Bass Numbers and Semidualizing Modules

Sean Sather-Wagstaff

Department of Mathematics
North Dakota State University

20 March 2011
AMS Central Section Meeting
University of Iowa

Semidualizing Modules

Assumption

R is a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$, and $E = E_R(k)$.

Definition (Vasconcelos, Foxby)

A semidualizing R-module is a finitely generated R-module C such that $\mathrm{Hom}_R(C, C) \cong R$ and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geq 1$.

Example

(a) R is a semidualizing R-module.

(b) A dualizing R-module is semidualizing.

Facts

(a) If R is Gorenstein, i.e., $\mathrm{id}_R(R) < \infty$, then R is the only semidualizing R-module.

(b) The converse holds if R is complete and Cohen-Macaulay.

Facts

(a) Given a minimal injective resolution J of R and an integer $i \geq 0$, the ith Bass number of R is the number of copies of E occurring as summands of J^i:

$$\mu^i = \mu^i_R(R) = \dim_k(\operatorname{Ext}^i_R(k, R)).$$

(b) The following conditions are equivalent:
   (i) R is Gorenstein,
   (ii) $\mu^i = 0$ for all $i \neq \dim(R)$, and
   (iii) $\mu^i = 0$ for some $i > \dim(R)$.

Questions (Huneke)

(a) If the sequence $\{\mu^i\}$ is bounded or bounded above by a polynomial, must R be Gorenstein?

(b) If R is not Gorenstein, is the sequence $\{\mu^i\}$ unbounded, and must the sequence $\{\mu^i\}$ have exponential growth?

Semidualizing Modules and Bass Numbers

Remark

Semidualizing modules and Bass numbers both have the ability to detect the Gorenstein property, so it should come as no surprise that they provide information about each other.

Theorem (SSW, 2009)

If R has a semidualizing module that is neither dualizing nor free, then the sequence $\{\mu^i\}$ is unbounded.

Remark

This reduces Huneke's original question to the case where R has exactly two semidualizing modules.

Question

What about Huneke's other questions?

SFD Rings

Assumption

Assume that R is complete and Cohen-Macaulay.

Remark

This implies that R has a dualizing module D.

Definition

(a) Let $\mathfrak{S}(R)$ denote the set of isomorphism classes of semidualizing R-modules.

(b) The ring R is SFD if $\mathfrak{S}(R) = \{[R], [D]\}$.

Example (Cooper and SSW)

If R has codimension 2, then R is SFD.

Reflexivity

Definition

Let C be a semidualizing R-module. A finitely generated R-module G is totally C-reflexive if $\mathrm{Hom}_R(\mathrm{Hom}_R(G, C), C) \cong G$ and $\mathrm{Ext}^i_R(G, C) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(G, C), C)$ for all $i \geq 1$.

Example

$$\begin{aligned} \text{Finitely generated free} &\implies \text{totally } C\text{-reflexive} \\ &\implies \text{maximal Cohen-Macaulay} \\ &\implies \text{totally } D\text{-reflexive} \end{aligned}$$

Question

Totally R-reflexive $\implies$ totally C-reflexive?

Huneke's Other Questions

Theorem (SSW)

Assume that every module-finite Cohen-Macaulay R-algebra S satisfies the following: for every semidualizing S-module C, every cyclic totally S-reflexive S-module is totally C-reflexive.

(a) *Assume that for every module-finite Cohen-Macaulay SFD R-algebra S, the sequence $\{\mu^i_S(S)\}$ is eventually bounded below by a polynomial in i of degree d. Then the sequence $\{\mu^i_R(R)\}$ is eventually bounded below by a polynomial in i of degree d.*

(b) *Assume that for every module-finite Cohen-Macaulay SFD R-algebra S, the sequence $\{\mu^i_S(S)\}$ has exponential growth. Then the sequence $\{\mu^i_R(R)\}$ has exponential growth.*

Remark

This reduces Huneke's original question to SFD rings.

Sketch of Proof

For $[B], [C] \in \mathfrak{S}(R)$ define $[B] \leq [C]$ if B is totally C-reflexive.

(a) Argue by induction on $n = \dim(\mathfrak{S}(R))$.

If $n = 1$, then R is SFD, and we are done by assumption.

Assume that $n > 1$, and let C be a semidualizing R-module that is neither free nor dualizing. Consider the trivial extension $S = R \ltimes C$, i.e., the idealization of C.

Then S is a module-finite Cohen-Macaulay R-algebra such that $1 \leq \dim(\mathfrak{S}(S)) < n$. This uses the technical assumption we made about the class of R-algebras, and the fact that C is neither free nor dualizing.

By induction, the sequence $\{\mu^i_S(S)\}$ is bounded below by a polynomial of degree d.

For all $i \geq 0$, one has $\mu^i_R(R) \geq \mu^i_S(S)$, so the sequence $\{\mu^i_R(R)\}$ is bounded below by a polynomial of degree d. $\square$