Homological Algebra of Local Ring Homomorphisms

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Outline

Set-Up for the Talk

Let R be a commutative ring with identity.
Examples include:

- The rings of integers $\mathbb{Z}$ and p-adic integers $\mathbb{Z}_p$
- A field k like $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}/p\mathbb{Z}$
- Rings of polynomials $A[x_1, \ldots, x_n]$ or formal power series $A[\![x_1, \ldots, x_n]\!]$ where A is a commutative ring A with identity
- Quotient rings $A[x_1, \ldots, x_n]/I$ and $A[\![x_1, \ldots, x_n]\!]/J$

R-modules are objects that can be "acted upon" by the ring R.

- If R is a field, then "R-module" $=$ "R-vector space".
- M is a $\mathbb{Z}$-module if and only if it is an abelian group.
- An ideal I of R is an R-module and so is the quotient R/I.

In a sense, modules unify the notions of "vector space," "abelian group," "ideal," and "quotient by an ideal."

Modules and Free Presentations

An R-module is free if it has a basis. If F is free and has a finite basis $e_1, \ldots, e_r$ then $F \cong R^r$ and r is the rank of F.
Most R-modules are not free. However, every R-module M admits a free presentation:

- There is a free R-module F and a surjection $\tau \colon F \twoheadrightarrow M$.
- $\operatorname{Ker}(\tau)$ is an R-module which is usually not free. However, there is a free R-module G and a surjection $\pi \colon G \twoheadrightarrow \operatorname{Ker}(\tau)$.
- Hence, there is an exact sequence

  $$G \xrightarrow{\pi} F \xrightarrow{\tau} M \to 0$$

  meaning that the kernel of each map equals the image of the preceding map: τ is surjective, and $\operatorname{Ker}(\tau) = \operatorname{Im}(\pi)$.

Computation of a Free Presentation

Example. $R = A[x,y]$ and $I = (x,y)R$ and $M = R/I$.

- $\tau\colon R^1 \to R/I$ is the canonical surjection, and $\mathrm{Ker}(\tau) = I$.
- A surjection $\pi\colon R^2 \to I$ is given by

$$\pi\left(\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]\right) = f \cdot x + g \cdot y.$$

- This gives a free presentation

$$R^2 \xrightarrow{[x\ y]} R^1 \xrightarrow{\tau} R/I \to 0$$

If R is Noetherian (e.g., if $R = k[x_1, \ldots, x_n]/I$) and M is a finitely generated R-module, then M admits a finite free presentation

$$R^s \xrightarrow{\pi} R^r \xrightarrow{\tau} M \to 0$$

The map $\pi\colon R^s \to R^r$ is given by a matrix with entries from R.

Definitions and Examples of Ring Homomorphisms

A function $\varphi\colon R \to S$ between two rings is a homomorphism if it respects the additive and multiplicative structures of R and S:

- $\varphi(r + r') = \varphi(r) + \varphi(r')$ for each $r, r' \in R$
- $\varphi(rr') = \varphi(r)\varphi(r')$ for each $r, r' \in R$
- $\varphi(1_R) = 1_S$

Examples include the following:

- A field extension $k \to K$ or the natural map $\mathbb{Z} \to S$.
- The natural inclusions $A \to A[x_1, \ldots, x_n] \to A[\![x_1, \ldots, x_n]\!]$.
- If $(A, \mathfrak{m})$ is a local ring, consider the maximal ideal $\mathfrak{M} = (\mathfrak{m}, x_1, \ldots, x_n) \subset A[x_1, \ldots, x_n]$ and the natural maps $A[x_1, \ldots, x_n] \to A[x_1, \ldots, x_n]_{\mathfrak{M}} \to A[\![x_1, \ldots, x_n]\!]$.
- If $\varphi\colon R \to S$ is a ring homomorphism and $I \subseteq R$, consider the natural maps $R \to R/I \to S/IS$ where $IS = (\varphi(I))S$.

Restriction of Scalars

Let $\varphi\colon R \to S$ be a ring homomorphism. Each S-module N inherits a natural R-module structure from φ by setting

$$rn = \varphi(r)n \qquad \text{for each } r \in R \text{ and } n \in N.$$

Examples include the following:

- N is a $\mathbb{Z}$-module (i.e., abelian group) via the natural ring homomorphism $\mathbb{Z} \to S$.
- If $S = k[x_1, \ldots, x_n]/I$ or $S = k[\![x_1, \ldots, x_n]\!]/J$ then the map $k \to S$ gives N the structure of a k-vector space.

Not every R-module can be given the structure of an S-module.

Example. Let $\mathbb{Z}/p \to K$ be a nontrivial field extension. Each nonzero K-module (vector space) has at least p^2 elements. The $\mathbb{Z}/p$-module $\mathbb{Z}/p$ has p elements and therefore does not admit an K-module structure.

Extension of Scalars

Let $\varphi\colon R \to S$ be a ring homomorphism. If M is an R-module, then the tensor product $S \otimes_R M$ is an S-module via the formula

$$s\left(\sum_i s_i \otimes m_i\right) = \sum_i (ss_i) \otimes m_i.$$

There is an R-module homomorphism $\varphi^M\colon M \to S \otimes_R M$ given by $m \mapsto 1_S \otimes m$.
When M has a finite R-free presentation

$$R^s \xrightarrow{(x_{ij})} R^r \to M \to 0$$

the isomorphism $S \otimes_R R^t \cong S^t$ induces an S-free presentation

$$S^s \xrightarrow{(\varphi(x_{ij}))} S^r \to S \otimes_R M \to 0.$$

Example. $S \otimes_R R/I \cong S/IS$.

In this case, $S \otimes_R M$ is finitely generated over S, not over R.

Extension of Scalars and Flatness

Fix a ring homomorphism $\varphi\colon R \to S$. An exact sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

of R-modules induces and exact sequence of S-modules

$$S \otimes_R M' \xrightarrow{S\otimes_R f} S \otimes_R M \xrightarrow{S\otimes_R g} S \otimes_R M'' \to 0.$$

In general, though, the map $S \otimes_R f$ is not injective.

Definition. The homomorphism φ is flat if, for every injective R-module homomorphism f, the map $S \otimes_R f$ is injective.

Examples include the following:

- If S is projective over R (e.g., if R is a field) then φ is flat.
- If $S = R[x_1, \ldots, x_n]$ or $S = R[\![x_1, \ldots, x_n]\!]$, then φ is flat.
- The localization map $R \to R_U$ is flat.
- $A[x_1, \ldots, x_n]/I \to A[\![x_1, \ldots, x_n]\!]/IA[\![x_1, \ldots, x_n]\!]$ is flat.

Completions

Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and M a finitely generated R-module. For $m, m' \in M$ set

$$\operatorname{ord}(m) = \sup\{n \geq 0 \mid m \in \mathfrak{m}^n M\} \qquad \operatorname{dist}(m, m') = 2^{-\operatorname{ord}(m-m')}$$

The function $\operatorname{dist}(-,-)$ is a metric on M. The topological completion of M, denoted $\widehat{M}$, is an R-module equipped with a natural R-linear map $\varphi_M \colon M \to \widehat{M}$.

Example. If $R = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}/(f_1, \ldots, f_m)$, then $\widehat{R} \cong k[\![x_1, \ldots, x_n]\!]/(f_1, \ldots, f_m)$.

Definition. M is complete if φ_M is bijective.

$\widehat{R}$ is a noetherian local ring, and the map $\varphi_R \colon R \to \widehat{R}$ is a flat ring homomorphism. The maximal ideal of $\widehat{R}$ is $\mathfrak{m}\widehat{R}$, and the induced map $R/\mathfrak{m} \to \widehat{R}/\mathfrak{m}\widehat{R}$ is an isomorphism.

How to Characterize Completeness?

Let R be a noetherian local ring and M a finitely generated R-module.

Problem. Give criteria implying that M is complete.

The following conditions are equivalent:

(i) M is complete;

(ii) M admits an $\widehat{R}$-module structure that is compatible with its R-module structure via $\varphi_R\colon R \to \widehat{R}$;

(iii) The map $\varphi_R^M\colon M \to \widehat{R} \otimes_R M$ is bijective.

Problem. Give criteria implying that M has an $\widehat{R}$-module structure that is compatible with its R-module structure via φ_R.

Problem. Give criteria implying that $M \xrightarrow{\varphi_R^M} \widehat{R} \otimes_R M$ is bijective.

How to Characterize Ascent of Module-Structures?

Let $\varphi\colon R \to S$ be a ring homomorphism such that:

- φ is flat,
- R is a noetherian local ring with maximal ideal $\mathfrak{m}$,
- S is a noetherian local ring with maximal ideal $\mathfrak{m}S$,
- the induced map $k = R/\mathfrak{m} \to S/\mathfrak{m}S$ is bijective.

Shorthand. When the above conditions are satisfied, we write "$\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{m}S, k)$ is a flat local ring homomorphism".

Examples. $\varphi_R\colon R \to \widehat{R}$ and $R \to R^h$.

Let M be a finitely generated R-module.

Problem. Give criteria implying that the R-module structure on M ascends along φ, that is, that M has an S-module structure that is compatible with its R-module structure via φ.

Problem. Give criteria implying that $M \xrightarrow{\varphi^M} S \otimes_R M$ is bijective.

First Results on Ascent of Module Structures

Theorem. (AJF-SSW-RAW, '07) *Let* $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{m}S,k)$ *be a flat local ring homomorphism and M a finitely generated R-module. The following conditions are equivalent:*

(i) *The R-module structure on M ascends along* φ*;*
(ii) *The natural map* $\varphi^M\colon M \to S \otimes_R M$ *is bijective;*
(iii) $S \otimes_R M$ *is finitely generated over R.*

Corollary. (AJF-SSW-RAW, '07) *Let* $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{m}S,k)$ *be a flat local ring homomorphism. TFAE:*

(i) *The R-module structure on R ascends along* φ*;*
(ii) *S is finitely generated over R;*
(iii) φ *is bijective.*

Corollary. (AJF-SSW-RAW, '07) *Let R be a noetherian local ring and M a finitely generated R-module. Then M is complete if and only if* $\widehat{M}$ *is finitely generated over R.*

Motivation for the Second Characterization

Theorem. (Buchweitz-Flenner, '06) *Let R be a local ring with modules F and M such that F is flat. If M is complete, then* $\operatorname{Ext}^i_R(F, M) = 0$ *for each* $i \geq 1$.

Question. If $\operatorname{Ext}^i_R(F, M) = 0$ for each $i \geq 1$ and for each flat R-module F, must M be complete? What if we only assume that $\operatorname{Ext}^i_R(\widehat{R}, M) = 0$ for each $i \geq 1$?

Example. Let R be a local noetherian domain that is not a field. The field of fractions K of R is an injective R-module, and so $\operatorname{Ext}^i_R(F, K) = 0$ for each $i \geq 1$ and for each (flat) R-module F. However, K is not complete.

The problem here is that K is not finitely generated.

Further Results on Ascent of Module Structures

Theorem. (AJF-SSW-RAW, '07) *Let $\varphi\colon (R,\mathfrak{m},k)\to(S,\mathfrak{m}S,k)$ be a flat local ring homomorphism and M a finitely generated R-module. The following conditions are equivalent:*

(i) *The R-module structure on M ascends along φ;*

(ii) *The evaluation map $\operatorname{Hom}_R(S,M)\to M$ is bijective;*

(iii) *$\operatorname{Ext}^i_R(S,M)$ is finitely generated over R for each $i\geq 1$;*

(iv) *$\operatorname{Ext}^i_R(S,M)=0$ for each $i\geq 1$.*

Corollary. (AJF-SSW-RAW, '07) *Let R be a noetherian local ring and M a finitely generated R-module. TFAE:*

(i) *M is complete;*

(ii) *The evaluation map $\operatorname{Hom}_R(\widehat{R},M)\to M$ is bijective;*

(iii) *$\operatorname{Ext}^i_R(\widehat{R},M)$ is finitely generated over R for each $i\geq 1$;*

(iv) *$\operatorname{Ext}^i_R(\widehat{R},M)=0$ for each $i\geq 1$.*

Structural Implications of the Ascent Results

The prime spectrum of a noetherian ring R is

$$\operatorname{Spec}(R) = \{\text{prime ideals } \mathfrak{p} \subset R\}.$$

The support of a finitely generated R-module M is

$$\operatorname{Supp}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$$

which determined by its (finite) set of minimal elements

$$\operatorname{Min}_R(M) = \{\text{minimal elements of } \operatorname{Supp}_R(M)\}.$$

Theorem. (AJF-SSW-RAW, '07) *Let $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{m}S, k)$ be a flat local ring homomorphism and M a finitely generated R-module. The following conditions are equivalent:*

(i) *The R-module structure on M ascends along φ;*

(ii) *The map $R/\mathfrak{p} \xrightarrow{\overline{\varphi}} S/\mathfrak{p}S$ is bijective for each $\mathfrak{p} \in \operatorname{Supp}_R(M)$;*

(iii) *The map $R/\mathfrak{p} \xrightarrow{\overline{\varphi}} S/\mathfrak{p}S$ is bijective for each $\mathfrak{p} \in \operatorname{Min}_R(M)$.*

Consequences of the Structural Results

Corollary. (AJF-SSW-RAW, '07) *Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{m}S,k)$ be a flat local ring homomorphism and M, N finitely generated R-modules such that $\operatorname{Supp}_R(N) \subseteq \operatorname{Supp}_R(M)$. If the R-module structure on M ascends along φ, then the R-module structure on N ascends along φ.*

Corollary. (AJF-SSW-RAW, '07) *Let R be a noetherian local ring and M a finitely generated R-module. The following conditions are equivalent:*

(i) *M is complete;*

(ii) *$R/\mathfrak{p}$ is complete for each $\mathfrak{p} \in \operatorname{Supp}_R(M)$;*

(iii) *$R/\mathfrak{p}$ is complete for each $\mathfrak{p} \in \operatorname{Min}_R(M)$.*

Corollary. (AJF-SSW-RAW, '07) *Let R be a noetherian local ring and M, N finitely generated R-modules such that $\operatorname{Supp}_R(N) \subseteq \operatorname{Supp}_R(M)$. If M is complete, then so is N.*