A local ring has only finitely many semidualizing modules up to isomorphism

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Semidualizing Modules

Assumption

$(R, \mathfrak{m}, k)$ is a local ring

Definition (Foxby '72, Vasconcelos '74)

A finitely generated R-module is semidualizing if
$R \cong \mathrm{Hom}_R(C, C)$ and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geq 1$.

Example

1. R is a semidualizing R-module.
2. D is dualizing for R if and only if it is semidualizing for R and $\mathrm{id}_R(D) < \infty$.

Notation

$\mathfrak{S}(R) = \{$isomorphism classes of semidualizing R-modules$\}$.

A Conjecture and Partial Solution

Fact (Base-change)

If $R \to S$ is a local homomorphism of finite flat dimension, then $\mathfrak{S}(R) \hookrightarrow \mathfrak{S}(S)$ by $C \mapsto S \otimes_R C$.

Conjecture (Vasconcelos '74)

If R is Cohen-Macaulay, then $\mathfrak{S}(R)$ is finite.

Theorem (Christensen and Sather-Wagstaff '08)

If R is Cohen-Macaulay and contains a field, then $\mathfrak{S}(R)$ is finite.

Outline of Proof.

There is a flat local ring homomorphism $R \to (R', \mathfrak{m}R', \overline{k})$.
Let $\mathbf{x} \in \mathfrak{m}R'$ be a maximal R'-sequence.
Then $R'/(\mathbf{x})$ is artinian and $\mathfrak{S}(R) \hookrightarrow \mathfrak{S}(R') \hookrightarrow \mathfrak{S}(R'/(\mathbf{x}))$.
A result of Happel essentially shows that $\mathfrak{S}(R'/(\mathbf{x}))$ is finite. □

Tools for the Complete Solution: DG Algebras

Definition

A commutative differential graded (DG) R-algebra is

1. a graded commutative R-algebra $A = \oplus_{i=0}^{\infty} A_i$ with
2. a differential, i.e., a sequence of R-linear maps $\partial_i^A \colon A_i \to A_{i-1}$ such that $\partial_i^A \partial_{i+1}^A = 0$ for all i, such that
3. ∂^A satisfies the Leibniz Rule: for all $a_i \in A_i$ and $a_j \in A_j$

$$\partial_{i+j}^A(a_i a_j) = \partial_i^A(a_i)a_j + (-1)^i a_i \partial_j^A(a_j).$$

Example (The ground ring)

R is a DG R-algebra

Example (The Koszul complex)

$K = K^R(\mathbf{x})$ is a DG R-algebra for each sequence $\mathbf{x} \in R$.

DG Modules

Definition

A DG A-module is a graded A-module $M = \oplus_{i=i_0}^{\infty} M_i$ with a differential $\partial_i^M : M_i \to M_{i-1}$ that satisfies the Leibniz Rule.

Example (The ground ring)

A DG R-module is a bounded below R-complex,
e.g., a projective resolution of an R-module.

Example (The Koszul complex)

$K \otimes_R M$ is a DG K-module for each DG R-module M.

Semi-free DG Modules

Definition

Let A be a DG R-algebra. A DG A-module M is semi-free if the underlying $A^\natural$-module $M^\natural$ has a graded basis.

Note

The boundedness condition on M is important here.

Example (The ground ring)

A semi-free DG R-module is a bounded below complex of free R-modules.

Example (The Koszul complex)

$K \otimes_R M$ is a semi-free DG K-module for each semi-free DG R-module M.

Semidualizing DG Modules

Definition

A semi-free DG A-module C is semidualizing if it is homologically finite and the natural map $A \to \operatorname{Hom}_A(C, C)$ is a quasi-isomorphism.

Notation

$\mathfrak{S}_{\text{dg}}(A)$ is the set of shift-quasiisomorphism classes of semidualizing DG A-modules.

Example (The ground ring)

A projective resolution of a semidualizing R-module is a semidualizing DG R-module: $\mathfrak{S}(R) \hookrightarrow \mathfrak{S}_{\text{dg}}(R)$.

Example (The Koszul complex)

$K \otimes_R C$ is a semidualizing DG K-module for each semidualizing DG R-module C: $\mathfrak{S}_{\text{dg}}(R) \hookrightarrow \mathfrak{S}_{\text{dg}}(K)$.

Solution to Vasconcelos' Conjecture

Theorem (Nasseh and Sather-Wagstaff '12)

The sets $\mathfrak{S}(R)$ and $\mathfrak{S}_{dg}(R)$ are finite.

Outline of Proof.

It suffices to prove that $\mathfrak{S}_{\mathrm{dg}}(R)$ is finite since $\mathfrak{S}(R) \hookrightarrow \mathfrak{S}_{\mathrm{dg}}(R)$.

$$R \to R' \to K \cong R' \otimes_Q \widetilde{K} \xleftarrow{\simeq} A \otimes_Q \widetilde{K} \xrightarrow{\simeq} A \otimes_Q \overline{k}$$

$$\mathfrak{S}_{\mathrm{dg}}(R) \hookrightarrow \mathfrak{S}_{\mathrm{dg}}(R') \approx \mathfrak{S}_{\mathrm{dg}}(K) \approx \mathfrak{S}_{\mathrm{dg}}(R' \otimes_Q \widetilde{K})$$

There is a flat local ring homomorphism $R \to (R', \mathfrak{m}R', \overline{k})$ such that R' is complete.

Let $\mathbf{x} \in \mathfrak{m}R'$ be minimal generating sequence and $K = K^{R'}(\mathbf{x})$.

Let Q be a regular local ring surjecting onto R'.

Let $\widetilde{\mathbf{x}} \in Q$ be a lift of $\mathbf{x}$, and set $\widetilde{K} = K^Q(\widetilde{\mathbf{x}})$.

Let A be a DG algebra resolution of R' over Q.

$\widetilde{K}$ is a minimal Q-free resolution of $\overline{k}$.

$A \otimes_Q \overline{k}$ is a finite dimensional DG $\overline{k}$-algebra, and $\mathfrak{S}_{\mathrm{dg}}(A \otimes_Q \overline{k})$ is finite. □