Ascent of Test Modules

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Take-home points

1. To understand topology and geometry, use algebra (rings).
2. To understand rings, use modules and homology.
3. To understand modules and homology, use complexes.
4. To understand all of these, use DG algebras.

Topology and Geometry - Take-Home Point 1

Big Picture

Homological algebra provides tools for deciding when something can't be done, e.g., when two things are not equivalent.

Example (How to prove that $\mathbb{R}^2$ and $\mathbb{R}$ are not homeomorphic)

If they were homeomorphic, then $\mathbb{R}^2 - \{0\}$ and $\mathbb{R} - \{0\}$ would be homeomorphic, but one is connected and the other is not.

Example ($\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic for $m > n \geq 2$)

If they were homeomorphic, then $\mathbb{R}^m - \{0\} \cong \mathbb{R}^n - \{0\}$, but

$$\mathrm{H}_{n-1}(\mathbb{R}^n - \{0\}) \cong \mathbb{Z} \neq 0 = \mathrm{H}_{n-1}(\mathbb{R}^m - \{0\}).$$

Smoothness - Take-Home Point 1, cont.

Big Picture

Homological algebra provides tools for deciding when geometric objects are smooth.

Grothendieck

If $V \subseteq \mathbb{R}^n$ is the vanishing locus of a set of polynomial equations (e.g., conic section) then $\mathbb{R}[V] = \mathbb{R}[X_1, \ldots, X_n]/I(V)$ is the coordinate ring of V; smoothness of V is detected by an ideal theoretic property of $\mathbb{R}[V]$: being regular.

Auslander-Buchsbaum, Serre

A ring R being regular is detected by homological properties of the R-modules.

Set-up

Assumption

R is a commutative noetherian ring with identity.

Example

$R = k[X_1, \ldots, X_n]/I$ or $k[\![X_1, \ldots, X_n]\!]/I$ where k is a field or a DVR and I is an ideal

Assumption

Let M and N be finitely generated R-modules, e.g., $M = R^n$.

Analogies

ring : module :: field : vector space
ring : finitely generated mod :: field : finite dimensional vct spc

Projective Dimension

Most modules do not have a basis...

Every R-module has a basis if and only if R is a field.

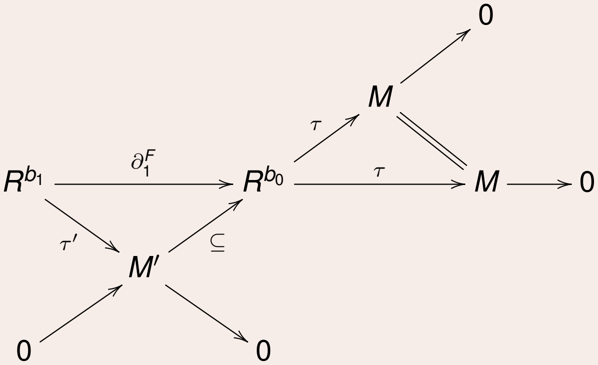
But

Every R-module can be approximated by a module with a basis.

Construction

Surject onto M by a free module: $R^{b_0} \xrightarrow{\tau} M \to 0$
R noetherian implies that $M' = \operatorname{Ker} \tau$ is finitely generated
Surject onto M' by a free module: $R^{b_1} \xrightarrow{\tau'} M' \to 0$

Free Presentations



The horizontal sequence (finite free presentation) is exact,
i.e., $M \cong R^{b_0} / \operatorname{Im} \partial_1^F$
∂_1^F is represented by a matrix; understand M via linear algebra

Free Resolutions

Fact

There exists an exact sequence

$$F^+ = \quad (\cdots \xrightarrow{\partial_3^F} R^{b_2} \xrightarrow{\partial_2^F} R^{b_1} \xrightarrow{\partial_1^F} R^{b_0} \xrightarrow{\tau} M \to 0)$$

and the associated free resolution *of M is*

$$F = \quad (\cdots \xrightarrow{\partial_3^F} R^{b_2} \xrightarrow{\partial_2^F} R^{b_1} \xrightarrow{\partial_1^F} R^{b_0} \to 0).$$

Example

$R = k[X, Y]$ and $M = R/\langle X, Y\rangle \cong k$:

$$F^+ = \quad (0 \to R \xrightarrow{\begin{pmatrix} -Y \\ X \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} X & Y \end{pmatrix}} R \xrightarrow{\tau} M \to 0)$$

Free Resolutions, cont.

Example

$R = k[X, Y]/\langle XY \rangle \ni x, y$ and $M = R/\langle x \rangle$:

$$F^{+} = (\cdots\cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{\tau} M \to 0)$$

Definition

M has finite projective dimension, denoted $\operatorname{pd}_R M < \infty$, if M has a bounded free resolution

$$F = \quad (0 \to R^{b_n} \xrightarrow{\partial_n^F} \cdots \xrightarrow{\partial_3^F} R^{b_2} \xrightarrow{\partial_2^F} R^{b_1} \xrightarrow{\partial_1^F} R^{b_0} \to 0).$$

Characterizing Regular Rings - Take-Home Point 2

Assumption

Assume R is local with maximal ideal $\mathfrak{m}$, and set $k = R/\mathfrak{m}$.

Auslander-Buchsbaum, Serre

- If R is regular, then every finitely generated R-module has fpd.
- Conversely, if $\mathrm{pd}_R\, k < \infty$, then R is regular

Auslander-Buchsbaum, Serre

If R is a regular and P is a prime ideal of R, then the localization R_P is regular.

Question

How to detect finiteness of projective dimension?

Detect Finiteness of Projective Dimension

Definition (Tor)

If F is a free resolution of M, then $\operatorname{Tor}_i^R(M,N) = \operatorname{H}_i(F \otimes_R N)$.

Theorem

- *If* $\operatorname{pd}_R M < \infty$, *then* $\operatorname{Tor}_i^R(M,N) = 0$ *for all* $i \gg 0$ *for all* N.
- *Conversely, if* $\operatorname{Tor}_i^R(M,k) = 0$ *for all* $i \gg 0$, *then* $\operatorname{pd}_R M < \infty$.

Definition (test module)

N is a test module over R if for all M:
$\operatorname{Tor}_i^R(M,N) = 0$ for all $i \gg 0 \implies \operatorname{pd}_R M < \infty$

Example

k is a test module over R.
R is test over R if and only if R is regular.

Ascent Questions

N is test if $\forall M$: $\operatorname{Tor}_i^R(M,N)=0$ for all $i\gg 0 \implies \operatorname{pd}_R M<\infty$

Question (Assume that N is a test module over R.)

1. Must $\widehat{N}$ be a test module over $\widehat{R}$?
2. If $R\to S$ is a flat local homomorphism with regular closed fibre $S/\mathfrak{m}S$, must $S\otimes_R N$ be a test module over S?

Example (Conclusion of Q 2 fails if closed fibre not regular)

Set $R=k\to k[X]/\langle X^2\rangle=S$.
Since R is regular, $N=R$ is test over R.
Since S is not regular, $S\otimes_R N=S\otimes_R R\cong S$ is not test over S.

First Ascent Result - Take-Home Point 3

N is test if $\forall M$: $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \gg 0 \implies \operatorname{pd}_R M < \infty$

Theorem (O. Celikbas and SSW)

If N is a test module over R, then $\widehat{N}$ is a test module over $\widehat{R}$.

Main idea of the proof: consider test complexes

Let K be the Koszul complex on a min gen seq for $\mathfrak{m}$.
Then $\operatorname{pd}_R M < \infty$ iff $\operatorname{pd}_R K \otimes_R M < \infty$
and N is test over R iff $K \otimes_R N$ is test over R.
If N is test over R, then so is $K \otimes_R N$.
$\mathfrak{m}\,\mathrm{H}(K \otimes_R N) = 0$, so $K \otimes_R N \simeq \widehat{K} \otimes_{\widehat{R}} \widehat{N}$ is test over $\widehat{R}$.
So $\widehat{N}$ is test over $\widehat{R}$. $\square$

Ascent for Test Modules

Theorem (SSW)

Assume that $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ is a flat local homomorphism with regular closed fibre and N is a test R-module. If the field extension $k \to \ell$ is algebraic, then $S \otimes_R N$ is a test S-module.

Special Case: R is a finite dim'l k-algebra and $S = \ell \otimes_k R$

Let M be a fg S-mod s.t. $0 = \mathrm{Tor}^S_{\gg 0}(M, S \otimes_R N) \cong \mathrm{Tor}^R_{\gg 0}(M, N)$.
Need to show that $\mathrm{pd}_S(M) < \infty$.
Sub-case 1: $k \to \ell$ is finite.
Then S is module finite over R, hence so is M.
As N is test over R, so $\mathrm{pd}_R(M) < \infty$.
Hence $\mathrm{pd}_S(M) < \infty$,
since $\mathrm{Tor}^S_{\gg 0}(M, \ell) \cong \mathrm{Tor}^S_{\gg 0}(M, S \otimes_R k) \cong \mathrm{Tor}^R_{\gg 0}(M, N) = 0$.

Ascent for Test Modules, Special Case, cont.

Theorem (SSW)

Assume that $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ is a flat local homomorphism with regular closed fibre and N is a test R-module. If the field extension $k \to \ell$ is algebraic, then $S \otimes_R N$ is a test S-module.

Special Case: R is a finite dim'l k-algebra and $S = \ell \otimes_k R$

Let M be fg/S s.t. $0 = \mathrm{Tor}^S_{\gg 0}(M, S \otimes_R N)$. NTS $\mathrm{pd}_S(M) < \infty$.
Sub-case 2: $k \to \ell$ is algebraic.

Consider a finite free presentation $S^a \xrightarrow{\partial} S^b \to M \to 0$.
The matrix ∂ is expressed using finitely many elements of ℓ.

These define an intermediate field extension $k \xrightarrow{\text{finite}} k' \to \ell$.
Set $R' = k' \otimes_k R$. Then $M \cong S \otimes_{R'} \widetilde{M}$ for some fg R'-module $\widetilde{M}$.
By Sub-case 1, $N' = R' \otimes_R N$ is test over R'.
As above it follows that $\mathrm{pd}_{R'}(\widetilde{M}) < \infty$ and hence $\mathrm{pd}_S(M) < \infty$.

Ascent for Test Modules: Take-Home Point 4

Theorem (SSW)

Assume that $(R,\mathfrak{m},k) \to (S,\mathfrak{n},\ell)$ is a flat local homomorphism with regular closed fibre and N is a test R-module. If the field extension $k \to \ell$ is algebraic, then $S \otimes_R N$ is a test S-module.

Outline of General Proof (Avramov's Hammer)

Routine reduction: R and S are complete with $S/\mathfrak{m}S = \ell$.
Cohen Structure Theorem: $\exists$ RLR $P \twoheadrightarrow R$ minimally.
Tate: $\exists$ finite DG algebra resolution F of R over P.

$$R \longrightarrow K^R \xleftarrow{\cong} R \otimes_P K^P \xleftarrow{\simeq} F \otimes_P K^P \xrightarrow{\simeq} F \otimes_P \ell = U \text{ fd DGA}/k$$

$$N \mapsto K^R \otimes_R N \longleftarrow\!\!\shortmid\; K^R \otimes_R N \longmapsto \overline{N} \text{ test over } U$$

$$S \longrightarrow K^S \xleftarrow{\cong} S \otimes_Q K^Q \xleftarrow{\simeq} G \otimes_Q K^Q \xrightarrow{\simeq} G \otimes_Q \ell \cong G/\mathfrak{m}_Q G = V$$

$$\cong \ell \otimes_k U$$

Now argue as in the special case. □

Concluding Remarks

Question 2 is still open in general

It essentially reduces to the case of a simple purely transcendental field extension $\ell = k(x)$.

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Thanks!