It follows that $\mu^i = \sum_{j=0}^{i} a_j b_i \geq i$. □

Chains of semidualizing modules

Definition. Let C be a semidualizing R-module. A finitely generated R-module M is totally C-reflexive if:

(1) The natural biduality map $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism; and
(2) $\mathrm{Ext}^i_R(M, C) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C)$ for all $i \geq 1$.

Example. Every finite rank free module R^n is totally C-reflexive.

Every finitely generated R-module is totally E-reflexive.

Definition. Given semidualizing R-modules B and C, write $C \trianglelefteq B$ whenever B is totally C-reflexive.

Example. For every C, one has $E \trianglelefteq C \trianglelefteq R$.

Fact. This ordering is reflexive and antisymmetric.

Theorem. (SSW 2008) *If R has a chain of semidualizing modules $C_0 \triangleleft C_1 \triangleleft \cdots \triangleleft C_{d+1}$, then the sequence $\{\mu^i\}_i$ is bounded below by a polynomial in i of degree d.*

Conclusion

These ideas generalize to non-artinian rings and to more general semidualizing complexes.

The paper `arXiv:0812.0643` contains these results, with a survey of the derived category notions needed for the proofs of the results.