Generalized Zero-Divisor Graphs

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Assumptions and Definitions

Assumption

R is a commutative ring with identity, and k is a field.

Notation

$Z^*(R)$ is the set of (non-zero) zero-divisors in R.

Definition (D.F. Anderson and P. Livingston)

$\Gamma(R)$ is the simple graph with vertex set $Z^*(R)$ such that two vertices $x, y \in Z^*(R)$ are adjacent provided that $xy = 0$.

Definition (S. Mulay)

Define an equivalence relation on $Z^*(R)$: given $x, y \in Z^*(R)$, we have $x \sim y$ provided that $\mathrm{Ann}_R(x) = \mathrm{Ann}_R(y)$.
$\Gamma_E(R)$ is the simple graph with vertex set $Z^*(R)/\sim$ such that vertices $[x], [y] \in Z^*(R)/\sim$ are adjacent provided that $xy = 0$.

Examples

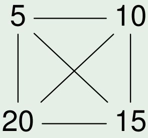
Remark

Let x be a zero-divisor in R, and let $u \neq 1$ be a non-zero-divisor in R. In $\Gamma(R)$, the vertices x and ux are distinct. In $\Gamma_E(R)$, we have $[x] = [ux]$. So $\Gamma_E(R)$ has less "noise" than $\Gamma(R)$.

Example

$R = \mathbb{Z}/(25)$

$\Gamma(R)$:



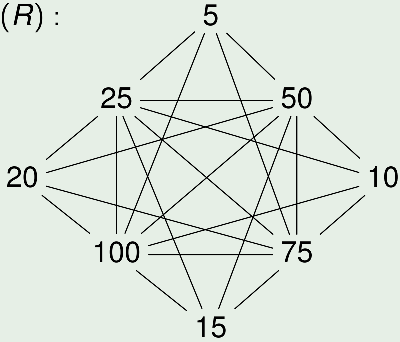
$\Gamma_E(R)$: $[5]$

More generally, if p is prime, then $\Gamma(\mathbb{Z}/(p^2))$ is a complete graph on $p - 1$ vertices, and $\Gamma_E(\mathbb{Z}/(p^2))$ is a single vertex.

Example

$R = \mathbb{Z}/(125)$

$\Gamma(R)$:



$\Gamma_E(R)$:

More generally, if p is prime, then $\Gamma(\mathbb{Z}/(p^3))$ is a mess, and $\Gamma_E(\mathbb{Z}/(p^3))$ is a single edge. The graph $\Gamma(k[X]/(X^n))$ is even messier, but $\Gamma_E(k[X]/(X^n))$ is straightforward to describe.

Pros and Cons

Remark

1. $\Gamma_E(R)$ is less noisy than $\Gamma(R)$.
2. Deciding whether $[x]$ and $[y]$ are equal can be a pain.
3. Many favorite properties of $\Gamma(R)$ are satisfied by $\Gamma_E(R)$.

Theorem (D.F. Anderson and P. Livingston)

The graph $\Gamma(R)$ *is connected with* $\operatorname{diam}(\Gamma(R)) \leq 3$.

Theorem (S. Spiroff and C. Wickham)

The graph $\Gamma_E(R)$ *is connected with* $\operatorname{diam}(\Gamma_E(R)) \leq 3$.

Finiteness

Theorem (D.F. Anderson and P. Livingston)

$\Gamma(R)$ is finite if and only if either R is finite or a domain.

Example

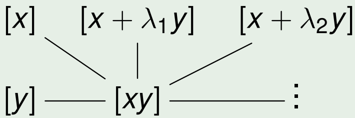
The graph $\Gamma_E(k[X]/(X^n))$ is finite.

Question

Is $\Gamma_E(R)$ always finite? No, not even when R is artinian.

Example (JC-SSW-SS-LS)

Assume that $\mathrm{char}(k) = 2$. The ring $R = k[X, Y]/(X^2, Y^2)$ is artinian of length 4, and $\Gamma_E(R)$ is a star with $|\Gamma_E(R)| = |k| + 2$:



Finiteness, continued

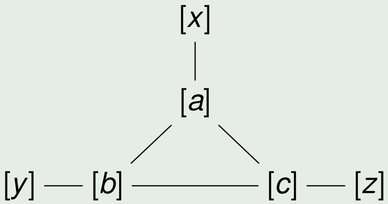
Remark

This example is minimal with respect to length.

Proposition (JC-SSW-SS-LS)

If $\operatorname{len}(R) \leq 3$, *then* $\Gamma_E(R)$ *has one of the following forms:*

$$\emptyset \quad \textit{or} \quad [x] \quad \textit{or} \quad [x] - [y] \quad \textit{or} \quad [x] - [y] - [z] \quad \textit{or}$$



Cliques

Theorem (D.F. Anderson and P. Livingston)

$\Gamma(R)$ *is complete if and only if* $R \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2)$ *or* $xy = 0$ *for all* $x, y \in Z^*(R)$.

Example

$\Gamma(k[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^2)$ and $\Gamma(\mathbb{Z}/(p^2))$ are complete.

Proposition (S. Spiroff and C. Wickham)

$\Gamma_E(R)$ *is complete if and only if* $|\Gamma_E(R)| \leq 2$.

Question

What size of complete subgraphs can $\Gamma_E(R)$ have?

Cliques, continued

Definition

$\omega_E(R)$ is the clique number of $\Gamma_E(R)$, that is

$$\omega_E(R) = \sup\{n \mid \exists K_n \subseteq \Gamma_E(R)\}.$$

Remark

This is related to the coloring number $\omega_E(R) \leq \chi_E(R)$.

Question

Must $\omega_E(R)$ be finite? No, not even when R is artinian.

Example (JC-SSW-SS-LS)

Set $A = k[X, Y]/(X, Y)^2$ and $D = \mathrm{Hom}_k(A, k)$. Consider the "trivial extension" or "idealization" $R = A \ltimes D$. Then R is artinian with length 6 such that $\Gamma_E(R)$ contains a complete subgraph on $|k| + 1$ vertices.

Cliques, continued

Question

Is this example minimal with respect to length? We don't know.

Theorem (JC-SSW-SS-LS)

If (a) $\mathrm{len(R)} \leq 4$ *or (b) R is non-local and* $\mathrm{len(R)} = 5$*, then* $\omega_E(R) < \infty$*.*

Question

If R is local of length 5, then $\omega_E(R) < \infty$?

Stars

Theorem (D.F. Anderson and P. Livingston)

If R is finite and $\Gamma(R)$ is a star, then $|\Gamma(R)| = p^n$. Conversely, if G is a star graph with $|G| = p^n$, then there is a finite ring R such that $\Gamma(R) \cong G$.

Remark

We have seen how to build a ring R such that $\Gamma_E(R)$ is a star, either infinite or with $|\Gamma_E(R)| = 2^n + 2$.

Question

Which star graphs can be realized as $\Gamma_E(R)$? We don't know.

Stars, continued

Theorem (JC-SSW-SS-LS)

There exist artinian local rings R such that $\Gamma_E(R)$ is a star with c vertices where c is any positive number of the following form

$$\begin{array}{rrrr}
2^n-4, & 2^n-3, & 2^n-2, & 2^n-1, \\
2^n, & 2^n+1, & 2^n+2, & 2^n+3, \\
2^n\cdot 3-2, & 2^n\cdot 3-1, & 2^n\cdot 3, & 2^n\cdot 3+1, \\
2^n\cdot 3+2, & 2^n\cdot 3+3, & 2^n\cdot 7-4, & 2^n\cdot 7-3, \\
2^n\cdot 7-2, & 2^n\cdot 7-1, & 2^n\cdot 7, & 2^n\cdot 7+1, \\
2^n\cdot 7+2, & 2^n\cdot 7+3, & 2^n\cdot 15-12, & 2^n\cdot 15-11, \\
2^n\cdot 15-6, & 2^n\cdot 15-5, & 2^n\cdot 15-4, & 2^n\cdot 15-3 \\
2^n\cdot 15, & 2^n\cdot 15+1, & 2^n\cdot 15+2, & 2^n\cdot 15+3.
\end{array}$$

Remark

The smallest star we do not know how to construct has 36 vertices.

Girth

Definition

The girth of a simple graph G is the length of the shortest cycle in G (if G has a cycle) or ∞ (if G is acyclic).

Theorem (S. Mulay)

If $\Gamma(R)$ *has finite girth, then* $\operatorname{girth}(\Gamma(R)) \leq 4$.

Theorem (JC-SSW-SS-LS)

If R *is noetherian and* $\Gamma_E(R)$ *has finite girth, then* $\operatorname{girth}(\Gamma_E(R)) \leq 3$.

Example

$\Gamma(\mathbb{Z}/(3) \times \mathbb{Z}/(3))$ is a 4-cycle, which has girth 4.
$\Gamma_E(\mathbb{Z}/(3) \times \mathbb{Z}/(3))$ is an edge.

Coloring Number

Definition

The coloring number $\chi_E(R)$ of the graph $\Gamma_E(R)$ is the minimal number of colors which can be assigned to the vertices of $\Gamma_E(R)$ so that no adjacent vertices have the same color.

Remark

We always have $\omega_E(R) \leq \chi_E(R)$.

Proposition (JC-SSW-SS-LS)

If $\omega_E(R) \leq 2$*, then* $\omega_E(R) = \chi_E(R)$*.*

Question

What if $\omega_E(R) = 3$? No, not even if R is artinian.

Coloring Number, continued

Example (JC-SSW-SS-LS)

The ring

$$R = \frac{k[X_1, \ldots, X_5]}{(X_1X_2, X_2X_3, X_3X_4, X_4X_5, X_1X_5) + (X_1, \ldots, X_5)^3}.$$

has $\omega_E(R) = 3 < 4 = \chi_E(R)$.