Homological Algebra Book

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Part I

Preface

Acknowledge that Part II sort of originates in Todd Morra's thesis.

Part II

Homological Algebra

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CHAPTER II.A

Preliminaries

apter062921a

It will be assumed that the reader is already familiar with introductory-level abstract algebra as well as the following definitions and results. Assume R is a commutative ring with identity throughout.

II.A.1. Exact Sequences and Projective Modules

ection062921f

Some basic properties of modules and (short) exact sequences will be essential in this book, so we present a number of them here.

def091517a DEFINITION II.A.1.1. Let M_1, M_2, and M_3 be R-modules. Then a sequence of R-module homomorphisms

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

is exact if $\operatorname{Im} f = \operatorname{Ker} g$. More generally, a sequence of R-module homomorphisms

$$\cdots \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \xrightarrow{d_{i-1}} \cdots$$

is exact if $\operatorname{Im} d_{i+1} = \operatorname{Ker} d_i$, for all relevant i.

fac091517b FACT II.A.1.2. *Let U, V, W be R-modules.*

fac091517b.a (a) *The following sequence is exact if and only if α is injective.*

$$0 \xrightarrow{\varepsilon} U \xrightarrow{\alpha} V$$

fac091517b.b (b) *The following sequence is exact if and only if β is surjective.*

$$V \xrightarrow{\beta} W \xrightarrow{\rho} 0$$

fac091517b.c (c) *The following sequence is exact if and only if α is injective, β in surjective, and* $\operatorname{Im}\alpha = \operatorname{Ker}\beta$.

$$0 \longrightarrow U \xrightarrow{\alpha} V \xrightarrow{\beta} W \longrightarrow 0$$

PROOF. By Definition II.A.1.1, the sequence is exact if and only if $\ker\alpha = \operatorname{Im}\varepsilon = \{0\}$, which proves (a). Part (b) also holds by Definition II.A.1.1, since the sequence is exact if and only if $\operatorname{Im}\beta = \ker\rho = W$. Part (c) is a corollary of parts (a) and (b). □

091517c DEFINITION II.A.1.3. When the sequence in Fact II.A.1.2c above is exact, it is a short exact sequence.

ex092918a EXAMPLE II.A.1.4. If M and N are R-modules, then so is $M \oplus N$. We claim the sequence

$$0 \longrightarrow M \xrightarrow{\varepsilon} M \oplus N \xrightarrow{\pi} N \longrightarrow 0$$

is a short exact sequence, where ε and π are the natural injection and surjection, respectively. To see this, let $(m,n) \in M \oplus N$. Then $\pi(m,n) = 0$ if and only if $n = 0$, which holds if and and only if $(m,n) \in \operatorname{Im}\varepsilon$. Therefore the sequence is exact in the center. By Fact II.A.1.2(c), the given sequence is a short exact sequence since ε and π are injective and surjective, respectively.

091517d FACT II.A.1.5. *Let A, B, C be R-modules.*

091517d.a (a) *The following sequence is exact if and only if α is an isomorphism.*

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

091517d.b (b) *The following sequence is exact if and only if C is the zero module.*

$$0 \longrightarrow C \longrightarrow 0$$

Proof. Both parts follow from Fact II.A.1.2. For part a, note α is injective if and only if the sequence is exact and α is surjective if and only if the sequence is exact. For part b, note the sequence is exact if and only if the map $0 \longrightarrow C$ is surjective, i.e., if and only if $C = 0$. ☐

091517e Definition II.A.1.6. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

and

$$0 \longrightarrow A' \xrightarrow{f'} B' \xrightarrow{g'} C' \longrightarrow 0$$

be two short exact sequences. A homomorphism of short exact sequences is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

where α, β, and γ are R-module homomorphisms. The homomorphism is an isomorphism if α, β, and γ are isomorphisms. This is an equivalence if $A = A'$, $C = C'$, $\alpha = id_A$, and $\gamma = id_C$. That is, we have equivalence if our diagram can be written

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "id_A"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d, "id_C"] & 0 \\
0 \arrow[r] & A \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C \arrow[r] & 0
\end{tikzcd}$$

Note in this case β is necessarily an isomorphism (see Fact II.A.1.9).

091517f Fact II.A.1.7. *Given any R-module homomorphism $g : A \longrightarrow B$, there exists an exact sequence*

$$0 \longrightarrow \operatorname{Ker} g \xrightarrow{\varepsilon} A \xrightarrow{g} B \xrightarrow{\tau} \operatorname{Coker}(g) \longrightarrow 0$$

where ε is the natural injection, τ is the natural surjection, and

$$\operatorname{Coker}(g) := \frac{B}{\operatorname{Im} g}$$

091517g Definition II.A.1.8. Given R-modules A, B, and C, the short exact sequence

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$$

is said to be split if there is an R-module complement to $\psi(A)$ in B. In this case $B \cong A \oplus C$, or to be precise, $B = \psi(A) \oplus C'$ where $C' \subseteq B$ is a submodule and $\phi(C') \cong C$. The module B is said to be a split extension of C by A.

An equivalent definition is to say that the above short exact sequence splits if there exists an equivalence

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\varepsilon"] \arrow[d, "\mathrm{id}_A"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & A \oplus C \arrow[r, "\rho"] \arrow[d, "\Gamma"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d, "\mathrm{id}_C"', "\cong"] & 0 \\
0 \arrow[r] & A \arrow[r, "\psi"'] & B \arrow[r, "\phi"'] & C \arrow[r] & 0
\end{tikzcd}$$

where ε and ρ are the natural injection and surjection, respectively.

fact011818a Fact II.A.1.9. *In the following commutative diagram with exact rows and with isomorphisms α and γ, the R-module homomorphism β must be an isomorphism.*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r, "g"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d, "\gamma"', "\cong"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

PROOF. To show β is injective, let $b \in \ker(\beta)$ be given and we want to show $b = 0$. By the commutivity of the diagram, $0 = g'(\beta(b)) = \gamma(g(b))$, so $g(b) \in \ker(\gamma) = \{0\}$. Since $b \in \ker(g) = \operatorname{Im} f$, let $a \in A$ such that $f(a) = b$. By the commutivity of the diagram, $f'(\alpha(a)) = \beta(f(a)) = \beta(b) = 0$, so $\alpha(a) \in \ker(f') = \{0\}$. Since α is injective, $a = 0$ and therefore $b = f(a) = 0$.

To show β is surjective, let $b' \in B'$ be given and we want to find a lift of this element in B. Since both γ and g are surjective, let $b \in B$ such that $(\gamma \circ g)(b) = g'(b')$. By the commutivity of the diagram it also holds that $(g' \circ \beta)(b) = g'(b')$, so the element $b' - \beta(b) \in \ker(g')$. Since the rows are exact and α an isomorphism, we may lift to some $a \in A$ such that $(f' \circ \alpha)(a) = b' - \beta(b)$ and the commutivity of the diagram implies $(\beta \circ f)(a) = b' - \beta(b)$, whereby we conclude

$$\beta(f(a) + b) = (\beta \circ f)(a) + \beta(b) = b'$$

as desired. □

091517h

FACT II.A.1.10. *A short exact sequence as in Definition II.A.1.8 splits if and only if there exists an R-module homomorphism $\mu : C \longrightarrow B$ such that $\phi \circ \mu = \mathrm{id}_C$. In this case, μ is called a splitting homomorphism for the sequence.*

PROOF. First assume an equivalence of short exact sequences exists as in Definition II.A.1.8 and define

$$\begin{aligned} \mu : C &\longrightarrow B \\ c &\longmapsto \Gamma(0, c). \end{aligned}$$

This is a well-defined R-module homomorphism because Γ is a well-defined R-module homomorphism. For an arbitrary element $c \in C$, the commutativity of the diagram gives

$$(\phi \circ \mu)(c) = (\phi \circ \Gamma)(0, c) = (\mathrm{id}_C \circ \rho)(0, c) = c.$$

Second, assume instead there exists a homomorphism $\mu : C \to B$ such that $\phi \circ \mu = \mathrm{id}_C$ (see Definition II.A.1.8). Define the following map.

$$\begin{aligned} \Gamma : A \oplus C &\longrightarrow B \\ (a, c) &\longmapsto \psi(a) + \mu(c) \end{aligned}$$

Since both ψ and μ are well-defined R-module homomorphisms, so is Γ. Moreover for any $a \in A$, we have

$$(\Gamma \circ \varepsilon)(a) = \Gamma(a, 0) = \psi(a)$$

and for any $(a, c) \in A \oplus C$ we have

$$(\phi \circ \Gamma)(a, c) = \phi(\psi(a) + \mu(c)) = (\phi \circ \psi)(a) + (\phi \circ \mu)(c) = 0 + c = (\mathrm{id}_C \circ \rho)(a, c).$$

Therefore the diagram commutes. By Fact II.A.1.9, Γ is also an isomorphism, so the bottom row is split. □

def082118c

DEFINITION II.A.1.11. Let R be a ring, let C be an R-module, and let $A \subseteq C$ be a submodule. We say A is a direct summand of C if there exists some R-submodule $B \subseteq C$ such that $C = A \oplus B$.

def082118e

DEFINITION II.A.1.12. A category consists of a collection of objects, a collection of morphisms for each pair of objects, and a binary operation on pairs of morphisms called composition (provided the morphisms have compatible domain and codomain). A functor is a map between categories that respects compositions and identity morphisms. A functor F is covariant if a morphism $\phi : A \longrightarrow B$ becomes

$$F(\phi) : F(A) \longrightarrow F(B)\ .$$

A functor G is contravariant if the morphism becomes

$$G(\phi) : G(B) \longrightarrow G(A)\ .$$

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REMARK II.A.1.13. Contravariant and covariant functors respect compositions differently. Let γ, ρ, and φ be morphisms in the same category such that $\varphi = \gamma \circ \rho$ and let F and G be covariant and contravariant functors, respectively, on this category. Then $F(\varphi) = F(\gamma) \circ F(\rho)$ and $G(\varphi) = G(\rho) \circ G(\gamma)$. In particular, if $\gamma \circ \rho = 0$, then $F(\gamma) \circ F(\rho) = 0$ and $G(\rho) \circ G(\gamma) = 0$.

091517i

DEFINITION II.A.1.14. An R-module P is projective if it satisfies any one (and therefore all) of the following equivalent conditions.

091517i.a (a) The covariant functor $\mathrm{Hom}_R(P,-)$ is exact. That is, for any R-modules L, M, and N, the exactness of the sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0$$

implies the following sequence is also exact.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R(P,L) \arrow[r, "\psi'"] & \mathrm{Hom}_R(P,M) \arrow[r, "\phi'"] & \mathrm{Hom}_R(P,N) \arrow[r] & 0 \end{tikzcd}$$

$$\rho \longmapsto \psi \circ \rho \qquad \gamma \longmapsto \phi \circ \gamma$$

091517i.b (b) For any R-modules M and N, if $M \xrightarrow{\phi} N \longrightarrow 0$ is exact, then every R-module homomorphism from P into N lifts to an R-module homomorphism into M. In other words, given $f \in \mathrm{Hom}_R(P,N)$ there is a lift $F \in \mathrm{Hom}_R(P,M)$ making the following diagram commute.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists F"'] \arrow[d, "f"] \arrow[dl, phantom, "\circlearrowleft", shift left=3] & \\ M \arrow[r, "\phi"'] & N \arrow[r] & 0 \end{tikzcd}$$

091517i.c (c) For any R-module M, if P is isomorphic to a quotient of M (i.e., $P \cong M/M'$ for some submodule $M' \subseteq M$), then P is isomorphic to a direct summand of M.

091517i.d (d) Every short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow P \longrightarrow 0$ splits.

091517i.e (e) P is a direct summand of a free R-module.

def111017c DEFINITION II.A.1.15. An R-module I is <u>injective</u> if it satisfies any one (and therefore all) of the following equivalent conditions.

def111017c.a (a) The contravariant functor $\mathrm{Hom}_R(-,I)$ is exact. That is, for any short exact sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0$$

the following sequence is exact as well.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R(L,I) \arrow[r, "\psi'"] & \mathrm{Hom}_R(M,I) \arrow[r, "\phi'"] & \mathrm{Hom}_R(N,I) \arrow[r] & 0 \end{tikzcd}$$

$$\rho \longmapsto \psi \circ \rho \qquad \gamma \longmapsto \phi \circ \gamma$$

def111017c.b (b) For any R-modules X, Y and any R-module homomorphisms $X \hookrightarrow Y$ and $X \longrightarrow I$, there exists an R-module homomorphism h such that the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, hook, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft", shift left=2] & Y \arrow[dl, dashed, "\exists h"] \\ & I & \end{tikzcd}$$

def111017c.c (c) For any R-module M, if I is isomorphic to a submodule $I' \subseteq M$, then I' is a direct summand of M.

def111017c.d (d) Every short exact sequence $0 \longrightarrow I \longrightarrow M \longrightarrow K \longrightarrow 0$ splits.

ection062921g

II.A.2. Localization

We briefly look at the construction of localized rings and modules and their properties. Of particular usefulness throughout this document will be the correspondence of prime ideals under localization given in Fact II.A.2.11. Assume M and N are R-modules throughout this section.

091517j DEFINITION II.A.2.1. A subset $U \subseteq R$ is <u>multiplicatively closed</u> if $1 \in U$ and the product $uv \in U$ for all $u, v \in U$.

091517k EXAMPLE II.A.2.2. For any element $s \in R$, the subset $S = \{s^\epsilon \mid \epsilon \in \mathbb{N}_0\}$ is multiplicatively closed. If $\mathfrak{p} \lneq R$ is a prime ideal, then $R \setminus \mathfrak{p}$ is multiplicatively closed as well.

091517l **Definition II.A.2.3.** Let $U \subseteq R$ be multiplicatively closed. We may define a relation on $M \times U$: let $(m, u) \sim (n, v)$ if there exists $w \in U$ such that $w(vm - un) = 0$. One can show that this is an equivalence relation. We therefore define

$$U^{-1}M := \{\text{equivalence classes from } M \times U \text{ under } \sim\}$$

and denote the equivalence class (m, u) as $\dfrac{m}{u}$ or m/u.

091517m **Fact II.A.2.4.** *In general $U^{-1}M$ is an abelian group by the operations*

$$\frac{m}{u} + \frac{n}{v} := \frac{vm + um}{uv} \qquad\qquad 0_{U^{-1}M} := \frac{0_M}{1_R} = \frac{0_M}{u},$$

an R-module by the operation

$$r \cdot \frac{m}{u} := \frac{rm}{u},$$

and a $U^{-1}R$-module by the operation

$$\frac{r}{v} \cdot \frac{m}{u} := \frac{rm}{vu}.$$

The special case when $M = R$ gives a commutative ring $U^{-1}R$ with the following operations and identities.

$$\frac{m}{u} + \frac{n}{v} := \frac{vm + un}{uv} \qquad\qquad \frac{m}{u} \cdot \frac{n}{v} := \frac{mn}{uv}$$

$$0_{U^{-1}R} := \frac{0_R}{u} = \frac{0_R}{1_R} \qquad\qquad 1_{U^{-1}R} := \frac{u}{u} = \frac{1_R}{1_R}$$

Moreover there exists a ring homomorphism

$$\begin{aligned} \psi : &R \longrightarrow U^{-1}R \\ r &\longmapsto \frac{r}{1} = \frac{ur}{u}. \end{aligned}$$

notn082318e **Notation II.A.2.5.** Let $R^\times$ denote the collection of all units in R.

091517n **Theorem II.A.2.6** (Universal Mapping Property). *Let R and S be commutative rings with identity. Given any ring homomorphism $\phi : R \longrightarrow S$ such that $\phi(U) \subseteq S^\times$, there exists a unique ring homomorphism $\widetilde{\phi} : U^{-1}R \longrightarrow S$ such that $\widetilde{\phi} \circ \psi = \phi$. This is summed up by a commutative diagram.*

$$\begin{tikzcd} U \subseteq R \arrow[r, "\psi"] \arrow[d, "\phi"'] & U^{-1}R \arrow[dl, dashed, "\exists!\widetilde{\phi}"] \arrow[dl, phantom, bend right, "\circlearrowleft"] \\ S^\times \subseteq S & \end{tikzcd}$$

091517o **Example II.A.2.7.** If R is an integral domain, then $0 \lneq R$ is a prime ideal and $R \setminus \{0\}$ is multiplicatively closed. We call $(R \setminus \{0\})^{-1}R$ the <u>field of fractions</u> of R, denoted $Q(R)$, and $(R \setminus \{0\})^{-1}M$ is a vector space over the field of fractions.

091517p **Notation II.A.2.8.** Recall Example II.A.2.2. If $s \in R$, then $M_s := S^{-1}M$. If $\mathfrak{p} \lneq R$ is prime then $M_\mathfrak{p} := (R \setminus \mathfrak{p})^{-1}M$. Notice that in the M_s case, the multiplicatively closed subset <u>does contain</u> the element s, but in the $M_\mathfrak{p}$ case, the multiplicatively closed subset <u>does not</u> contain $\mathfrak{p}$.

091517q **Fact II.A.2.9.** *There is a one-to-one correspondence of prime ideals under this localization process. Explicitly, if $U \subseteq R$ is a multiplicatively closed subset and ψ is the ring homomorphism from Theorem II.A.2.6, then we have*

$$\{\textit{prime ideals of } U^{-1}R\} \rightleftarrows \{\textit{prime ideals } \mathfrak{q} \lneq R \mid \mathfrak{q} \cap U = \emptyset\}$$

$$Q \longmapsto \psi^{-1}(Q) = \{x \in R \mid \psi(x) \in Q\}$$

$$(x/1 \mid x \in \mathfrak{q})\, U^{-1}R = \mathfrak{q}(U^{-1}R) \longleftarrow\!\shortmid\; \mathfrak{q}$$

and the isomorphic relations

$$\frac{U^{-1}R}{\mathfrak{q}(U^{-1}R)} \cong U^{-1}(R/\mathfrak{q}) \qquad\qquad \begin{aligned} (U^{-1}R)_{\mathfrak{q}(U^{-1}R)} &\xleftarrow{\cong} R_{\mathfrak{q}} \, . \\ \frac{r/1}{z/1} &\longleftarrow\!\!\shortmid \frac{r}{z} \end{aligned}$$

091517r EXAMPLE II.A.2.10. Let $\mathfrak{p}$ be a prime ideal. The correspondence for $R_{\mathfrak{p}}$ under the description in Fact II.A.2.9 is

$$\{\text{prime ideals of } R_{\mathfrak{p}}\} \leftrightharpoons \{\text{prime ideals } \mathfrak{q} \lneq R \mid \mathfrak{q} \subseteq \mathfrak{p}\}$$

$$\frac{R_{\mathfrak{p}}}{\mathfrak{q}R_{\mathfrak{p}}} \cong (R/\mathfrak{q})_{\mathfrak{p}}$$

$$(R_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}} \cong R_{\mathfrak{q}}$$

Considering the special case $\mathfrak{q} = \mathfrak{p}$, we have two ways of thinking about a field.

$$\frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}} \cong (R/\mathfrak{p})_{\mathfrak{p}} \cong Q(R/\mathfrak{p})$$

Under the correspondence we know $\mathfrak{p}R_{\mathfrak{p}}$ is the unique maximal ideal of $\mathfrak{p}R_{\mathfrak{p}}$, so on the left-hand side we have a local ring modulo the unique maximal ideal, which must be a field. On the right-hand side, we have the field of fractions on the integral domain $R/\mathfrak{p}$.

091517s FACT II.A.2.11. *Given any R-module homomorphism $f : M \longrightarrow N$, this induces the following well-defined $U^{-1}R$-module homomorphism.*

$$U^{-1}f : U^{-1}M \longrightarrow U^{-1}N$$
$$\frac{m}{u} \longmapsto \frac{f(m)}{u}$$

PROOF. We need to check well-definedness and $U^{-1}R$-linearity. If $m/u, m'/u' \in U^{-1}M$ such that $m/u = m'/u'$, then there exists some $v \in U$ such that $vu'm = vum'$. Therefore

$$v \cdot u'f(m) = f(vu'm) = f(vum') = v \cdot uf(m')$$

which implies $f(m)/u = f(m')/u'$, so $U^{-1}f$ preserves equality. Since it also lands well by construction, it is well-defined. Letting $m/u, x/w \in U^{-1}M$ and $r/u \in U^{-1}R$, we verify linearity as follows.

$$\begin{aligned}
(U^{-1}f)\left(\frac{m}{u} + \frac{x}{w}\right) &= (U^{-1}f)\left(\frac{wm+ux}{uw}\right) \\
&= \frac{f(wm+ux)}{uw} \\
&= \frac{w \cdot f(m) + u \cdot f(x)}{uw} \\
&= \frac{w \cdot f(m)}{uw} + \frac{u \cdot f(x)}{uw} \\
&= \frac{f(m)}{u} + \frac{f(x)}{w} \\
&= (U^{-1}f)\left(\frac{m}{u}\right) + (U^{-1}f)\left(\frac{x}{w}\right)
\end{aligned}$$

$$\begin{aligned}
(U^{-1}f)\left(\frac{r}{u} \cdot \frac{m}{v}\right) &= (U^{-1}f)\left(\frac{rm}{uv}\right) \\
&= \frac{f(rm)}{uv} \\
&= \frac{r \cdot f(m)}{uv} \\
&= \frac{r}{u} \cdot \frac{f(m)}{v} \\
&= \frac{r}{u} \cdot (U^{-1}f)\left(\frac{m}{v}\right)
\end{aligned}$$

$\square$

091517t FACT II.A.2.12. *The operation* $U^{-1}(-)$ *is a covariant functor. Therefore it respects function composition and* $U^{-1}(\mathrm{id}_M) = \mathrm{id}_{U^{-1}M}$.

Motivating Ext

apter062921b

In this chapter we motivate our study of Ext modules by discussing three applications in abstract algebra. We also give a few major results that will be explored more fully in later chapters.

II.B.1. Application 1: Long Exact Sequence

ection062921h

Given a short exact sequence of R-modules and R-module homomorphisms

$$0 \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0$$

and given an R-module N, the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, M_1) \xrightarrow{f_{1*}} \mathrm{Hom}_R(N, M_2) \xrightarrow{f_{2*}} \mathrm{Hom}_R(N, M_3) \tag{II.B.1.0.1}$$ eqn082118d

is exact, where f_{i*} denotes $\mathrm{Hom}_R(N, f_i)$ and is defined as follows.

$$\begin{aligned} f_{i*} : \mathrm{Hom}_R(N, M_i) &\longrightarrow \mathrm{Hom}_R(N, M_{i+1}) \\ \phi &\longmapsto f_i \circ \phi \end{aligned}$$

A similar sequence was seen previously in Definition II.A.1.10.

Here is demonstrated why we say Hom is left-exact. Writing the zero on the left in Equation (II.B.1.0.1) maintains the exactness of the sequence. The contravariant sequence below is exact as well.

$$0 \longrightarrow \mathrm{Hom}_R(M_3, N) \xrightarrow{f_2^*} \mathrm{Hom}_R(M_2, N) \xrightarrow{f_1^*} \mathrm{Hom}_R(M_1, N)$$

Here f_i^* functions analogously to f_{i*} above.

$$\begin{aligned} f_i^* : \mathrm{Hom}_R(M_{i+1}, N) &\longrightarrow \mathrm{Hom}_R(M_i, N) \\ \psi &\longmapsto \psi \circ f_i \end{aligned}$$

If we were to put the zero module on the right of either the covariant sequence or the contravariant sequence, the exactness would fail in general at that point of the sequence. We can, however, compute something else on the right for a longer exact sequence. This is one of the first great achievements of homological algebra and the application from the title of this section. We will prove this in Section II.F.2 (see Theorem II.F.2.1).

091517v

Theorem II.B.1.1 (Long Exact Sequences). *Given the short exact sequence*

$$0 \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0$$

and an R-module N as above, there exist exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R(N, M_1) \arrow[r] & \mathrm{Hom}_R(N, M_2) \arrow[r] & \mathrm{Hom}_R(N, M_3) \arrow[dll, rounded corners] \\ & \mathrm{Ext}^1_R(N, M_1) \arrow[r] & \mathrm{Ext}^1_R(N, M_2) \arrow[r] & \mathrm{Ext}^1_R(N, M_3) \arrow[dll, rounded corners] \\ & \mathrm{Ext}^2_R(N, M_1) \arrow[r] & \cdots & \end{tikzcd}$$

and

$$0 \longrightarrow \operatorname{Hom}_R(M_3,N) \longrightarrow \operatorname{Hom}_R(M_2,N) \longrightarrow \operatorname{Hom}_R(M_1,N) \longrightarrow \operatorname{Ext}^1_R(M_3,N) \longrightarrow \operatorname{Ext}^1_R(M_2,N) \longrightarrow \operatorname{Ext}^1_R(M_1,N) \longrightarrow \operatorname{Ext}^2_R(M_3,N) \longrightarrow \cdots$$

where $\operatorname{Ext}^i_R(-,-)$ *will be defined after some discussion. We will simply say colloquially here that* Ext^1_R *measures the lack of right exactness of* Hom.

091517w DISCUSSION II.B.1.2. Given an R-module M, there exists a projective R-module P_0 and a surjective homomorphism $P_0 \overset{\tau}{\twoheadrightarrow} M$, because every R-module is a homomorphic image of a projective R-module. The sequence

$$P_0 \overset{\tau}{\twoheadrightarrow} M \longrightarrow 0$$

can be thought of as approximating M by the projective module P_0 where the error of the approximation is $\operatorname{Ker}\tau$. The sequence can be lengthened into the short exact sequence

$$0 \longrightarrow \operatorname{Ker}\tau \overset{\subseteq}{\hookrightarrow} P_0 \overset{\tau}{\twoheadrightarrow} N \longrightarrow 0\ .$$

The R-module $\operatorname{Ker}\tau$ can likewise be approximated by a projective R-module. That is there exists a sequence

$$P_1 \overset{\tau_1}{\twoheadrightarrow} \operatorname{Ker}\tau \longrightarrow 0$$

and the short exact sequence

$$0 \longrightarrow \operatorname{Ker}\tau_1 \overset{\subseteq}{\hookrightarrow} P_1 \overset{\tau_1}{\twoheadrightarrow} \operatorname{Ker}\tau \longrightarrow 0\ .$$

Inductively there exists a short exact sequence

$$0 \longrightarrow \operatorname{Ker}\tau_i \overset{\subseteq}{\hookrightarrow} P_i \overset{\tau_i}{\twoheadrightarrow} \operatorname{Ker}\tau_{i-1} \longrightarrow 0$$

for any $i \geq 2$, giving us diagram (II.B.1.5.1). Moreover, a standard diagram chase shows the infinite sequence

$$\ldots \longrightarrow P_4 \xrightarrow{\partial_4^P} P_3 \xrightarrow{\partial_3^P} P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \longrightarrow 0$$

is exact by virtue of the exactness of the short exact sequences that compose it, as we show next.

PROOF. For any $i \geq 1$ we want to show $\operatorname{Im}\partial^P_{i+1} = \operatorname{Ker}\partial^P_i$ by mutual containment. For any $b \in \operatorname{Im}\partial^P_{i+1}$, there exists some $a \in P_{i+1}$ such that $\partial^P_{i+1}(a) = b$ and by the commutivity of diagram (II.B.1.5.1), $b = \tau_{i+1}(a) \in \operatorname{Ker}\tau_i$, so $\tau_i(b) = 0$. Again by the commutivity of the diagram $\partial^P_i(b) = 0$, so $b \in \operatorname{Ker}\partial^P_i$ and thus $\operatorname{Im}\partial^P_{i+1} \subseteq \operatorname{Ker}\partial^P_i$.

For any $d \in \operatorname{Ker}\partial^P_i$, the commutivity of the diagram implies $d \in \operatorname{Ker}\tau_i$. Since τ_{i+1} surjective we let $c \in P_{i+1}$ such that $\tau_{i+1}(c) = d$ and by the commutivity of the diagram $\partial^P_{i+1}(c) = d$, so $d \in \operatorname{Im}\partial^P_{i+1}$ and therefore $\operatorname{Ker}\partial^P_i \subseteq \operatorname{Im}\partial^P_{i+1}$, which establishes equality. The proof at the $i = 0$ step using τ is just as straightforward. □

From this construction we define some new notation.

091517x DEFINITION II.B.1.3. Every R-module M has an associated exact sequence, called an augmented projective resolution,

$$P_\bullet^+ \ = \qquad \cdots \longrightarrow P_4 \xrightarrow{\partial_4^P} P_3 \xrightarrow{\partial_3^P} P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \longrightarrow 0$$

where each module P_i is projective and τ is a surjection, an associated (truncated) projective resolution (not typically exact),

$$P_\bullet \ = \qquad \cdots \longrightarrow P_4 \xrightarrow{\partial_4^P} P_3 \xrightarrow{\partial_3^P} P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \longrightarrow 0$$

and an associated Hom sequence

$$P_\bullet^* \;=\; \mathrm{Hom}_R(P_\bullet, N) \;=\; \qquad 0 \longrightarrow P_0^* \xrightarrow{(\partial_1^P)^*} P_1^* \xrightarrow{(\partial_2^P)^*} P_2^* \xrightarrow{(\partial_3^P)^*} P_3^* \xrightarrow{(\partial_4^P)^*} P_4^* \longrightarrow \cdots.$$

The maps ∂_i^P are the <u>differentials</u> of the resolution.

091517y Fact II.B.1.4. *In the notation of II.B.1.3, we have*

$$\left(\partial_{n+1}^P\right)^* \circ \left(\partial_n^P\right)^* = \left(\partial_n^P \circ \partial_{n+1}^P\right)^* = 0^* = 0.$$

In other words, since $\mathrm{Hom}_R(-, N)$ *is a functor, we have* $\mathrm{Im}\,\partial_i^{P*} \subseteq \mathrm{Ker}\,\partial_{i+1}^{P*}$ *by Remark II.A.1.13.*

091517z Definition II.B.1.5. Given a projective resolution of an R-module M in the notation of Definition II.B.1.3 and given an arbitrary R-module N, we define

$$\mathrm{Ext}_R^i(M, N) = \frac{\mathrm{Ker}\left(\partial_{i+1}^P\right)^*}{\mathrm{Im}\left(\partial_i^P\right)^*}.$$

Colloquially,

$$\mathrm{Ext}_R^i(M, N) = \frac{\mathrm{Ker}\,\text{outgoing from } i^{th} \text{ position}}{\mathrm{Im}\,\text{incoming to } i^{th} \text{ position}}.$$

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & & & \\
& \operatorname{Ker}\tau_4 \arrow[dr, "\subseteq"] & & & & \operatorname{Ker}\tau_2 \arrow[ur] \arrow[dr, "\subseteq"] \arrow[d, phantom, "\circlearrowleft"] & & & & \operatorname{Ker}\tau \arrow[ur] \arrow[dr, "\subseteq"] \arrow[d, phantom, "\circlearrowleft"] & & & & 0 & \\
\cdots \arrow[rr] & & P_4 \arrow[rr, "\partial_4^P"] \arrow[dr, "\tau_4"'] & & P_3 \arrow[rr, "\partial_3^P"'] \arrow[ur, "\tau_3"] & & P_2 \arrow[rr, "\partial_2^P"] \arrow[dr, "\tau_2"'] & & P_1 \arrow[rr, "\partial_1^P"'] \arrow[ur, "\tau_1"] & & P_0 \arrow[rr, "\tau"] \arrow[dr, "\tau"'] & & M \arrow[ur] \arrow[rr] & & 0 \\
& & & \operatorname{Ker}\tau_3 \arrow[ur, "\subseteq"'] \arrow[dr] \arrow[u, phantom, "\circlearrowleft"] & & & & \operatorname{Ker}\tau_1 \arrow[ur, "\subseteq"'] \arrow[dr] \arrow[u, phantom, "\circlearrowleft"] & & & & M \arrow[ur, "="'] \arrow[dr] \arrow[u, phantom, "\circlearrowleft"] & & & \\
& & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd} \tag{II.B.1.5.1}$$

$\boxed{\texttt{eqn070118a}}$

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EXAMPLE II.B.1.6. Let N be an R-module. Then

$$\operatorname{Ext}^i_R(R,N) \cong \begin{cases} N & i=0 \\ 0 & i \neq 0 \end{cases}.$$

Indeed, since R is projective (consider Definition II.A.1.14(e)), we have the augmented projective resolution of R

$$P_\bullet^+ \quad = \qquad 0 \longrightarrow R \xrightarrow{\mathrm{id}} R \longrightarrow 0$$

which is exact by Fact II.A.1.5. The corresponding projective resolution is therefore

$$P_\bullet \quad = \qquad 0 \longrightarrow R \longrightarrow 0\ .$$

To compute Ext we need the following sequence $\operatorname{Hom}_R(P_\bullet, N)$:

$$0 \xrightarrow{f} \operatorname{Hom}_R(R,N) \xrightarrow{g} 0.$$

From position $i=0$ we have

$$\operatorname{Ext}^0_R(R,N) = \frac{\operatorname{Ker} g}{\operatorname{Im} f} = \frac{\operatorname{Hom}_R(R,N)}{0} \cong \operatorname{Hom}_R(R,N) \cong N$$

and for any $i \neq 0$ we have

$$\operatorname{Ext}^i_R(R,N) = \frac{0}{0} \cong 0.$$

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NOTE II.B.1.7. In general, if $P_i = 0$, then $\operatorname{Hom}_R(P_i, N) = 0$ and therefore $\operatorname{Ext}^i_R(M,N) = 0$.

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NOTATION II.B.1.8. For an R-module M and any $r \in R$, the multiplication map

$$\begin{aligned} \mu_r \colon M &\longrightarrow M \\ m &\longmapsto rm \end{aligned}$$

is a well-defined R-module homomorphism by the axioms for R- modules. Unless otherwise noted, we will let μ_x denote a multiplication map by the element x.

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LEMMA II.B.1.9. *Consider a commutative diagram of R-modules and R-module homomorphisms*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r, "g"] \arrow[d, "\beta"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "\gamma"', "\cong"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C \end{tikzcd}$$

and assume $g \circ f = 0$ (and consequently $g' \circ f' = 0$). Then there is a well-defined R-module isomorphism

$$\begin{aligned} \overline{\beta} \colon \frac{\operatorname{Ker} g}{\operatorname{Im} f} &\longrightarrow \frac{\operatorname{Ker} g'}{\operatorname{Im} f'} \\ b + \operatorname{Im} f &\longmapsto \beta(b) + \operatorname{Im} f'. \end{aligned}$$

Proof. We give here only a sketch via a commutative diagram.

$$\begin{tikzcd}
& A \arrow[rrrr, "f"] \arrow[dr, "\tilde{f}"] \arrow[ddd, "\alpha"', "\cong"] & & & & B \arrow[rrr, "g"] \arrow[ddd, "\beta"', "\cong"] & & & C \arrow[ddd, "\gamma"', "\cong"] \\
0 \arrow[rr] & & \operatorname{Im} f \arrow[rr, "\subseteq"] \arrow[dr] \arrow[ddd, dashed, "\exists\beta''"'] & & \operatorname{Ker} g \arrow[rr, "\pi"] \arrow[ur, "\subseteq"] \arrow[ddd, dashed, "\exists\beta'"'] & & \frac{\operatorname{Ker} g}{\operatorname{Im} f} \arrow[r] \arrow[ddd, dashed, "\therefore\exists\overline{\beta}"', "\cong"] & 0 & \\
& & & 0 \arrow[ur] & & & & & \\
& A' \arrow[rrrr, "f'"] \arrow[dr, "\tilde{f}'"] & & & & B' \arrow[rrr, "g'"] & & & C' \\
0 \arrow[rr] & & \operatorname{Im} f' \arrow[rr, "\subseteq"] \arrow[dr] & & \operatorname{Ker} g' \arrow[rr, "\pi'"] \arrow[ur, "\subseteq"] & & \frac{\operatorname{Ker} g'}{\operatorname{Im} f'} \arrow[r] & 0 & \\
& & & 0 \arrow[ur] & & & & &
\end{tikzcd}$$

□

091517ac Example II.B.1.10. Let A be a non-zero commutative ring with identity and set $R = A[x]$ and $\mathfrak{a} = (x)R$. Note $R/\mathfrak{a} \cong A$ and therefore A is an R-module. Then we will show

$$\operatorname{Ext}^i_R(A,R) \cong \begin{cases} A & i = 1 \\ 0 & i \neq 1 \end{cases} \qquad \operatorname{Ext}^i_R(A,A) \cong \begin{cases} A & i = 0,1 \\ 0 & \text{else.} \end{cases}$$

We begin with an augmented projective resolution of A from the diagram

$$\begin{tikzcd}
& 0 \arrow[dr] & & & & \\
0 \arrow[rr] & & R \arrow[rr, "\mu_x"] \arrow[dr, "\mu_x"'] & & R \arrow[r, two heads, "\tau"] & R/\mathfrak{a} \cong A \\
& & & \mathfrak{a} \arrow[ur, "\subseteq"'] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

where μ_x is multiplication by x and τ is the natural surjection. Define $P_\bullet^+$ to be the row from the above diagram. Hence

$$P_\bullet = \qquad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow R \xrightarrow{\mu_x} R \longrightarrow 0$$

and

$$P_\bullet^* = \qquad \begin{tikzcd}
0 \arrow[r] \arrow[d, "\cong"] & \operatorname{Hom}_R(R,R) \arrow[r, "\mu_x^*"] \arrow[d, "\cong"] & \operatorname{Hom}_R(R,R) \arrow[r] \arrow[d, "\cong"] & \operatorname{Hom}_R(0,R) \arrow[r] \arrow[d, "\cong"] & \cdots \\
0 \arrow[r, "f"] & \underset{i=0}{R} \arrow[r, "\mu_x"] & \underset{i=1}{R} \arrow[r, "g"] & \underset{i=2}{0} \arrow[r] & \cdots
\end{tikzcd}$$

where the vertical isomorphisms are by Hom-cancellation. We can now calculate $\operatorname{Ext}^i_R(A,R)$ from the bottom row of this diagram because of Lemma II.B.1.9.

$$\operatorname{Ext}^i_R(A,R) = \begin{cases} 0/0 = 0 & i \neq 0,\ i \neq 1 \\ \operatorname{Ker}\mu_x/\operatorname{Im} f = 0/0 & i = 0 \\ \operatorname{Ker} g/\operatorname{Im}\mu_x = R/\mathfrak{a} \cong A & i = 1 \end{cases}$$

We calculate $\operatorname{Ext}^i_R(A, A)$ similarly.

$$\operatorname{Hom}_R(P_\bullet, A) \cong \qquad 0 \xrightarrow{h} A \xrightarrow[0]{\mu_x^A} A \xrightarrow{k} 0 \longrightarrow 0 \longrightarrow \cdots$$

which implies that

$$\operatorname{Ext}^i_R(A, A) = \begin{cases} 0/0 = 0 & i \neq 0, i \neq 1 \\ \operatorname{Ker} \mu_x^A / \operatorname{Im} h = A/0 \cong A & i = 0 \\ \operatorname{Ker} k / \operatorname{Im} \mu_x^A = A/0 \cong A & i = 1 \end{cases}$$

Note in this case μ_x^A is the zero map since $\operatorname{Im} \mu_x^A = \mathfrak{a}/\mathfrak{a} = \{0\} \subset R/\mathfrak{a} \cong A$.

One might wonder why we did not write $\operatorname{Ext}^0_R(N, M_\ell)$ in Example II.B.1.1, so here we give a reason in the form of a proposition.

091517ad PROPOSITION II.B.1.11. *For any two R-modules M and N*

$$\operatorname{Ext}^0_R(M, N) \cong \operatorname{Hom}_R(M, N).$$

PROOF. Let M and N be two R-modules and let

$$P_\bullet^+ \ = \qquad \ldots \longrightarrow P_4 \xrightarrow{\partial_4^P} P_3 \xrightarrow{\partial_3^P} P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \longrightarrow 0$$

be an augmented projective resolution of M. Since Hom is left-exact, the following piece of the sequence $\operatorname{Hom}_R(P_\bullet^+, N)$ is exact as well.

$$0 \longrightarrow \operatorname{Hom}_R(M, N) \xrightarrow{\tau^*} \operatorname{Hom}_R(P_0, N) \xrightarrow{(\partial_1^P)^*} \operatorname{Hom}_R(P_1, N)$$

This exactness yields

$$\operatorname{Ker} \partial_1^{P*} = \operatorname{Im} \tau^* \overset{(1)}{\cong} \frac{\operatorname{Hom}_R(M, N)}{\operatorname{Ker} \tau^*} \overset{(2)}{=} \frac{\operatorname{Hom}_R(M, N)}{\{0\}} \cong \operatorname{Hom}_R(M, N)$$

where (1) holds by the First Isomorphism Theorem and (2) holds since

$$\operatorname{Ker} \tau^* = \operatorname{Im} \ 0 \longrightarrow \operatorname{Hom}_R(M, N) \ = \{0\}.$$

On the other hand, from the definition of Ext we have

$$\operatorname{Ext}^0_R(M, N) = \frac{\operatorname{Ker} \left(\partial_1^P\right)^*}{\operatorname{Im} 0 \to \operatorname{Hom}_R(P_0, N)} = \frac{\operatorname{Ker} \left(\partial_1^P\right)^*}{\{0\}} \cong \operatorname{Ker} \left(\partial_1^P\right)^*.$$

□

prop110417k PROPOSITION II.B.1.12. *Given R-modules and a projective resolution as in the above discussion, we have the following.*

(a) $\operatorname{Ext}^i_R(M, N) = 0$ *for all* $i < 0$
(b) $\operatorname{Ext}^i_R(M, 0) = 0$ *for all* $i \in \mathbb{Z}$
(c) $\operatorname{Ext}^i_R(0, N) = 0$ *for all* $i \in \mathbb{Z}$

PROOF. (a) We have $(P_\bullet^*)_i = 0$ for all $i < 0$. Therefore $(\partial_i^P)^* \colon 0 \longrightarrow 0$ for all $i < 0$ and $(\partial_0^P)^* \colon 0 \longrightarrow P_0^*$. It follows that

$$\operatorname{Ext}^i_R(M, N) = \frac{\operatorname{Ker} (\partial_{i+1}^P)^*}{\operatorname{Im} (\partial_i^P)^*} = \frac{0}{0} = 0$$

for all $i < 0$.

(b) For any $i \in \mathbb{Z}$ we have

$$\operatorname{Hom}_R(P_\bullet, 0)_{-i} = \operatorname{Hom}_R(P_i, 0) = 0.$$

Then $(\partial_i^P)^* \colon 0 \longrightarrow 0$ and therefore

$$\operatorname{Ext}^i_R(M, 0) = \frac{0}{0} = 0$$

for all $i \in \mathbb{Z}$.

(c) We can define a projective resolution of the R-module $M = 0$.

$$P_\bullet^+ = P_\bullet = \qquad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

Therefore

$$\operatorname{Hom}_R(P_\bullet, N)_{-i} = \operatorname{Hom}_R(0, N) = 0$$

for all $i \in \mathbb{Z}$ and hence

$$\operatorname{Ext}^i_R(0, N) = \frac{0}{0} = 0$$

again for all $i \in \mathbb{Z}$. □

091517ae FACT II.B.1.13. *Ext is well-defined. That is, the calculation of* $\operatorname{Ext}^i_R(M, N)$ *is independent (up to isomorphism) of our choice of projective resolution of* M*.*

Establishing the Fact II.B.1.13 is the main point of Chapter II.F. See Theorem II.F.5.2.

II.B.2. Application 2: Depth

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Depth is a nice tool on which to perform induction arguments. One thing that makes it so versatile is that it has strong ties to Ext modules.

091517af DEFINITION II.B.2.1. Let M be an R-module. An element $x \in R$ is a non-zero-divisor on M if the sequence $0 \longrightarrow M \xrightarrow{\mu_x} M$ is exact (i.e., for all $m \in M$, $xm = 0$ implies $m = 0$). We say x is M-regular if x is a non-zero-divisor on M and $xM \neq M$ (i.e., $M/xM \neq 0$). A sequence $\underline{x} = x_1, \ldots, x_n \in R$ is M-regular if x_1 is M-regular and x_i is $M/(x_1, \ldots, x_{i-1})M$-regular for all $i = 2, \ldots, n$.

091517ah FACT/DEFINITION II.B.2.2. *Let* R *be noetherian and* $\mathfrak{a} \leq R$ *an ideal such that* $\mathfrak{a}M \neq M$*. Then there exists a maximal* M*-regular sequence in* $\mathfrak{a}$*. That is, there exists an* M*-regular sequence* $\underline{x} = x_1, \ldots, x_n \in \mathfrak{a}$ *such that for all* $y \in \mathfrak{a}$*, the sequence* $x_1, \ldots, x_n, y$ *is not* M*-regular. The longest length* n *of an* M*-regular sequence in* $\mathfrak{a}$ *is called the depth of* $\mathfrak{a}$ *on* M*, denoted*

$$n = \operatorname{depth}(\mathfrak{a}, M).$$

091517ai FACT II.B.2.3. *Depth is independent of our choice of maximal* M*-regular sequence as long as* M *is finitely generated. The proof of this fact requires* Ext*. One proves there exists some* $n \in \mathbb{N}_0$ *such that* $\operatorname{Ext}^i_R(R/\mathfrak{a}, M) = 0$ *whenever* $0 \leq i \leq n-1$ *and* $\operatorname{Ext}^n_R(R/\mathfrak{a}, M) \neq 0$*, in order to conclude*

$$\operatorname{depth}(\mathfrak{a}, M) = \inf\left\{ i \geq 0 \;\middle|\; \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \neq 0 \right\}.$$

Establishing Fact II.B.2.3 is the goal of Chapter II.C. See Theorem II.C.5.16.

II.B.3. Application 3: Localization Problem for Regular Local Rings

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Here we introduce regular rings. The question of whether regularity is preserved under localization (Question II.B.3.7) was one of the great open questions solved using homological methods. We give an answer immediately in Theorem II.B.3.8, which is seen again later (Theorem II.G.4.11). Throughout the section assume $(R, \mathfrak{m}, \mathfrak{K})$ is a local, noetherian ring. That is, assume $\mathfrak{m}$ is the unique maximal ideal and $\mathfrak{K} \cong R/\mathfrak{m}$.

091517aj DEFINITION II.B.3.1. The Krull dimension, or just dimension, of R can be said to measure the size of R and is defined

$$\dim(R) = \sup\left\{ n \geq 0 \mid \exists\, \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subsetneq R \text{ s.t. } \mathfrak{p}_i \text{ prime}, \forall\, i = 1, \ldots, n \right\}.$$

Under our local and noetherian assumptions, Krull dimension is finite.

091517ak DEFINITION II.B.3.2. The embedding dimension is defined as the dimension of a particular R-module as a $\mathfrak{K}$-vector space.

$$\operatorname{edim}(R) = \dim_{\mathfrak{K}}(\mathfrak{m}/\mathfrak{m}^2)$$

Since $\mathfrak{m}/\mathfrak{m}^2$ is an R-module satisfying $\mathfrak{m} \cdot (\mathfrak{m}/\mathfrak{m}^2) = 0$, it is also an $R/\mathfrak{m}$-module. That is, it is a $\mathfrak{K}$-vector space (since $\mathfrak{K}$ a field) and moreover since R is noetherian, $\mathfrak{m}/\mathfrak{m}^2$ is finitely generated over R and is consequently a finite dimensional vector space over $\mathfrak{K}$. In summary, the noetherian assumption on R again guarantees a finite dimension.

091517al THEOREM II.B.3.3. *One has*

$$\operatorname{depth}(\mathfrak{m}, R) \overset{(1)}{\leq} \dim(R) \overset{(2)}{\leq} \operatorname{edim}(R).$$

091517am DEFINITION II.B.3.4. R is <u>Cohen-Macauley</u> if (1) is an equality and R is <u>regular</u> if (2) is an equality.

091517an FACT II.B.3.5. *Every regular ring is Cohen-Macaulay.*

091517ao EXAMPLE II.B.3.6. For the localization ring

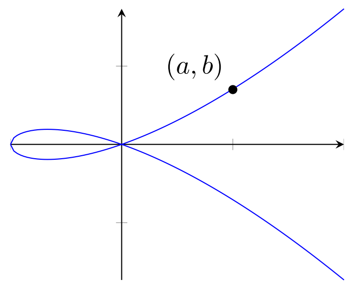
$$R = \mathbb{K}[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$$

with unique maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)R$, we have $\dim(R) = n$ and $\operatorname{edim}(R) = n$, so the ring is regular and one can think of R as the geometric object $\mathbb{K}^n$ (e.g., $\mathbb{R}^n$ or $\mathbb{C}^n$). There is more on the construction of localization rings in the preliminaries.

In many ways the quotient ring

$$R_0 = \frac{\mathbb{R}[x, y]}{(y^2 - x^2(x + 1))}$$

represents the curve $y^2 = x^2(x + 1)$, which we plot in the Cartesian coordinate plane below. This plot tells us a number of things about the ring R_0, though none of them are necessarily obvious.



- Points $p = (a, b)$ on the curve correspond to maximal ideals $\mathfrak{m}_p = (x - a, y - b)R_0$ and the local ring $(R_0)_{\mathfrak{m}_p}$ has the maximal ideal $(x - a, y - b)(R_0)_{\mathfrak{m}_p}$.
- All rings $(R_0)_{\mathfrak{m}_p}$ have Krull dimension 1, because the curve is 1-dimensional.
- If p is a smooth point of the curve, then the ring $(R_0)_{\mathfrak{m}_p}$ is regular.
- $\operatorname{edim}((R_0)_{\mathfrak{m}_p}) = \dim_{\mathbb{R}}(\text{tangent space at } p)$.
- At the origin $p = (0, 0)$, $\operatorname{edim}((R_0)_{\mathfrak{m}_p}) = 2$ and therefore $(R_0)_{\mathfrak{m}_p}$ is not regular.
- The localization in this example can be thought of as zooming in on some neighborhood of your point, so it should at least not make the singularity worse.

An important question from the early 1900's asked if regularity is preserved under localization, which is meaning of this section.

091517ap QUESTION II.B.3.7. If R is regular and $\mathfrak{p} \lneq R$ is prime, must $R_{\mathfrak{p}}$ necessarily be regular as well?

It turns out that the answer is 'yes'. This is highly nontrivial because while one can exert some control from $\dim(R)$ to $\dim(R_{\mathfrak{p}})$, controlling $\operatorname{edim}(R_{\mathfrak{p}})$ is harder and requires homological algebra. The essential point is in the following theorem.

091517aq THEOREM II.B.3.8 (Auslander, Buchsbaum, Serre). *The following are equivalent.*

(a) *R is regular.*
(b) *For any two finitely generated modules M and N, $\operatorname{Ext}^i_R(M, N) = 0$ for all $i > \dim(R)$.*
(c) $\operatorname{Ext}^{\dim(R)+1}_R(\mathfrak{K}, \mathfrak{K}) = 0$.
(d) *There exists some $d \geq 0$ such that $\operatorname{Ext}^d_R(\mathfrak{K}, \mathfrak{K}) = 0$.*

The proof of this result, unfortunately, is outside the scope of this book.

Exercises

exer170827a EXERCISE II.B.3.9. Let A be a non-zero commutative ring with identity, and set $R = A[x, y]$, the polynomial ring over A on two variables. Set $\mathfrak{a} = (x, y)R \subseteq R$, and note that $R/\mathfrak{a} \cong A$. in particular, A is an R-module.

item170828a (a) Prove that the following sequence is exact

$$0 \to R \xrightarrow{\binom{y}{-x}} R^2 \xrightarrow{(x\ y)} R \xrightarrow{\tau} A \to 0$$

where τ is the natural surjection.

item170828b (b) Use the exact sequence from part (a) with the isomorphism $\operatorname{Hom}_R(R, *) \cong *$ to prove that

$$\operatorname{Ext}^i_R(A, R) \cong \begin{cases} 0 & \text{for } i \neq 2 \\ A & \text{for } i = 2 \end{cases}$$

$$\operatorname{Ext}^i_R(A, A) \cong A^{\binom{2}{i}} \quad \text{for all } i$$

item170828c (c) Prove that $\operatorname{Ext}^0_R(A, R) = 0 = \operatorname{Hom}_R(A, R)$ and $\operatorname{Ext}^0_R(A, A) \cong A \cong \operatorname{Hom}_R(A, A)$.

item170828d (d) Compute $\operatorname{depth}_R(\mathfrak{a}; R)$ and $\operatorname{depth}_R(\mathfrak{a}, A)$.

exer170828a EXERCISE II.B.3.10. Let R be a non-zero commutative ring with identity, and let M be an R-module. Prove that the following conditions are equivalent.

exer170828a1 (i) Every submodule of M is finitely generated over R.

exer170828a2 (ii) (ACC on submodules) Every chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq M$ of R-submodules stabilizes, that is, for every such chain there is an integer n such that $M_n = M_{n+1} = M_{n+2} = \cdots$.

exer170828a3 (iii) (maximum condition) Every nonempty set S of R-submodules of M has a maximal element (with respect to set inclusion), that is, there is an element $N \in S$ such that for all $N' \in S$ if $N \subseteq N'$, then $N = N'$.

If M satisfies these equivalent conditions, then M is a noetherian R-module. (Hint: model your proof on the proof of the corresponding result for rings and ideals.) Observe that R is a noetherian ring if and only if R is noetherian as an R-module.

exer170828b EXERCISE II.B.3.11. Let R be a non-zero commutative ring with identity. Consider an exact sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

of R-modules and R-module homomorphisms. Prove that M is noetherian over R if and only if M' and M'' are noetherian over R.

exer170828c EXERCISE II.B.3.12. Let R be a non-zero commutative ring with identity. Use Exercise II.B.3.11 to show that the following conditions are equivalent for an R-module M:

(i) M is noetherian over R,
(ii) M^n is noetherian over R for all $n \in \mathbb{N} = \{1, 2, 3 \ldots\}$, and
(iii) M^n is noetherian over R for some $n \in \mathbb{N}$.

exer170828d EXERCISE II.B.3.13. Let R be a non-zero commutative ring with identity. Use Exercise II.B.3.12 to show that the following conditions are equivalent.

(i) R is a noetherian ring,
(ii) R^n is noetherian over R for all $n \in \mathbb{N} = \{1, 2, 3 \ldots\}$, and
(iii) R^n is noetherian over R for some $n \in \mathbb{N}$.

exer170828e EXERCISE II.B.3.14. Let R be a non-zero commutative noetherian ring with identity, and let M be an R-module. Use the above exercises to show that the following conditions are equivalent.

(i) M is finitely generated over R.
(ii) M is noetherian over R.
(iii) M has a degree-wise finite free resolution, that is, there is an exact sequence

$$\cdots \to R^{\beta_2} \to R^{\beta_1} \to R^{\beta_0} \to M \to 0$$

with each $\beta_i \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.

CHAPTER II.C

Depth by Ext

apter062021c

In this chapter we build the tools we need to characterize depth in terms of Ext (Fact II.B.2.3), which is given with proof as Theorem II.C.5.16 at the end of the chapter.

ction062921k

II.C.1. Hom and Direct Sums of Modules

In this section we observe that direct sums of modules interact very intuitively with functors like $\operatorname{Hom}_R(-,N)$ and $U^{-1}(-)$. We conclude the section by proving in Proposition II.C.1.8 that with a few assumptions, the two functors interact with one another exactly as one might like them to.

091517at

Fact II.C.1.1. *If M and M' are two R-modules, then there is a split short exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\epsilon"] & M \oplus M' \arrow[r, "\tau'"] \arrow[l, dashed, bend left, "\tau"] & M' \arrow[r] \arrow[l, dashed, bend left, "\epsilon'"] & 0 \end{tikzcd}$$

where $\tau \circ \epsilon = \mathrm{id}_M$ and $\tau' \circ \epsilon' = \mathrm{id}_{M'}$, and we have

$$\begin{tikzcd} \operatorname{Hom}_R(M \oplus M', N) \arrow[r, "\omega", "\cong"'] & \operatorname{Hom}_R(M,N) \oplus \operatorname{Hom}_R(M',N) \\ \psi \arrow[r, mapsto] & (\psi \circ \epsilon, \psi \circ \epsilon') = (\epsilon^*(\psi), \epsilon'^*(\psi)) \end{tikzcd}$$

Proof. Applying $\operatorname{Hom}_R(-,N)$ to the split exact sequence above we get

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_R(M',N) \arrow[r, "\tau'^*"] & \operatorname{Hom}_R(M \oplus M', N) \arrow[r, "\epsilon^*"] & \operatorname{Hom}_R(M,N) \arrow[r, dashed] \arrow[l, dashed, bend left, "\tau^*"] & 0 \end{tikzcd} \tag{II.C.1.1.1}$$ eqn082218e

tacking on a zero on the right-hand side. We claim this is a short exact sequence. Indeed since Hom is left-exact and

$$\epsilon^* \circ \tau^* = (\tau \circ \epsilon)^* = (\mathrm{id}_M)^* = \mathrm{id}_{\operatorname{Hom}_R(M,N)}$$

we know ϵ^* is surjective and therefore (II.C.1.1.1) is a short exact sequence.

From here we can take one of two approaches to reach the desired conclusion. On the one hand, note that we now have a split exact sequence in (II.C.1.1.1), so the desired isomorphism follows immediately from the definition of a split sequence in Definition II.A.1.8. On the other hand, we can also prove directly that the map ω is an isomorphism as follows.

We claim the following is a homomorphism of short exact sequences, for which it suffices to show ω is a well-defined R-module homomorphism and the proposed maps make the diagram commute. This will complete the proof by the Short-Five Lemma.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "="'] & \operatorname{Hom}_R(M',N) \arrow[r, "\tau'^*"] \arrow[d, "="'] \arrow[dr, phantom, "\circlearrowright ?"] & \operatorname{Hom}_R(M\oplus M',N) \arrow[r, "\epsilon^*"] \arrow[d, "\omega"] \arrow[dr, phantom, "\circlearrowright ?"] & \operatorname{Hom}_R(M,N) \arrow[r] \arrow[d, "="] & 0 \arrow[d, "="] \\
0 \arrow[r] & \operatorname{Hom}_R(M',N) \arrow[r, "E'"'] & \operatorname{Hom}_R(M,N)\oplus\operatorname{Hom}_R(M',N) \arrow[r, "T"'] & \operatorname{Hom}_R(M,N) \arrow[r] & 0
\end{tikzcd}$$

$$\alpha \xmapsto{\quad E' \quad} (0,\alpha)$$

$$(\beta,\gamma) \xmapsto{\quad T \quad} \beta$$

For an arbitrary pair of elements $\psi_1,\psi_2 \in \operatorname{Hom}_R(M\oplus M',N)$ and for any $r\in R$ we have

$$\begin{aligned}
\omega(r\psi_1+\psi_2) &= ((r\psi_1+\psi_2)\circ\varepsilon\,,\,(r\psi_1+\psi_2)\circ\varepsilon') \\
&= ((r\psi_1)\circ\varepsilon+\psi_2\circ\varepsilon\,,\,(r\psi_1)\circ\varepsilon'+\psi_2\circ\varepsilon') \\
&= (r(\psi_1\circ\varepsilon)+\psi_2\circ\varepsilon\,,\,r(\psi_1\circ\varepsilon')+\psi_2\circ\varepsilon') \\
&= (r(\psi_1\circ\varepsilon)\,,\,r(\psi_1\circ\varepsilon'))+(\psi_2\circ\varepsilon\,,\,\psi_2\circ\varepsilon') \\
&= r(\psi_1\circ\varepsilon\,,\,\psi_1\circ\varepsilon')+(\psi_2\circ\varepsilon\,,\,\psi_2\circ\varepsilon') \\
&= r\cdot\omega(\psi_1)+\omega(\psi_2)
\end{aligned}$$

Thus ω is a well-defined R-module homomorphism. Consider an arbitrary $\alpha\in\operatorname{Hom}_R(M',N)$ and we have

$$(\omega\circ\tau'^*)(\alpha)=\omega(\alpha\circ\tau')=(\alpha\circ\tau'\circ\epsilon,\alpha\circ\tau'\circ\epsilon')\overset{(1)}{=}(\alpha\circ 0,\alpha\circ\mathrm{id}_{M'})=(0,\alpha)=E'(\alpha)$$

where (1) holds since $\operatorname{Im}\varepsilon=\ker(\tau')$. Now taking an arbitrary $\psi\in\operatorname{Hom}_R(M\oplus M',N)$ we have

$$(T\circ\omega)(\psi)=T(\psi\circ\epsilon,\psi\circ\epsilon')=\psi\circ\epsilon=\epsilon^*(\psi).$$

So the diagram commutes and ω must be an isomorphism by the Short-Five Lemma. $\square$

091517au

Example II.C.1.2. Using II.C.1.1 as a base case, one can prove inductively that

$$\begin{aligned}
\operatorname{Hom}_R\left(\bigoplus_{i=1}^n M_i, N\right) &\longrightarrow \bigoplus_{i=1}^n \operatorname{Hom}_R(M_i,N) \\
\psi &\longmapsto \begin{pmatrix}\epsilon_1^*(\psi)\\ \vdots \\ \epsilon_n^*(\psi)\end{pmatrix}
\end{aligned}$$

is an isomorphism, where $\varepsilon_j\colon M_j \longrightarrow \bigoplus_{i=1}^n M_i$ is the standard injection. In particular the map

$$\begin{aligned}
\omega_n : \operatorname{Hom}_R(R^n,R) &\longrightarrow R^n \\
\psi &\longmapsto \begin{pmatrix}\psi(e_1)\\ \vdots \\ \psi(e_n)\end{pmatrix}
\end{aligned}$$

is an isomorphism, where $e_1,\ldots,e_n\in R^n$ is the standard basis for R^n. Note in this case the base case is simply Hom-cancellation, i.e., $\operatorname{Hom}_R(R,R)\cong R$. Moreover, if we let $v_1,\ldots,v_m\in R^m$ be the standard basis vectors of R^m, let $e_1^*,\ldots,e_n^*$ and $v_1^*,\ldots,v_m^*$ be the respective dual basis vectors, and let $\phi: R^m\longrightarrow R^n$ be an R-module homomorphism represented by a matrix A, where the jth column of A is $\phi(v_j)$, then $\operatorname{Hom}_R(-,R)$ yields

$$\begin{tikzcd}
\{e_1^*,\ldots,e_n^*\}\subset\operatorname{Hom}_R(R^n,R) \arrow[r, "\phi^*"] \arrow[d, "\omega_n", "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & \operatorname{Hom}_R(R^m,N)\supset\{v_1^*,\ldots,v_m^*\} \arrow[d, "\omega_m"', "\cong"] \\
\{e_1,\ldots,e_n\}\subset R^n \arrow[r, "A^T"'] & R^m\supset\{v_1,\ldots,v_m\}.
\end{tikzcd}$$

One can prove the diagram commutes using the basis vectors and the dual basis vectors, which in conjunction with linearity, proves the entire diagram commutes. The first direction looks like

$$(\omega_m \circ \phi^*)(e_i^*) = \omega_m(e_i^* \circ \phi) = \begin{pmatrix} (e_i^* \circ \phi)(v_1) \\ \vdots \\ (e_i^* \circ \phi)(v_m) \end{pmatrix} \overset{(2)}{=} \begin{pmatrix} a_{i1} \\ \vdots \\ a_{im} \end{pmatrix} = \left(i^{th} \text{ row of } A\right)^T$$

where (2) holds since $e_i^*(\phi(v_j))$ is simply e_i^* applied to the j^{th} column of A, which is a_{ij}. The second direction looks like

$$(A^T \circ \omega_n)(e_i^*) = A^T \cdot \begin{pmatrix} e_i^*(e_1) \\ \vdots \\ e_i^*(e_i) \\ \vdots \\ e_i^*(e_n) \end{pmatrix} = i^{th} \text{ column of } (A^T).$$

Therefore the diagram commutes.

We now state an even more general version of Fact II.C.1.1 without proof.

fact082218a FACT II.C.1.3. *For a direct sum of an arbitrary collection of R-modules, denoted $\bigoplus_{\lambda\in\Lambda} M_\lambda$, we have*

$$\operatorname{Hom}_R\left(\bigoplus_{\lambda\in\Lambda} M_\lambda, N\right) \cong \prod_{\lambda\in\Lambda} \operatorname{Hom}_R(M_\lambda, N).$$

091517ar THEOREM II.C.1.4. *The functor $U^{-1}(-)$ is exact, i.e., $U^{-1}(-)$ respects short exact sequences (and therefore exact sequences).*

PROOF. Let

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

be a short exact sequence and consider

$$0 = U^{-1}0 \longrightarrow U^{-1}M \xrightarrow{U^{-1}f} U^{-1}N \xrightarrow{U^{-1}g} U^{-1}P \longrightarrow U^{-1}0 = 0\ .$$

First, and most straightforward to show, is the containment $\operatorname{Im} U^{-1}f \subseteq \operatorname{Ker} U^{-1}g$.

$$(U^{-1}g) \circ (U^{-1}f) = U^{-1}(g \circ f) = U^{-1}0 = 0$$

Second, to verify the reverse containment we let $n/u \in \operatorname{Ker} U^{-1}g$ and show it has a preimage under $U^{-1}f$. Residing in the kernel implies $g(u)/n = 0$, i.e., there exists some $v \in U$ such that $0 = v \cdot g(n) = g(vn)$. Since $\operatorname{Ker} g \subseteq \operatorname{Im} f$, we have $f(m) = vn$ for some $m \in M$ and we consider the element $m/uv \in U^{-1}M$.

$$(U^{-1}f)\left(\frac{m}{uv}\right) = \frac{f(m)}{uv} = \frac{vn}{uv} = \frac{n}{u}$$

Third, we want to show $U^{-1}f$ is injective. Let $m/u \in \operatorname{Ker} U^{-1}f$ and similar to the previous part this implies there exists some $v \in U$ such that $v \cdot f(m) = 0$. This also implies $f(vm) = v \cdot f(m) = 0$ and since f is injective, $vm = 0$. Therefore we have

$$\frac{m}{u} = \frac{vm}{vu} = \frac{0}{vu} = 0.$$

So $U^{-1}f$ has trivial kernel and is therefore injective.

Finally, let $p/u \in U^{-1}P$ and note $p = g(n)$ for some $n \in N$ since g is surjective. The immediate implication is

$$\frac{p}{u} = \frac{g(n)}{u} = U^{-1}g\left(\frac{n}{u}\right) \in \operatorname{Im} U^{-1}g$$

Hence $U^{-1}(-)$ preserves short exact sequences. To expand to the arbitrary sequence suppose

$$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$$

is exact. Around this sequence we build four short exact sequences as in Diagram (II.C.1.6.1). The point in this construction is applying $U^{-1}(-)$ to it will preserve commutivity of the diagram and exactness of the diagonals. Then a standard diagram chase (omitted) shows the exactness of the row in which we are interested is also preserved. □

091517aw FACT II.C.1.5. *We have results similar to those in Fact II.C.1.1 and Example II.C.1.2 for localizations. For $U \subseteq R$ a multiplicatively closed set and for R-modules M and M' we have the following isomorphism.*

$$\begin{tikzcd}
U^{-1}(M \oplus M') \arrow[r, "\cong"] & U^{-1}(M) \oplus U^{-1}(M') \\
\dfrac{(m,m')}{u} \arrow[r, mapsto] & \left(\dfrac{m}{u}, \dfrac{m'}{u}\right) \\
\dfrac{(u'm, um')}{uu'} & \left(\dfrac{m}{u}, \dfrac{m'}{u'}\right) = \left(\dfrac{u'm}{uu'}, \dfrac{um'}{uu'}\right) \arrow[l, mapsto]
\end{tikzcd}$$

More generally we write

$$\begin{tikzcd}
U^{-1}\left(\bigoplus_{i=1}^{n} M_i\right) \arrow[r, "\cong"] & \bigoplus_{i=1}^{n} U^{-1} M_i \\
\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} / u \arrow[r, mapsto] & \begin{pmatrix} m_1/u \\ \vdots \\ m_n/u \end{pmatrix}
\end{tikzcd}$$

REMARK II.C.1.6. Replacing M_i above with copies of R shows that the notation $U^{-1}R^n$ is not ambiguous, because $U^{-1}(R^n)$ is isomorphic to $(U^{-1}R)^n$. Thus homomorphisms between modules in the form of the former induce homomorphisms between modules in the form of the latter. We summarize this relationship in the following commutative diagram, where (a_{ij}) is an $n \times m$ matrix over R.

$$\begin{tikzcd}
U^{-1}(R^m) \arrow[r, "U^{-1}(a_{ij})"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & U^{-1}(R^n) \arrow[d, "\cong"] \\
(U^{-1}R)^m \arrow[r, "(a_{ij}/1)"'] & (U^{-1}R)^n
\end{tikzcd}$$

$$
\begin{tikzcd}
 & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & & & \\
 & & \operatorname{Ker}\psi \arrow[ur] \arrow[dr, "i"] \arrow[d, phantom, "\circlearrowright"] & & & & \operatorname{Im}\psi = \operatorname{Ker} p \arrow[ur] \arrow[dr, "k"] \arrow[d, phantom, "\circlearrowright"] & & & & 0 & \\
0 \arrow[r] & \operatorname{Ker}\phi \arrow[ur, "="] \arrow[rr, "i"'] & & X \arrow[rr, "\phi"] \arrow[dr, "\tilde{\phi}"'] & & Y \arrow[rr, "\psi"'] \arrow[ur, "\tilde{\psi}"] & & Z \arrow[rr, "p"] \arrow[dr, "p"'] & & \operatorname{Coker}(\psi) \arrow[rr] \arrow[ur] & & 0 \\
0 \arrow[ur] & & & & \operatorname{Ker}\psi = \operatorname{Im}\phi \arrow[ur, "j"'] \arrow[dr] \arrow[u, phantom, "\circlearrowright"] & & & & \operatorname{Coker}(\psi) \arrow[ur, "="'] \arrow[dr] \arrow[u, phantom, "\circlearrowright"] & & & \\
 & & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
\tag{II.C.1.6.1}
$$

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091617a DEFINITION II.C.1.7. An R-module M is finitely presented if there exists an exact sequence

$$R^m \xrightarrow{f} R^n \xrightarrow{g} M \longrightarrow 0 .$$

091617b PROPOSITION II.C.1.8. *Let R be a non-zero commutative ring with identity, let M and N be R-modules, and let $U \subseteq R$ be a multiplicatively closed subset.*

091617b.a (a) *For all $\phi/u \in U^{-1}\operatorname{Hom}_R(M,N)$, the map ϕ_u below is a well-defined $U^{-1}R$-module homomorphism.*

$$\phi_u : U^{-1}M \longrightarrow U^{-1}N$$

$$m/v \longmapsto \phi(m)/(uv)$$

091617b.b (b) *The function $\Theta_{U,M,N}$ below is a well-defined $U^{-1}R$-module homomorphism.*

$$\Theta_{U,M,N} : U^{-1}\operatorname{Hom}_R(M,N) \longrightarrow \operatorname{Hom}_{U^{-1}R}\big(U^{-1}M, U^{-1}N\big)$$

$$\phi/u \longmapsto \phi_u$$

091617b.c (c) *If M is finitely presented, then $\Theta_{U,M,N}$ is an isomorphism.*

091617b.d (d) *If R is noetherian and M is finitely generated, then*

$$\operatorname{Hom}_{U^{-1}R}\big(U^{-1}M, U^{-1}N\big) \cong U^{-1}\operatorname{Hom}_R(M,N)$$

as $U^{-1}R$-modules (via $\Theta_{U,M,N}$).

PROOF. (a) We prove this part in two steps. First let $u \in U$ and $\phi \in \operatorname{Hom}_R(M,N)$. For any $m/v \in U^{-1}M$ we have

$$\phi_u\left(\frac{m}{v}\right) = \frac{\phi(m)}{uv} = \frac{1}{u}\cdot (U^{-1}\phi)\left(\frac{m}{v}\right) = (\mu_{\frac{1}{u}} \circ U^{-1}\phi)\left(\frac{m}{v}\right)$$

where $\mu_{1/u}$ is the standard product map (see Example II.B.1.10). Thus ϕ_u is the composition of two well-defined $U^{-1}R$-module homomorphisms, so it is itself a well-defined $U^{-1}R$-module homomorphism. The second question of well-definedness has to do with our choice of representative from $U^{-1}\operatorname{Hom}_R(M,N)$, so let $\phi/u = \phi'/u'$. This means there exists some $u'' \in U$ such that $uu''\phi' = u'u''\phi$, so for any $m/v \in U^{-1}M$ we have

$$uu''\cdot\phi'(m) = (uu''\phi')(m) = (u'u''\phi)(m) = u'u''\cdot\phi(m)$$

and therefore

$$\phi_u\left(\frac{m}{v}\right) = \frac{\phi(m)}{uv} = \frac{u'u''\phi(m)}{u'u''uv} = \frac{uu''\phi'(m)}{u'u''uv} = \frac{\phi'(m)}{u'v} = \phi'_{u'}\left(\frac{m}{v}\right).$$

(b) The well-definedness of $\Theta_{U,M,N}$ is a consequence of part (a), so we need only show it is $U^{-1}R$-linear. Let $\phi/u, \phi'/u' \in U^{-1}\operatorname{Hom}_R(M,N)$ be given and note that showing Θ respects sums is equivalent to showing

$$(u'\phi + u\phi')_{uu'} = \phi_u + \phi'_{u'}$$

since

$$\frac{\phi}{u} + \frac{\phi'}{u'} = \frac{u'\phi + u\phi'}{uu'}.$$

To this end, for any $m/v \in U^{-1}M$ we have

$$\begin{aligned}
(u'\phi + u\phi')_{uu'}\left(\frac{m}{v}\right) &= \frac{(u'\phi + u\phi')(m)}{uu'v} \\
&= \frac{(u'\phi)(m) + (u\phi')(m)}{uu'v} \\
&= \frac{u'\cdot\phi(m)}{uu'v} + \frac{u\cdot\phi'(m)}{uu'v} \\
&= \frac{\phi(m)}{uv} + \frac{\phi'(m)}{u'v} \\
&= \phi_u\left(\frac{m}{v}\right) + \phi'_{u'}\left(\frac{m}{v}\right).
\end{aligned}$$

To complete the proof of part (b) let $r/t \in U^{-1}R$ be given and we observe for any $m/v \in U^{-1}M$

$$\begin{aligned}\Theta_{U,M,N}\left(\frac{r\phi}{tu}\right)\left(\frac{m}{v}\right) &= (r\phi)_{tu}\left(\frac{m}{v}\right)\\ &= \frac{(r\phi)(m)}{tuv}\\ &= \frac{r}{t}\cdot\frac{\phi(m)}{uv}\\ &= \frac{r}{t}\cdot\phi_u\left(\frac{m}{v}\right)\\ &= \frac{r}{t}\cdot\Theta_{U,M,N}\left(\frac{\phi}{u}\right)\left(\frac{m}{v}\right).\end{aligned}$$

(c) We complete this part in four steps. First we claim $\Theta_{U,M\oplus M',N}$ is an isomorphism if and only if both $\Theta_{U,M,N}$ and $\Theta_{U,M',N}$ are isomorphisms. We prove this by showing Diagram (II.C.1.8.3) of $U^{-1}R$-modules and homomorphisms commutes. To make clear some of our notation we define the following homomorphisms.

$$\gamma : U^{-1}(M\oplus M') \longrightarrow (U^{-1}M)\oplus(U^{-1}M') \qquad \gamma^{-1} : (U^{-1}M)\oplus(U^{-1}M') \longrightarrow U^{-1}(M\oplus M')$$

$$\frac{(m,m')}{v} \longmapsto \left(\frac{m}{v},\frac{m'}{v}\right) \qquad \left(\frac{m}{v},\frac{m'}{v'}\right) \longmapsto \frac{(v'm, vm')}{vv'}$$

Consider the map ω as defined in Fact II.C.1.1 and from the same fact, consider the standard injections ε and ε' along with the standard projections τ and τ', all of which we reproduce below.

$$\varepsilon : U^{-1}M \longrightarrow (U^{-1}M)\oplus(U^{-1}M') \qquad \varepsilon' : U^{-1}M' \longrightarrow (U^{-1}M)\oplus(U^{-1}M')$$

$$\frac{m}{u} \longmapsto \left(\frac{m}{u},0\right) \qquad \frac{m'}{u} \longmapsto \left(0,\frac{m'}{u}\right)$$

$$\tau : (U^{-1}M)\oplus(U^{-1}M') \longrightarrow U^{-1}M \qquad \tau' : (U^{-1}M)\oplus(U^{-1}M') \longrightarrow U^{-1}M'$$

$$\left(\frac{m}{u},\frac{m'}{u'}\right) \longmapsto \frac{m}{u} \qquad \left(\frac{m}{u},\frac{m'}{u'}\right) \longmapsto \frac{m'}{u'}$$

The maps Γ and Ω will be defined implicitly in the diagram chase.

For any $\psi/u \in U^{-1}\operatorname{Hom}_R(M\oplus M',N)$ we have

$$\frac{\psi}{u} \xmapsto{U^{-1}\omega} \frac{\omega(\psi)}{u} = \frac{(\psi\circ\tau,\psi\circ\tau')}{u} \xmapsto{\Gamma} \left(\frac{\psi\circ\tau}{u},\frac{\psi\circ\tau'}{u}\right) \xmapsto{\Theta_{U,M,N}\oplus\Theta_{U,M',N}} ((\psi\circ\tau)_u,(\psi\circ\tau')_u)\ . \tag{II.C.1.8.1}$$

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Tracking along the other half of the diagram we find

$$\frac{\psi}{u} \xmapsto{\Theta_{U,M\oplus M',N}} \psi_u \xmapsto{(\gamma^{-1})^*} \psi_u\circ\gamma^{-1} \xmapsto{\Omega} (\psi_u\circ\gamma^{-1}\circ\varepsilon,\psi_u\circ\gamma^{-1}\circ\varepsilon')\ . \tag{II.C.1.8.2}$$

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Now it is a matter of showing the resulting maps in (II.C.1.8.1) and (II.C.1.8.2) are equivalent.

For any $m/v \in U^{-1}M$ and any $m'/v' \in U^{-1}M'$, (II.C.1.8.1) produces

$$\left((\psi\circ\tau)_u\left(\frac{m}{v}\right),(\psi\circ\tau')_u\left(\frac{m'}{v'}\right)\right) = \left(\frac{(\psi\circ\tau)(m)}{uv},\frac{(\psi\circ\tau')(m')}{uv'}\right) = \left(\frac{\psi(m,0)}{uv},\frac{\psi(0,m')}{uv'}\right).$$

and likewise (II.C.1.8.2) produces

$$\begin{aligned}
\left((\psi_u \circ \gamma^{-1} \circ \varepsilon)\left(\frac{m}{v}\right), (\psi_u \circ \gamma^{-1} \circ \varepsilon')\left(\frac{m'}{v'}\right)\right) &= \left(\psi_u\left(\gamma^{-1}\left(\frac{m}{v}, 0\right)\right), \psi_u\left(\gamma^{-1}\left(0, \frac{m'}{v'}\right)\right)\right) \\
&= \left(\psi_u\left(\gamma^{-1}\left(\frac{m}{v}, \frac{0}{v}\right)\right), \psi_u\left(\gamma^{-1}\left(\frac{0}{v'}, \frac{m'}{v'}\right)\right)\right) \\
&= \left(\psi_u\left(\frac{(m,0)}{v}\right), \psi_u\left(\frac{(0,m')}{v'}\right)\right) \\
&= \left(\frac{\psi(m,0)}{uv}, \frac{\psi(0,m')}{uv'}\right)
\end{aligned}$$

Hence the diagram commutes and our first claim follows from a standard diagram chase.

Next we claim $\Theta_{U,\oplus_{i=1}^n M_i,N}$ is an isomorphism if and only if $\Theta_{U,M_i,N}$ is an isomorphism for every $i = 1, \ldots, n$. The base case is our first claim, so assume our second claim holds for R-modules $M_1, \ldots, M_{n-1}$ and let M_n be another R-module. By our first claim we have $\Theta_{U,\oplus_{i=1}^n M_i,N}$ is an isomorphism if and only if both $\Theta_{U,\oplus_{i=1}^{n-1} M_i,N}$ and $\Theta_{U,M_n,N}$ are isomorphisms, so our second claim follows from our induction hypothesis.

Third we claim $\Theta_{U,R^n,N}$ is an isomorphism, for which it suffices to show $\Theta_{U,R,N}$ is an isomorphism (by our second claim). Consider the diagram

$$\begin{tikzcd}
U^{-1}\operatorname{Hom}_R(R,N) \arrow[r, "\Theta_{U,R,N}"] \arrow[d, "U^{-1}f"', "\cong"] & \operatorname{Hom}_{U^{-1}R}\left(U^{-1}R, U^{-1}N\right) \arrow[dl, "\cong", "F"'] \\
U^{-1}N &
\end{tikzcd}$$

where f and F are the evaluation maps at 1_R and $1_{U^{-1}R}$, respectively. The diagram commutes since for any $\frac{\psi}{u} \in U^{-1}\operatorname{Hom}_R(R,N)$ we have the following.

$$\begin{aligned}
(U^{-1}f)\left(\frac{\psi}{u}\right) &= \frac{f(\psi)}{u} = \frac{\psi(1)}{u} \\
(F \circ \Theta_{U,R,N})\left(\frac{\psi}{u}\right) &= F(\psi_u) = \psi_u(1) = \psi_u\left(\frac{1}{1}\right) = \frac{\psi(1)}{1 \cdot u}
\end{aligned}$$

Since the evaluation maps $U^{-1}f$ and F are known isomorphisms, a standard diagram chase shows $\Theta_{U,R,N}$ is an isomorphism also.

To finish the proof of part (c), assume the sequence

$$R^m \xrightarrow{f} R^n \xrightarrow{g} M \longrightarrow 0$$

is exact (f no longer an evaluation map). Since Hom is left-exact the sequence

$$0 \longrightarrow \operatorname{Hom}_R(M,N) \xrightarrow{g^*} \operatorname{Hom}_R(R^n,N) \xrightarrow{f^*} \operatorname{Hom}_R(R^m,N)$$

is exact as well, where $*$ is defined as

$$(-)^* := \operatorname{Hom}_R(-,N).$$

Localization is also exact, so

$$\begin{tikzcd}
0 \arrow[r] & U^{-1}\operatorname{Hom}_R(M,N) \arrow[r, "U^{-1}(g^*)"] \arrow[d, "\Theta_{U,M,N}"] \arrow[dr, phantom, "\circlearrowleft"] & U^{-1}\operatorname{Hom}_R(R^n,N) \arrow[r, "U^{-1}(f^*)"] \arrow[d, "\Theta_{U,R^n,N}", "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & U^{-1}\operatorname{Hom}_R(R^m,N) \arrow[d, "\Theta_{U,R^m,N}", "\cong"'] \\
0 \arrow[r] & \operatorname{Hom}_{U^{-1}R}\left(U^{-1}M, U^{-1}N\right) \arrow[r, "(U^{-1}g)^\star"'] & \operatorname{Hom}_{U^{-1}R}\left(U^{-1}R^n, U^{-1}N\right) \arrow[r, "(U^{-1}f)^\star"'] & \operatorname{Hom}_{U^{-1}R}\left(U^{-1}R^m, U^{-1}N\right)
\end{tikzcd}$$

is a homomorphism of exact sequences, where the commutivity of the diagram is verified as above, the isomorphisms therein follow from our third claim, and $\star$ is defined as

$$(-)^\star := \operatorname{Hom}_{U^{-1}R}\left(-, U^{-1}N\right).$$

Another diagram chase allows us to conclude that $\Theta_{U,M,N}$ is an isomorphism as desired, completing the proof of (c).

(d) Since R noetherian and M finitely generated, there exists an exact sequence

$$\dots \longrightarrow R^b \longrightarrow R^a \longrightarrow R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0\ .$$

Therefore

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

is exact and M is finitely presented. Part (d) then follows from part (c). $\square$

$$\begin{tikzcd}
U^{-1}\operatorname{Hom}_R(M \oplus M', N) \arrow[r, "\Theta_{U,M\oplus M',N}"] \arrow[d, "U^{-1}\omega"', "\cong"] & \operatorname{Hom}_{U^{-1}R}\big(U^{-1}(M \oplus M'), U^{-1}N\big) \arrow[d, "(\gamma^{-1})^*"] \\
U^{-1}(\operatorname{Hom}_R(M,N) \oplus \operatorname{Hom}_R(M',N)) \arrow[d, "\Gamma"', "\cong"] & \operatorname{Hom}_{U^{-1}R}\big((U^{-1}M) \oplus (U^{-1}M'), U^{-1}N\big) \arrow[d, "\Omega", "\cong"'] \\
U^{-1}\operatorname{Hom}_R(M,N) \oplus U^{-1}\operatorname{Hom}_R(M',N) \arrow[r, "\Theta_{U,M,N}\oplus\Theta_{U,M',N}"] & \operatorname{Hom}_{U^{-1}R}\big(U^{-1}M, U^{-1}N\big) \oplus \operatorname{Hom}_{U^{-1}R}\big(U^{-1}M', U^{-1}N\big)
\end{tikzcd} \tag{II.C.1.8.3}$$

eqn070118b

ection0629211

II.C.2. Modules and Prime Spectra

The prime spectrum of a ring and related constructs are used heavily throughout the remainder of the chapter and we introduce them here. Remark II.C.2.11 in particular will get a lot of use and will be used directly in the proof of Theorem II.C.5.16, the ultimate goal of the chapter.

notn012018h NOTATION II.C.2.1. For any natural number $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

def092217a DEFINITION II.C.2.2. Let I be an ideal of the ring R. The prime spectrum of R is

$$\operatorname{Spec}(R) = \{\mathfrak{p} \leq R \mid \mathfrak{p} \text{ prime}\}.$$

The variety of I is

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}.$$

The radical of I is

$$\operatorname{rad}(I) = \{x \in R \mid \exists n \in \mathbb{N} \text{ s.t. } x^n \in I\},$$

also denoted $r(I)$ or $\sqrt{I}$.

rmk092217e REMARK II.C.2.3. We have the following properties of the radical ideal and the variety of an ideal.

rmk092217e.a (a) $\operatorname{rad}(I) \leq R$

rmk092217e.b (b) $I \subseteq \operatorname{rad}(I)$

rmk092217e.c (c) $I \supseteq J \leq R \implies \operatorname{rad}(J) \subseteq \operatorname{rad}(I)$

rmk092217e.d (d) $\operatorname{rad}(\operatorname{rad}(I)) = \operatorname{rad}(I)$

rmk092217e.e (e) $I \subseteq J \implies V(I) \supseteq V(J)$

rmk092217e.f (f) $I = R \iff \operatorname{rad}(I) = R$

ex092217b EXAMPLE II.C.2.4. Let R be a principal ideal domain. For any $x \in R \setminus \{0\}$ there exists a unit $u \in R^{\times}$, prime elements $p_1, \ldots, p_n \in R$, and positive $e_1, \ldots, e_n \in \mathbb{N}$ such that

$$x = up_1^{e_1} \cdots p_n^{e_n}$$

and $p_iR \neq p_jR$ whenever $i \neq j$. If we define $I = xR$, then $V(I) = \{p_1R, \ldots, p_nR\}$. This is because $qR \in \operatorname{Spec}(R)$ is such that qR contains xR if and only if $q|x = up_1^{e_1} \cdots p_n^{e_n}$. That is, $qR \in V(I)$ if and only if $q \sim p_i$ for some i.

We can also show $\operatorname{rad}(I) = p_1 \cdots p_nR$. Note by Remark II.C.2.3(f) above, we may assume without loss of generality that $n \geq 1$ (i.e., x is not a unit). Define $e = \max_i(e_i)$ and we have

$$(p_1 \cdots p_n)^e = p_1^e \cdots p_n^e \in p_1^{e_1} \cdots p_n^{e_n}R = xR.$$

So the product $p_1 \cdots p_n \in \operatorname{rad}(I)$ and hence $p_1 \cdots p_nR \subseteq \operatorname{rad}(I)$, because $\operatorname{rad}(I)$ is an ideal. For the reverse containment let $y \in \operatorname{rad}(I)$ and let $m \in \mathbb{N}$ such that $y^m \in I$. This implies $up_1^{e_1} \cdots p_n^{e_n}|y^m$. For each $i \in [n]$, $e_i \geq 1$ so $p_i|y^m$ and $p_i|y$. Moreover $p_i \not\sim p_j$ whenever $i \neq j$ implies $p_1 \cdots p_n|y$ and therefore $y \in p_1 \cdots p_nR$. Hence $\operatorname{rad}(I) \subseteq p_1 \cdots p_nR$, concluding the proof.

To give a more explicit example, consider $x = 2^5 3^{17} 19 \in \mathbb{Z}$. By what we have shown above $V(x\mathbb{Z}) = \{2\mathbb{Z}, 13\mathbb{Z}, 19\mathbb{Z}\}$ and $\operatorname{rad}(x\mathbb{Z}) = 2 \cdot 13 \cdot 19\mathbb{Z}$.

fact092217d FACT II.C.2.5. *If $I \leq R$, then $V(\operatorname{rad}(I)) = V(I)$.*

PROOF. The forward containment follows from parts (b) and (e) in Remark II.C.2.3 above. For the reverse containment, let $\mathfrak{p} \in V(I)$. For any $x \in \operatorname{rad}(I)$ with $x^n \in I \subseteq \mathfrak{p}$, we know $x \in \mathfrak{p}$, implying $\operatorname{rad}(I) \subseteq \mathfrak{p}$. Having shown an arbitrary prime ideal containing I must also contain $\operatorname{rad}(I)$, we conclude $V(\operatorname{rad}(I)) \supseteq V(I)$. $\square$

prop092217c PROPOSITION II.C.2.6. *If $I \leq R$, then*

$$\operatorname{rad}(I) = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}.$$

PROOF. First we deal with a special case. If $V(I) = \emptyset$, then we have the empty intersection on the right, which is defined to be all of R. Moreover in this case I must actually be the entire ring, since if $I \lneq R$, then I must be contained in some maximal (and therefore prime) ideal, violating the emptiness of $V(I)$. This gives $\operatorname{rad}(I) = \operatorname{rad}(R) = R$, so the proposition holds in this case.

Now assume without loss of generality that $V(I)$ is nonempty and therefore $I \neq R$. For any $x \in \operatorname{rad}(I)$ with $x^n \in I$ for some $n \in \mathbb{N}$, if I lies in some prime ideal $\mathfrak{p}$, then $x^n \in I \subseteq \mathfrak{p}$ and therefore $x \in \mathfrak{p}$. Having shown that $\operatorname{rad}(I)$ is contained in an arbitrary element of $V(I)$, we conclude

$$\operatorname{rad}(I) \subseteq \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}.$$

For the other containment, we use a clever application of localization. Let $x \in R \setminus \operatorname{rad}(I)$ and define the multiplicatively closed subset $S = \{1, x, x^2, x^3, \dots\} \subseteq R$. Since $\operatorname{rad}(I) \cap S = \emptyset$, it follows that $S^{-1}\operatorname{rad}(I)$ contains no units of $S^{-1}R$ and therefore is a proper ideal of $S^{-1}R$ (see Fact II.C.2.10). Then we may let $S^{-1}\mathfrak{q} \lneq S^{-1}R$ be a maximal ideal containing $S^{-1}\operatorname{rad}(I)$. By this we know $\mathfrak{q} \in \operatorname{Spec}(R)$ satisfies $\mathfrak{q} \cap S = \emptyset$ and $\operatorname{rad}(I) \subseteq \mathfrak{q}$, so $\mathfrak{q} \in V(\operatorname{rad}(I)) = V(I)$ by Fact II.C.2.5. Since $\mathfrak{q} \cap S = \emptyset$, we know $x^n \notin \mathfrak{q}$ for any integer $n \geq 0$ and in particular

$$x \notin \mathfrak{q} \supseteq \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}.$$

Hence we have proven the reverse containment by contraposition. □

lem092217f LEMMA II.C.2.7. *If $I, J \leq R$ and $V(J) \subseteq V(I)$, then $I \subseteq \operatorname{rad}(J)$. If I is also finitely generated over R, then $I^n \subseteq J$, for all sufficiently large $n > 0$.*

PROOF. The first implication is a corollary of Remark II.C.2.3 and Proposition II.C.2.6:

$$I \subseteq \operatorname{rad}(I) = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in V(J)} \mathfrak{p} = \operatorname{rad}(J).$$

For the second part, let $x_1, \dots, x_n \in R$ be such that $(x_1, \dots, x_n)R = I \subseteq \operatorname{rad}(J)$. By definition of the radical there exist $e_1, \dots, e_n \in \mathbb{N}$ such that $x_1^{e_1}, \dots, x_n^{e_n} \in J$ and we define $e = \sum_{i=1}^n e_i$. We then have

$$I^e = \left\langle x_1^{f_1} \cdots x_n^{f_n} \;\middle|\; \sum_{i=1}^n f_i = e \right\rangle.$$

Since for any generator of I^e above, the f_i's and e_i's both sum to e, we know $f_i \geq e_i$ for some i and therefore

$$x_1^{f_1} \cdots x_i^{f_i} \cdots x_n^{f_n} \in (x_i^{f_i})R \subseteq (x^{e_i})R \subseteq J.$$

Hence $I^e \subseteq J$ and therefore $I^t \subseteq I^e \subseteq J$ for all $t \geq e$. □

def092217g DEFINITION II.C.2.8. For all $m \in M$, the annihilator of m is

$$\operatorname{Ann}_R(m) = \{r \in R \mid rm = 0\}.$$

Similarly we may define the annihilator of M as

$$\operatorname{Ann}_R(M) = \{r \in R \mid rM = 0\} = \{r \in R \mid rm = 0, \ \forall m \in M\} = \bigcap_{m \in M} \operatorname{Ann}_R(m).$$

The support of M is the set of all prime ideals for which M "survives the localization process"; formally we write

$$\operatorname{supp}(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}.$$

ex092217h EXAMPLE II.C.2.9. Let $U \subseteq R$ be multiplicatively closed.

ex092217h.a (a) For any $m \in M$ the following are equivalent.

x092217h.a.i (i) $m/1 = 0 \in U^{-1}M$.

092217h.a.ii (ii) There exists some $u \in U$ such that $um = 0$.

92217h.a.iii (iii) $U \cap \operatorname{Ann}_R(m) \neq \emptyset$.

ex092217h.b (b) If M is finitely generated, then the following are equivalent.

x092217h.b.i (i) $U^{-1}M = 0$.

092217h.b.ii (ii) There exists some $u \in U$ such that $uM = 0$.

92217h.b.iii (iii) $U \cap \operatorname{Ann}_R(M) \neq \emptyset$.

The majority of the above implications are simply restatements of definitions, so we will only prove (bi) implies (bii) in part (b).

PROOF. If there exists $u \in U$ such that $uM = 0$, then for any $m/v \in U^{-1}M$ we have

$$\frac{m}{v} = \frac{um}{uv} = \frac{0}{uv} = 0.$$

Therefore $U^{-1}M = 0$. This proves one direction and we point out here that we did not need the finitely generated assumption. If $M = (m_1, \ldots, m_n)R$ and $U^{-1}M = 0$, then notice $m_i/1 = 0$ for each $i = 1, \ldots, n$. By part (a) this means there exist $u_1, \ldots, u_n \in U$ such that $u_i m_i = 0$ for $i = 1, \ldots, n$. Define $u = \prod_{i=1}^n u_i$ and let $m = \sum_{i=1}^n r_i m_i$ be given. It follows that

$$um = u\left(\sum_{i=1}^n r_i m_i\right) = \sum_{i=1}^n \left(r_i \prod_{j\neq i} u_j\right)(u_i m_i) = 0$$

implying $uM = 0$. □

In order to be explicit in our reasoning in Remark II.C.2.11, we prove a fact about ideals under localization.

fact092817a

FACT II.C.2.10. *Let $I \leq R$ be an ideal and let $U \subseteq R$ be a multiplicatively closed subset. Then $U^{-1}I = U^{-1}R$ if and only if $I \cap U \neq \emptyset$.*

PROOF. If we first assume there exists an element $u \in I \cap U$, then we write

$$1_{U^{-1}R} = \frac{u}{u} \in U^{-1}I$$

and therefore $U^{-1}I = U^{-1}R$. On the other hand if we assume $U^{-1}I = U^{-1}R$, then $1_{U^{-1}R} \in U^{-1}I$ and we have

$$1_{U^{-1}R} = \frac{1}{1} = \frac{a}{u}$$

for some $a \in I$ and some $u \in U$. By the definition of equality in $U^{-1}R$, there exists an element $v \in U$ such that

$$\underbrace{va}_{\in I} = \underbrace{vu}_{\in U}$$

and we conclude $I \cap U \neq \emptyset$. □

rmk092217i

REMARK II.C.2.11. We have the following relationships between annihilators, supports, and prime spectra.

rmk092217i.a (a) $\operatorname{Ann}_R(m), \operatorname{Ann}_R(M) \leq R$

rmk092217i.b (b) $\operatorname{supp}(R) = \operatorname{Spec}(R)$

rmk092217i.c (c) $\operatorname{supp}(0) = \emptyset$

rmk092217i.d (d) $\operatorname{supp}(R/I) = V(I)$

rmk092217i.e (e) M finitely generated $\implies$ $\operatorname{supp}(M) = V(\operatorname{Ann}_R(M))$

PROOF. (a) The annihilators are non-empty since they each contain 0. They are closed under addition and subtraction as a result of the distributive property. They contain additive inverses, because

$$rm = 0 \implies (-r)m = (-1)rm = 0.$$

Finally, they absorb multiplication from R as a result of the associative property.

(b) Supports are special sets of prime ideals and spectra contain all prime ideals of the particular ring, so $\operatorname{supp}(R) \subseteq \operatorname{Spec}(R)$ from the definitions. On the other hand, for any $\mathfrak{p} \in \operatorname{Spec}(R)$, $1 \notin \mathfrak{p}$ so 1 is an allowable denominator and we write

$$0 \neq \frac{1}{1} \in R_{\mathfrak{p}}$$

implying $R_{\mathfrak{p}} \neq 0$. Thus $\mathfrak{p} \in \operatorname{supp}(R)$.

(c) This holds simply because there is no localization under which zero is 'resurrected' to something non-zero. That is, $0_{\mathfrak{p}} = 0$ for any $\mathfrak{p} \in \operatorname{Spec}(R)$.

(d) For any $\mathfrak{p} \in \operatorname{Spec}(R)$, by Fact II.C.2.10 above we have $I_\mathfrak{p} = R_\mathfrak{p}$ if and only if $I \cap (R \setminus \mathfrak{p}) \neq \emptyset$. This is equivalent to $I \not\subseteq \mathfrak{p}$ which is equivalent to $\mathfrak{p} \notin V(I)$. Therefore

$$I_\mathfrak{p} \subsetneq R_\mathfrak{p} \iff \mathfrak{p} \in V(I). \tag{II.C.2.11.1}$$ eqn082318b

Consider the short exact sequence

$$0 \longrightarrow I \xrightarrow{\subseteq} R \xrightarrow{\pi} R/I \longrightarrow 0$$

where π is the canonical surjection. Since localization is exact by Theorem II.C.1.4, the sequence

$$0 \longrightarrow I_\mathfrak{p} \xrightarrow{i} R_\mathfrak{p} \xrightarrow{\pi_\mathfrak{p}} (R/I)_\mathfrak{p} \longrightarrow 0$$

is also exact. The First Isomorphism Theorem for modules applied to $\pi_\mathfrak{p}$ yields

$$(R/I)_\mathfrak{p} = \operatorname{Im} \pi_\mathfrak{p} \cong \frac{R_\mathfrak{p}}{\ker(\pi_\mathfrak{p})} = \frac{R_\mathfrak{p}}{\operatorname{Im} i} = R_\mathfrak{p}/I_\mathfrak{p}. \tag{II.C.2.11.2}$$ eqn082318c

Our application of short exact sequences shortens the proof immensely. By the definition of a support, $\mathfrak{p} \in \operatorname{supp}(R/I)$ if and only if $(R/I)_\mathfrak{p} \neq 0$. Then by Equation (II.C.2.11.2), this is if and only if $R_\mathfrak{p}/I_\mathfrak{p}$ or $I_\mathfrak{p} \subsetneq R_\mathfrak{p}$. Then by Equation (II.C.2.11.1), we get that $\mathfrak{p} \in \operatorname{supp}(R/I)$ if and only if $\mathfrak{p} \in V(I)$.

(e) This requires only definitions and Example II.C.2.9. First, we use the definition of a support to see that $\mathfrak{p} \in \operatorname{supp}(M)$ if and only if $M_\mathfrak{p} \neq 0$. Then by Example II.C.2.9 and the definition of a variety, it follows that $M_\mathfrak{p} \neq 0$ if and only if $(R \setminus \mathfrak{p}) \cap \operatorname{Ann}_R(M) = \emptyset$ if and only if $\operatorname{Ann}_R(M) \subseteq \mathfrak{p}$ if and only if $\mathfrak{p} \in V(\operatorname{Ann}_R(M))$. ☐

ex092217j EXAMPLE II.C.2.12. Let $\mathbb{K}$ be a field and define the ring $R = \mathbb{K}[x, y]$.

ex092217j.a (a) For every polynomial $f \in R$

$$\operatorname{supp}(R/fR) = V(fR) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid f \in \mathfrak{p}\}.$$

ex092217j.b (b) For every $m, n \in \mathbb{N}$ we have

$$\operatorname{supp}\left(\frac{R}{(x^m, y^n)R}\right) = \{(x, y)R\} = \operatorname{supp}\left(\frac{R}{((x, y)R)^m}\right).$$

ex092217j.c (c) For the ideal $L = (x^2, xy)R \leq R$ we have

$$\operatorname{supp}(R/L) = V(xR) = \operatorname{supp}(R/xR)$$

and

$$\operatorname{rad}(L) = xR.$$

PROOF. Here we will only justify part (b) of the example, as the other parts follow more or less similarly. By Remark II.C.2.11(d), to prove (b) it suffices to show

$$V((x^m, y^n)R) \overset{(1)}{=} \{(x, y)R\} \overset{(2)}{=} V(((x, y)R)^m).$$

If $\mathfrak{p} \in \operatorname{Spec}(R)$ and $(x^m, y^n)R \subseteq \mathfrak{p} \subsetneq R$, then $x^m, y^n \in \mathfrak{p}$ and thus $x, y \in \mathfrak{p}$ since $\mathfrak{p}$ prime. It follows that $(x, y)R \subseteq \mathfrak{p}$ and the strictness of $\mathfrak{p}$ implies $(x, y)R = \mathfrak{p}$, because $(x, y)R$ is maximal. Therefore

$$V((x^m, y^n)R) \subseteq \{(x, y)R\}.$$

On the other hand, $(x, y)R \in \operatorname{Spec}(R)$ and $x^m, y^n \in (x, y)R$, so $(x^m, y^n)R \subseteq (x, y)R$. Hence equality (1) holds by mutual containment.

Now for equality (2). Since $(x, y)R \in \operatorname{Spec}(R)$ and $x^a y^b \in (x, y)R$ for any $a, b \geq 0$, we know

$$((x, y)R)^m = \left\{x^a y^b \mid a, b \geq 0,\ a + b = m\right\} \subseteq (x, y)R$$

implying $(x, y)R \in V(((x, y)R)^m)$, so we have containment in one direction. For the reverse, let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $((x, y)R)^m \subseteq \mathfrak{p}$. Then $x^m, y^m \in \mathfrak{p}$ a prime ideal, so $x, y \in \mathfrak{p}$ and therefore $(x, y)R \subseteq \mathfrak{p}$. It follows that

$$V(((x, y)R)^m) \subseteq \{(x, y)R\}$$

so equality (2) holds by mutual containment. ☐

def092217l DEFINITION II.C.2.13. A prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ is associated to M if there exists $m \in M$ such that $\mathfrak{p} = \operatorname{Ann}_R(m)$. The set of all such ideals is the set of associated primes, denoted as follows.

$$\begin{aligned}\operatorname{Ass}(M) &= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \text{ is associated to } M\} \\ &= \{\operatorname{Ann}_R(m) \leq R \mid m \in M\} \cap \operatorname{Spec}(R) \\ &= \{\operatorname{Ann}_R(m) \leq R \mid m \in M,\ \operatorname{Ann}_R(m) \text{ is a prime ideal}\}\end{aligned}$$

In other words, $\operatorname{Ass}(M)$ is the set of prime ideals of R that are also the annihilator of some element of M.

ex092217m EXAMPLE II.C.2.14. Let $\mathfrak{p} \in \operatorname{Spec}(R)$, $r \in R$, and $r + \mathfrak{p} \in R/\mathfrak{p}$.

$$\operatorname{Ann}_R(r+\mathfrak{p}) = \begin{cases} R & r \in \mathfrak{p} \\ \mathfrak{p} & r \notin \mathfrak{p}\end{cases}$$

The first case follows because $r + \mathfrak{p} = 0_{R/\mathfrak{p}}$ and the second case follows because $r + \mathfrak{p} \neq 0_{R/\mathfrak{p}}$ and $R/\mathfrak{p}$ is a domain. In general, determining the set of associated primes of R/I is difficult, but in this case we have just shown that

$$\operatorname{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}.$$

ex092217n EXAMPLE II.C.2.15. Assume R is a principal ideal domain and let $I = xR$. If $x \in R^\times$, then $xR = R$ and $R/xR = 0$, implying

$$\operatorname{Ass}(R/xR) = \operatorname{Ass}(0) = \{\operatorname{Ann}_R(0)\} \cap \operatorname{Spec}(R) = \{R\} \cap \operatorname{Spec}(R) = \emptyset.$$

If $x = 0$, then xR is the (prime) zero ideal and therefore by Example II.C.2.14 we have

$$\operatorname{Ass}(R/xR) = \{xR\} = \{0\}.$$

So let $x \in R \setminus (R^\times \cup \{0\})$ and $p_1, \ldots, p_n \in R$ primes (not necessarily distinct) such that $x = p_1 \cdots p_n$. We claim

$$\operatorname{Ass}(R/xR) = \{p_1R, \ldots, p_nR\}.$$

PROOF. For the reverse containment, first define $x' = p_2 \cdots p_n$. Since prime factorizations are unique in R, this implies $\{r \in R \mid rx' \in xR\} = p_1R$. We can also write

$$\{r \in R \mid rx' \in xR\} = \{r \in R \mid r(x' + xR) = 0 \text{ in } R/xR\} = \operatorname{Ann}_R(x' + xR).$$

We have therefore shown

$$p_1R \in \operatorname{Spec}(R/xR) \cap \{\operatorname{Ann}_R(y + xR) \leq R \mid y + xR \in R/xR\} = \operatorname{Ass}(R/xR).$$

Since multiplication in R is commutative, we conclude $p_iR \in \operatorname{Ass}(R/xR)$ for all $i = 1, \ldots, n$, proving the reverse containment.

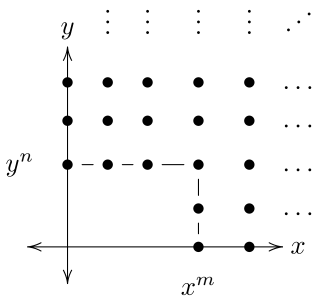
Let $y \in R$ such that $\operatorname{Ann}_R(y + xR) \in \operatorname{Spec}(R)$ and set $\operatorname{Ann}_R(y + xR) = pR$ for some prime $p \in R$. Then $p(y + xR) = \overline{0}$ or in other words $py = xr = p_1 \cdots p_n r$ for some $r \in R$. This implies $xr \in pR$ a prime ideal, so either $r \in pR$ or $p_i \in pR$ for some $i \in [n]$. Suppose $r \in pR$, so $r = pz$ for some $z \in R$. Therefore

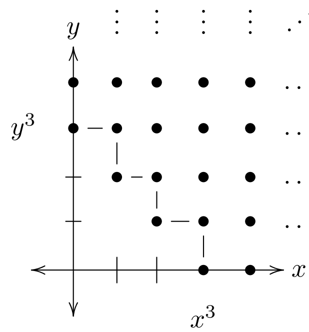
$$py = p_1 \cdots p_n r = p_1 \cdots p_n pz$$

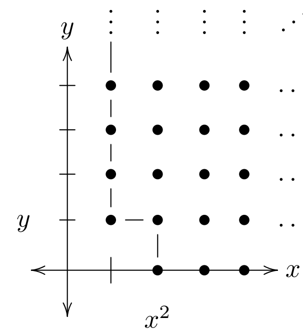
and since R is commutative $y = p_1 \cdots p_n z$ by cancellation. However, this implies $y \in xR$ and thus $pR = \operatorname{Ann}_R(\overline{0}) = R$, a contradiction. Therefore $p_i \in pR$ for some $i \in [n]$ and it follows that $pR = p_iR$ for some $i \in [n]$. $\square$

note092217k NOTE II.C.2.16. Given the polynomial ring $R = \mathbb{K}[x, y]$ as in Example II.C.2.12, we can plot graphic representations of ideals generated by monomials in the ring.

$(x^m, y^n)R$ $\qquad$ $((x, y)R)^3$ $\qquad$ $(x^2, xy)R$

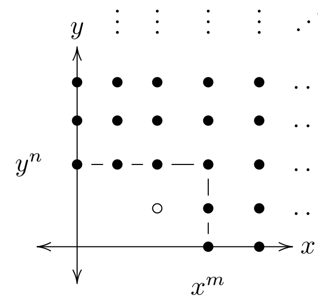






ex092217o

EXAMPLE II.C.2.17. Here we make use of lattice diagrams representing monomial ideals in the polynomial ring $R = \mathbb{K}[x, y]$, where $\mathbb{K}$ is a field. Given the lattice representation of an ideal from Note II.C.2.16, certain corners in the lattice give us information about the associated primes of the residual ring. Specifically, first consider $I = (x^m, y^n)R$ with lattice representation



where we have denoted the element $(x^{m-1}y^{n-1}) \in R$ with '$\circ$'. For the element $\overline{x^{m-1}y^{n-1}} \in R/I$, since $x, y \in \operatorname{Ann}_R\left(\overline{x^{m-1}y^{n-1}}\right)$ and $1 \notin \operatorname{Ann}_R\left(\overline{x^{m-1}y^{n-1}}\right)$, we have

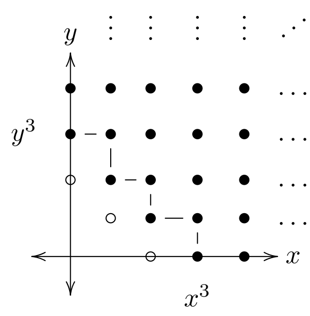
$$\operatorname{Ann}_R\left(\overline{x^{m-1}y^{n-1}}\right) = (x, y)R.$$

This implies

$$(x, y)R \in \operatorname{Ass}(R/I)$$

and in fact we will later show $\operatorname{Ass}(R/I) = \{(x, y)R\}$.

Next consider the ideal $J = ((x, y)R)^3 = (x^3, x^2y, xy^2, y^3)R$ with lattice representation



where $y^2, xy, x^2 \in R$ have been marked. Since $(x, y)R \subseteq \operatorname{Ann}_R(\overline{xy}) \subsetneq R$ for $\overline{xy} \in R/J$ and $(x, y)R$ is maximal, we have $(x, y)R = \operatorname{Ann}_R(\overline{xy})$ and an identical argument shows $\operatorname{Ann}_R(\overline{y^2}) = (x, y)R = \operatorname{Ann}_R(\overline{x^2})$. Hence

$$(x, y)R \in \operatorname{Ass}(R/J).$$

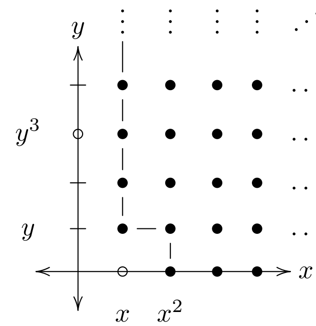
Lastly let $L = (x^2, xy)R$ and consider the elements $\overline{y^3}, \overline{x} \in R/L$ for which we have

$$(x)R = \operatorname{Ann}_R\left(\overline{y^3}\right)$$
$$(x, y)R = \operatorname{Ann}_R(\overline{x})$$

and therefore

$$\{(x)R, (x,y)R\} \subseteq \mathrm{Ass}(R/L).$$

As with the previous two ideals, the element $\overline{x}$ that get annihilated resides near the corner in our lattice diagram below.



Contrary to our first two examples however, notice $\overline{y^3}$ instead lies in the 'corridor' along the vertical axis and in fact could have been $\overline{y^t}$ for any $t \geq 1$. So when looking for associated primes, we look near corners and in the corridors of the corresponding lattice diagram.

rmk092217p

Remark II.C.2.18. For any $m \in M$ there exists a well-defined R-module homomorphism

$$\begin{aligned}\lambda_m \colon R &\longrightarrow M\\ r &\longmapsto rm.\end{aligned}$$

If we let $I = \mathrm{Ann}_R(m)$, then $\ker(\lambda_m) = I$, $\mathrm{Im}\,\lambda_m = Rm$, and the First Isomorphism Theorem gives $R/I \cong Rm \subseteq M$. Moreover there exists an injective R-module homomorphism

$$\begin{aligned}\overline{\lambda}_m \colon R/I &\hookrightarrow M\\ r + I &\longmapsto rm.\end{aligned}$$

In particular, if $\mathfrak{p} \in \mathrm{Ass}(M)$, then there exists an injective R-module homomorphism from $R/\mathfrak{p}$ into M.

Conversely, if $\varphi \colon R/\mathfrak{p} \hookrightarrow M$ is an injective R-module homomorphism, then for the element $m = \varphi\left(\overline{1}\right)$, we have $\mathrm{Ann}_R(m) = \mathfrak{p}$ and so $\mathfrak{p} \in \mathrm{Ass}(M)$. Hence $\mathfrak{p} \in \mathrm{Ass}(M)$ if and only if there exists an injection $\varphi \colon R/\mathfrak{p} \hookrightarrow M$.

prop092217q

Proposition II.C.2.19. *Assume R is noetherian and let M be a non-zero R-module.*

prop092217q.a

(a) *The set*

$$A_R(M) = \{\mathrm{Ann}_R(m) \mid m \in M \setminus \{0\}\}$$

has maximal elements, and every maximal element is prime. Therefore $\mathrm{Ass}(M) \neq \emptyset$.

prop092217q.b

(b) *If we define the set*

$$\mathrm{ZD}_R(M) = \{\textit{zero divisors on } M \textit{ in } R\},$$

then we have

$$\{0\} \cup \mathrm{ZD}_R(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}.$$

prop092217q.c

(c) *Independent of the noetherian assumption we have* $\mathrm{Ass}(M) \subseteq \mathrm{supp}(M)$.

Proof. (a) Since $A_R(M)$ is a nonempty set of ideals of R, the maximum condition for noetherian rings guarantees $A_R(M)$ has a maximal element $I = \mathrm{Ann}_R(m)$ for some $m \in M \setminus \{0\}$. Note $m \neq 0$ implies $I \neq R$. To show I is prime, let $a, b \in R$ such that $ab \in I$ and $a \notin I$. Since $a \notin I$, it follows that $am \neq 0$ and we have

$$I = \mathrm{Ann}_R(m) \subseteq \mathrm{Ann}_R(am) \in A_R(M)$$

where the set containment holds by the commutivity of R. Moreover, since $\mathrm{Ann}_R(am) \neq R$, the maximality of I implies

$$I = \mathrm{Ann}_R(m) = \mathrm{Ann}_R(am).$$

In particular, $b \in \mathrm{Ann}_R(am) = I$.

(b) By part (a) every element of $A_R(M)$ is contained in an associated prime of M. That is, for every $m \in M \setminus \{0\}$ there exists $\mathfrak{p} \in \mathrm{Ass}(M)$ such that $\mathrm{Ann}_R(m) \subseteq \mathfrak{p}$. Hence

$$\{0\} \cup \mathrm{ZD}_R(M) \subseteq \bigcup_{\mathfrak{p}\in \mathrm{Ass}(M)} \mathfrak{p}.$$

On the other hand, $\mathfrak{p} \in \mathrm{Ass}(M)$ means precisely that $\mathfrak{p} = \mathrm{Ann}_R(m)$ for some $m \in M$, so $(\mathfrak{p}\setminus\{0\}) \subseteq \mathrm{ZD}_R(M)$.

(c) Let $\mathfrak{p} = \mathrm{Ann}_R(\hat{m}) \in \mathrm{Ass}(M)$, for some $\hat{m} \in M$. Then we can define the following injective R-module homomorphism.

$$\begin{aligned} \pi_{\hat{m}} : R/\mathfrak{p} &\hookrightarrow M \\ r+\mathfrak{p} &\longmapsto r\hat{m} \end{aligned}$$

Localizing at $\mathfrak{p}$ (which preserves injectivity) we have

$$0 \neq Q(R/\mathfrak{p}) = (R/\mathfrak{p})_{\mathfrak{p}} \xhookrightarrow{(\pi_{\hat{m}})_{\mathfrak{p}}} M_{\mathfrak{p}}$$

where $Q(R/\mathfrak{p})$ denotes the field of fractions (recall Example II.A.2.7). So $M_{\mathfrak{p}}$ contains a non-zero submodule and therefore $M_{\mathfrak{p}} \neq 0$. Hence $\mathfrak{p} \in \mathrm{supp}(M)$. $\square$

rmk101117a

REMARK II.C.2.20. Recall Examples II.C.2.12 and II.C.2.17. Applying Proposition II.C.2.19(c), we can justify the equalities in the next display.

$$\begin{aligned} \{(x,y)R\} &= \mathrm{Ass}(R/I) \\ \{(x,y)R\} &= \mathrm{Ass}(R/J) \\ \{(x)R,\ (x,y)R\} &\subseteq \mathrm{Ass}(R/L) \end{aligned}$$

PROOF. Since $\mathrm{rad}(I) = (x,y)R$, justifying the first equality is done as follows and the second is proven almost identically.

$$\{(x,y)R\} \overset{II.C.2.17}{\subseteq} \mathrm{Ass}(R/I) \overset{II.C.2.19(c)}{\subseteq} \mathrm{supp}(R/I) \overset{II.C.2.11.4}{=} V(I) \overset{II.C.2.5}{=} V((x,y)R) = \{(x,y)R\}$$

Later in Example II.C.3.10 we will see we have equality in the third case as well. Right now we have

$$\{(x)R,(x,y)R\} \subseteq \mathrm{Ass}(R/L) \subseteq \mathrm{supp}(R/L) = V(xR)$$

but this does not give the desired equality. What we will later see is that we can greatly refine the list of primes to consider on the far right-hand side of this containment. $\square$

Since $M = \{0\}$ implies $\mathrm{Ass}(M) = \emptyset$, we may strengthen the conclusion of Proposition II.C.2.19(a).

cor092817b

COROLLARY II.C.2.21. *If R is noetherian, then an R-module M is non-zero if and only if* $\mathrm{Ass}(M) \neq \emptyset$.

ex092817c

EXAMPLE II.C.2.22. Let $\mathbb{K}$ be a field. The ring defined as

$$R = \prod_{i=1}^{\infty} \mathbb{K} = \{(a_1, a_2, \dots) \mid a_i \in \mathbb{K},\ \forall i \in \mathbb{N}\}$$

is a commutative ring with identity under component-wise operations, but it is not noetherian. Indeed, consider the proper ideal

$$I = \bigoplus_{i=1}^{\infty} \mathbb{K} = \{(a_1, a_2, \dots) \in R \mid a_i = 0,\ \forall i \gg 0\} \not\ni (1,1,\dots).$$

Some examples of maximal ideals of R include those of the form

$$\mathfrak{m}_i = \{(a_1, a_2, \dots) \in R \mid a_i = 0\} = \mathrm{Ker}\, \tau_i$$

where $\tau_i : R \twoheadrightarrow \mathbb{K}$ maps sequences from R to their i^{th} entry and the maximality of $\mathfrak{m}$ follows from the First Isomorphism Theorem for rings (i.e., $R/\mathfrak{m} \cong \underline{\text{field}}$ implies $\mathfrak{m}$ maximal). Since no $\mathfrak{m}_i$ contains I, there must be some other maximal ideal $\mathfrak{m} \lneq R$ such that $\underline{I \subseteq} \mathfrak{m}$. It is actually quite difficult to write down $\mathfrak{m}$ explicitly. In addition, it can be shown that $\mathrm{Ass}(R/I) = \emptyset$, even though $R/I \neq 0$, thereby demonstrating the necessity of the noetherian assumption in Corollary II.C.2.21.

fact092817d

Fact II.C.2.23. *If R is not noetherian, but M is a noetherian module over R, then the conclusion of Corollary II.C.2.21 still holds.*

Proof. The details are omitted here, but the crux of the proof is M noetherian lets us conclude after some work that $R/\operatorname{Ann}_R(M)$ is noetherian. $\square$

We give a fact that will be used to prove the proposition that follows.

fact011218a

Fact II.C.2.24. *Given a short exact sequence* $0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$ *of R-module homomorphisms, $A = 0$ if and only if $A' = 0 = A''$.*

Proof. Assume $A = 0$. Since f is injective, A' must be zero and since g is surjective, A'' must be zero as well. Conversely if we assume $A' = 0 = A''$, by the exactness of the sequence we have $0 = \operatorname{Im} f = \ker(g) = A$ as desired. $\square$

prop092817e

Proposition II.C.2.25. *Consider a short exact sequence of R-module homomorphisms.*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

(a) $\operatorname{supp}(M) = \operatorname{supp}(M') \cup \operatorname{supp}(M'')$
(b) $\operatorname{Ass}(M') \subseteq \operatorname{Ass}(M) \subseteq \operatorname{Ass}(M') \cup \operatorname{Ass}(M'')$

Proof. (a) First note by Fact II.C.2.24 that $A \neq 0$ if and only if either $A' \neq 0$ or $A'' \neq 0$. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and since localization is exact we have the short exact sequence

$$0 \longrightarrow M'_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow M''_{\mathfrak{p}} \longrightarrow 0$$

implying the following string of equivalent conditions.

$$\begin{aligned} \mathfrak{p} \in \operatorname{supp}(M) &\iff M_{\mathfrak{p}} \neq 0 \\ &\iff M'_{\mathfrak{p}} \neq 0 \text{ or } M''_{\mathfrak{p}} \neq 0 \\ &\iff \mathfrak{p} \in \operatorname{supp}(M') \text{ or } \mathfrak{p} \in \operatorname{supp}(M'') \\ &\iff \mathfrak{p} \in \operatorname{supp}(M') \cup \operatorname{supp}(M'') \end{aligned}$$

This completes the proof of this part.

(b) For the first containment let $\mathfrak{p} \in \operatorname{Ass}(M')$. By Remark II.C.2.18 there exists an injective R-module homomorphism $\pi : R/\mathfrak{p} \hookrightarrow M'$. Composing with f we conclude there exists another injective R-module homomorphism $f \circ \pi : R/\mathfrak{p} \hookrightarrow M$ and by the same remark $\mathfrak{p} \in \operatorname{Ass}(M)$.

For the second containment let $\mathfrak{q} \in \operatorname{Ass}(M)$. Then there exists a submodule $N \subseteq M$ such that $N \cong R/\mathfrak{q}$ by Remark II.C.2.18. For any $\overline{r} \in R/\mathfrak{q} \cong N$ such that $\overline{r} \neq 0$ (i.e., $r \notin \mathfrak{q}$), notice

$$\operatorname{Ann}_R(\overline{r}) = \{s \in R \mid s\overline{r} = 0 \in R/\mathfrak{q}\} = \mathfrak{q}.$$

We now consider two possibilities regarding the intersection of the image of f and the submodule N.

In the first case when $N \cap \operatorname{Im} f \neq \{0\}$, there is a non-zero element $\alpha \in N \cap \operatorname{Im} f$ with $\alpha = f(\beta)$ for some $\beta \in M'$. Since f is injective, $\operatorname{Ann}_R(\beta) = \operatorname{Ann}_R(\alpha) = \mathfrak{q}$ and hence $\mathfrak{q} \in \operatorname{Ass}(M')$.

In the second case we have

$$\{0\} = N \cap \operatorname{Im} f = N \cap \operatorname{Ker} g = \operatorname{Ker} g|_N : N \longrightarrow M''$$

so the restriction $g|_N$ is injective. This yields

$$R/\mathfrak{q} \cong N \cong g(N) \subseteq M''$$

implying $\mathfrak{q} \in \operatorname{Ass}(M'')$. $\square$

rmk083018a

Remark II.C.2.26. For any R-modules A and B, if there exists an injective R-module homomorphism $\phi \colon A \hookrightarrow B$, then by Proposition II.C.2.25 we have $\operatorname{Ass}(A) \subseteq \operatorname{Ass}(B)$.

lem100217a

Lemma II.C.2.27. *Let M be an R-module and assume there exists a finite filtration*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M.$$

(a) $\operatorname{supp}(M) = \displaystyle\bigcup_{i=1}^{n} \operatorname{supp}\left(\frac{M_i}{M_{i-1}}\right)$

(b) $\operatorname{Ass}(M_i) \subseteq \operatorname{Ass}(M) \subseteq \bigcup_{i=1}^{n} \operatorname{Ass}\left(\frac{M_i}{M_{i-1}}\right)$

PROOF. (a) We will induct on n. The base case $n = 0$ holds trivially. So we assume $n \geq 1$ and the result holds for all modules with filtrations of length $n-1$. Given a filtration of length n above, we know M_{n-1} has a filtration of length $n-1$. Therefore under our induction hypothesis

$$\operatorname{supp}(M_{n-1}) = \bigcup_{i=1}^{n-1} \left(\frac{M_i}{M_{i-1}}\right).$$

To the short exact sequence

$$0 \longrightarrow M_{n-1} \xrightarrow{\subseteq} M_n \xrightarrow{\pi} M_n/M_{n-1} \longrightarrow 0$$

with canonical epimorphism π, we apply Proposition II.C.2.25 to get

$$\begin{aligned}
\operatorname{supp}(M) = \operatorname{supp}(M_n) &= \operatorname{supp}(M_{n-1}) \cup \operatorname{supp}(M_n/M_{n-1}) \\
&= \left(\bigcup_{i=1}^{n-1} \operatorname{supp}\left(\frac{M_i}{M_{i-1}}\right)\right) \cup \operatorname{supp}(M_n/M_{n-1}) \\
&= \bigcup_{i=1}^{n} \operatorname{supp}\left(\frac{M_i}{M_{i-1}}\right).
\end{aligned}$$

(b) The first containment follows from Remark II.C.2.26 and the inclusion maps $M_i \xrightarrow{\varepsilon_i} M$. For the second containment we will again induct on n, skipping the trivial base cases when $n = 0$ or 1. Assume $n \geq 2$ and the result holds for modules with filtrations of length $n-1$. We again use the short exact sequence

$$0 \longrightarrow M_{n-1} \xrightarrow{\subseteq} M_n \longrightarrow M_n/M_{n-1} \longrightarrow 0$$

which yields

$$\begin{aligned}
\operatorname{Ass}(M_n) &\subseteq \operatorname{Ass}(M_{n-1}) \cup \operatorname{Ass}(M_n/M_{n-1}) && II.C.2.25 \\
&\subseteq \bigcup_{i=1}^{n-1} \operatorname{Ass}(M_i/M_{i-1}) \cup \operatorname{Ass}(M_n/M_{n-1}) && \text{induction hypothesis} \\
&= \bigcup_{i=1}^{n} \operatorname{Ass}(M_i/M_{i-1}).
\end{aligned}$$

□

lem100217b LEMMA II.C.2.28. *Let $M_1, \ldots, M_n$ be R-modules and set $M = \bigoplus_{i=1}^{n} M_i$.*

lem100217b.a (a) $\operatorname{supp}(M) = \bigcup_{i=1}^{n} \operatorname{supp}(M_i)$

lem100217b.b (b) $\operatorname{Ass}(M) = \bigcup_{i=1}^{n} \operatorname{Ass}(M_i)$

PROOF. (a) We can explicitly build the following finite filtration of M.

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq \bigoplus_{i=1}^{n-2} M_i \subseteq \bigoplus_{i=1}^{n-1} M_i \subseteq \bigoplus_{i=1}^{n} M_i = M$$

For any $j \in [n]$, applying the First Isomorphism Theorem for modules to the canonical projection $\pi_j : \bigoplus_{i=1}^{j} M_i \to M_j$ gives the following isomorphism.

$$\frac{\bigoplus_{i=1}^{j} M_i}{\bigoplus_{i=1}^{j-1} M_i} \cong M_j$$

This, along with Lemma II.C.2.27, allows us to write

$$\operatorname{supp}(M) = \bigcup_{j=1}^{n} \operatorname{supp}\left(\frac{\bigoplus_{i=1}^{j} M_i}{\bigoplus_{i=1}^{j-1} M_i}\right) = \bigcup_{j=1}^{n} \operatorname{supp}(M_j)$$

so part (a) holds.

(b) By Lemma II.C.2.27 we have

$$\operatorname{Ass}(M) \subseteq \bigcup_{j=1}^{n} \operatorname{Ass}\left(\frac{\bigoplus_{i=1}^{j} M_i}{\bigoplus_{i=1}^{j-1} M_i}\right) = \bigcup_{j=1}^{n} \operatorname{Ass}(M_j).$$

For the reverse containment, notice for any $j \in [n]$ we have canonical injection and surjection ε_j and π_j, respectively, by which we construct the exact sequence

$$0 \longrightarrow M_j \xrightarrow{\varepsilon_j} M \xrightarrow{\pi_j} M/M_j \longrightarrow 0$$

Applying II.C.2.25, we conclude $\operatorname{Ass}(M_j) \subseteq \operatorname{Ass}(M)$ for every $j \in [n]$ and hence so is the union of all such $\operatorname{Ass}(M_j)$. $\square$

ection062921m

II.C.3. Prime Filtrations

We will see in Theorem II.C.3.3 (and Corollary II.C.3.4) that in the noetherian setting, finite prime filtrations of modules guarantee the sets of associated primes are finite as well. Moreover, Theorem II.C.3.3 will be used either directly or indirectly in a number of future results (e.g., Corollary II.C.4.2 and Lemma II.C.4.19). Proposition II.C.3.13 is another significant result. Part (a) in particular leverages prime filtrations to give equality between three noteworthy sets of primes.

thm100217c

Theorem II.C.3.1. *Assume R is noetherian and let M be a finitely generated R-module. There exists a finite filtration*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$$

such that for all $i \in [n]$, there exists an ideal $\mathfrak{p}_i \in \operatorname{Spec}(R)$ for which $M_i/M_{i-1} \cong R/\mathfrak{p}_i$.

Proof. If $M = 0$, then the empty filtration will suffice with $n = 0$, so assume $M \neq 0$. Set $M_0 = 0$ and since R is noetherian, there exists an ideal $\mathfrak{p} \in \operatorname{Ass}(M)$ by Proposition II.C.2.19. By the First Isomorphism Theorem there exists a submodule $M_1 \subseteq M$ such that $M_1/0 \cong M_1 \cong R/\mathfrak{p}$. If $M_1 = M$, then stop here with a finite filtration of length one. Otherwise $M/M_1 \neq 0$ and there exists $\mathfrak{p}_2 \in \operatorname{Ass}(M/M_1)$ by the same proposition. Hence by the Fourth Isomorphism Theorem there exists a submodule $M_2 \subseteq M$ with $M_1 \subseteq M_2$ and $M_2/M_1 \cong R/\mathfrak{p}_2$. If $M_2 = M$, then stop here with a finite filtration of length two. Otherwise continue the process, which must terminate after finitely many steps since M is noetherian (finitely generated modules over a noetherian ring are themselves noetherian). $\square$

fact082418a

Fact II.C.3.2. *The conclusion of Theorem II.C.3.1 holds if we replace the noetherian assumption on the ring R with a noetherian assumption on the module M.*

thm100217d

Theorem II.C.3.3. *Assume M has a filtration as in Theorem II.C.3.1.*

(a) $\operatorname{Ass}(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subseteq \operatorname{supp}(M)$ *and therefore* $|\operatorname{Ass}(M)| < \infty$.

(b) *For any ideal $\mathfrak{p} \in \operatorname{Spec}(R)$, $\mathfrak{p} \in \operatorname{supp}(M)$ if and only if $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some $i \in [n]$. In other words*

$$\operatorname{supp}(M) = \bigcup_{i=1}^{n} V(\mathfrak{p}_i).$$

Proof. (a) From Lemma II.C.2.27 and Example II.C.2.14 we have

$$\operatorname{Ass}(M) \subseteq \bigcup_{i=1}^{n} \operatorname{Ass}(M_i/M_{i-1}) = \bigcup_{i=1}^{n} \operatorname{Ass}(R/\mathfrak{p}_i) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$$

thereby proving the first containment. For any $i \in [n]$ we have

$$0 \neq Q(R/\mathfrak{p}_i) \cong \left(\frac{R}{\mathfrak{p}_i}\right)_{\mathfrak{p}_i} \cong \left(\frac{M_i}{M_{i-1}}\right)_{\mathfrak{p}_i} \cong \frac{(M_i)_{\mathfrak{p}_i}}{(M_{i-1})_{\mathfrak{p}_i}}$$

where the second isomorphism is a consequence of Theorem II.C.1.4. Therefore $(M_i)_{\mathfrak{p}_i}$ is a non-zero submodule of $M_{\mathfrak{p}_i}$. Hence $M_{\mathfrak{p}_i}$ is non-zero, i.e., $\mathfrak{p}_i \in \operatorname{supp}(M)$.

(b) This part is a corollary of previous results.

$$
\begin{aligned}
\operatorname{supp}(M) &= \bigcup_{i=1}^{n} \operatorname{supp}\left(\frac{M_i}{M_{i-1}}\right) && II.C.2.27 \\
&= \bigcup_{i=1}^{n} \operatorname{supp}\left(\frac{R}{\mathfrak{p}_i}\right) && \frac{M_i}{M_{i-1}} \cong \frac{R}{\mathfrak{p}_i} \\
&= \bigcup_{i=1}^{n} V(\mathfrak{p}_i) && II.C.2.11
\end{aligned}
$$

□

cor100217e COROLLARY II.C.3.4. *If R is noetherian and M is a finitely generated R-module, then* $|\operatorname{Ass}(M)| < \infty$.

rmk100217f REMARK II.C.3.5. The finitely generated assumption for M in the above corollary is indeed necessary. There exist noetherian rings such as $R = \mathbb{K}[x]$ or $R = \mathbb{Z}$ for which $|\operatorname{Spec}(R)| = \infty$. In such cases we can define a module

$$M = \bigoplus_{i=1}^{\infty} \frac{R}{\mathfrak{p}_i}$$

such that $\mathfrak{p}_1, \mathfrak{p}_2, \dots \in \operatorname{Spec}(R)$ are all distinct. Since $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots\} \subseteq \operatorname{Ass}(M)$ by Proposition II.C.2.25, we know $|\operatorname{Ass}(M)| = \infty$.

We next use prime filtrations to give another verification of Example II.C.2.15.

ex100217g EXAMPLE II.C.3.6. Assume R is a unique factorization domain and let $x \in R \setminus R^{\times}$ be non-zero. Then $x = p_1 \cdots p_n$ for some primes $p_1, \dots, p_n \in R$. Defining the module $M = R/xR$ we have the filtration

$$0 \hookrightarrow \frac{R}{p_1 R} \hookrightarrow \cdots \hookrightarrow \frac{R}{p_1 \cdots p_{n-2} R} \xhookrightarrow{\phi_2} \frac{R}{p_1 \cdots p_{n-1} R} \xhookrightarrow{\phi_1} \frac{R}{p_1 \cdots p_n R} = \frac{R}{xR} = M$$

where $\phi_1(\overline{r}) := \overline{p_n r}$ and hence $\operatorname{Im}\phi_1 \leq R/xR$ is the ideal generated over R by $\overline{p_n}$. The maps ϕ_i for $i \geq 2$ are defined similarly. Using a clever re-write of the submodule $\operatorname{Im}\phi_1$ (cf. Note II.C.3.7), we apply the Third Isomorphism Theorem to write

$$\frac{R/xR}{\operatorname{Im}\phi_1} = \frac{R/p_1 \cdots p_n R}{\overline{p_n} \cdot R/(p_1 \cdots p_n R)} = \frac{R/p_1 \cdots p_n R}{p_n R/p_1 \cdots p_n R} \cong \frac{R}{p_n R}$$

and similarly

$$\frac{R/p_1 \cdots p_i R}{\operatorname{Im}\phi_{n-i+1}} \cong \frac{R}{p_i R}$$

for any $i \in [n-1]$. Therefore R/xR has a prime filtration with $\mathfrak{p}_i = p_i R$ and from Theorem II.C.3.3 we know

$$\operatorname{Ass}(R/xR) \subseteq \{p_1 R, \dots, p_n R\}. \tag{II.C.3.6.1}$$ eqn012018a

Moreover for any $i \in [n]$ we can define the injection

$$
\begin{aligned}
R/p_i R &\hookrightarrow R/xR \\
\overline{r} &\mapsto \overline{r p_1 \cdots p_{i-1} p_{i+1} \cdots p_n}
\end{aligned}
$$

implying by Remark II.C.2.18 that $p_i R \in \operatorname{Ass}(R/xR)$. Hence we have equality in (II.C.3.6.1).

note100417a

NOTE II.C.3.7. In the previous example we rely on the following general fact. Let $J \leq R$, M an R-module, and $N \subseteq M$ a submodule. Then we have

$$\begin{aligned} J \cdot \frac{M}{N} &= \langle j \cdot (m+N) \mid j \in J,\ m \in M \rangle \\ &= \langle jm + N \mid j \in J,\ m \in M \rangle \\ &= \langle (jm+n) + N \mid j \in J,\ m \in M,\ n \in N \rangle \\ &= \frac{JM+N}{N} \end{aligned} \tag{II.C.3.7.1}$$

eqn012018b

In the special case when $N \subseteq JM$ we have

$$JM + N = \langle jm + n \mid j \in J,\ m \in M,\ n \in N \rangle = \langle jm \mid j \in J,\ m \in M \rangle = JM$$

so Equation (II.C.3.7.1) above simplifies to give the equality

$$J \cdot \frac{M}{N} = \frac{JM}{N}.$$

We now introduce some notation and a lemma in order to simplify the proof of Example II.C.3.10.

def082618a

DEFINITION II.C.3.8. Let A be a commutative ring with identity and define the polynomial ring $R = A[X_1, \ldots, X_d]$. A monomial in R is an element of the form $X_1^{n_1} \cdots X_d^{n_d}$ where $n_1, \ldots, n_d \in \mathbb{N}_0$. The collection of all monomials in a subset $S \subseteq R$ is denoted $[\![S]\!]$ and an ideal $I \leq R$ is a monomial ideal if it there exists a set $T \subseteq [\![R]\!]$ such that $I = \langle T \rangle$. Let $\mathfrak{X} \leq R$ denote the ideal generated by the variables, i.e., $\mathfrak{X} = \langle X_1, \ldots, X_d \rangle$.

lem060318a

LEMMA II.C.3.9. *Let k be a field, let $R = k[X_1, \ldots, X_d]$ be a polynomial ring, and let I be a monomial ideal of R such that $X_i^{m_i} \in I$ for all $i \in [d]$. Then there exists a chain of ideals*

$$I = I_0 \subset I_1 \subset \cdots \subset I_A = R$$

where A is the dimension of R/I as a k-vector space, such that $I_j/I_{j-1} \cong R/\mathfrak{X}$ for all $j \in [A]$.

PROOF. We induct on A. If $A = 1$, then noting that $A = |[\![R]\!] \setminus [\![I]\!]|$, this implies $[\![R]\!] \setminus [\![I]\!] = \{1\}$, so $I = \mathfrak{X}$ and the chain $I \subset R$ satisfies the claim.

Assume $A > 1$ and the claim holds for any monomial ideal $J \supset \{X_1^{n_1}, \ldots, X_d^{n_d}\}$ for some non-zero $n_1, \ldots, n_d \in \mathbb{N}$ for which $\dim_k(R/J) = A - 1$. Let $f \in [\![R]\!]$ such that $f \notin I$, but $X_i f \in I$ for every $i \in [d]$. (These are called the corner elements of I.) We have an R-module homomorphism

$$\begin{aligned} \tau : R &\longrightarrow \frac{I + fR}{I} \\ r &\longmapsto \overline{0 + rf} \end{aligned}$$

which is non-zero and surjective since $\tau(1) = \overline{f}$, $f \notin I$, and $\overline{f}$ generates the codomain. However, $X_i \in \ker(\tau)$ for every $i \in [d]$ by definition of a corner element, so $X_1, \ldots, X_d \in \ker(\tau)$. Hence $\mathfrak{X} \subset \ker(\tau) \subsetneq R$ so $\ker(\tau) = \mathfrak{X}$ by the maximality of $\mathfrak{X}$ and therefore $R/\mathfrak{X} \cong (I + fR)/I$. We write $I \subsetneq I + fR$ to get the beginning of our chain.

Now we claim $\dim_k(R/(I + fR)) = A - 1$. We have a short exact sequence

$$0 \longrightarrow \frac{I + fR}{I} \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{I + fR} \longrightarrow 0$$

with the natural injection and surjection. This is exact as a sequence of k-modules (i.e., vector spaces), so it splits to yield

$$\frac{R}{I} \cong \frac{I + fR}{I} \oplus \frac{R}{I + fR}$$

and therefore to justify our claim we need only point out that

$$\frac{I + fR}{I} \cong \frac{R}{\mathfrak{X}} \cong k.$$

Therefore by our induction hypothesis we have a sequence

$$I + fR = I_1 \subset I_2 \subset \cdots \subset I_A = R$$

such that $I_j/I_{j-1} \cong R/\mathfrak{X}$ for $j = 2, \ldots, A$. Splicing on $I \subset I_1$ we have the desired chain. $\square$

ex100217h EXAMPLE II.C.3.10. Let k be a field and consider the unique factorization domain $R = k[x, y]$.

ex100217h.a (a) Define $I = (x^m, y^n)R$ where $m, n \in \mathbb{N}$ and consider R/I as a finitely generated R-module. From Example II.C.2.17 we already know

$$\mathrm{Ass}\left(\frac{R}{(x^m, y^n)R}\right) = \{(x, y)R\}$$

so we want to build a prime filtration such that each subquotient is isomorphic to $R/(x, y)R$. By Lemma II.C.3.9 we know there exists a chain of ideals

$$I = I_0 \subset I_1 \subset \cdots \subset I_A = R$$

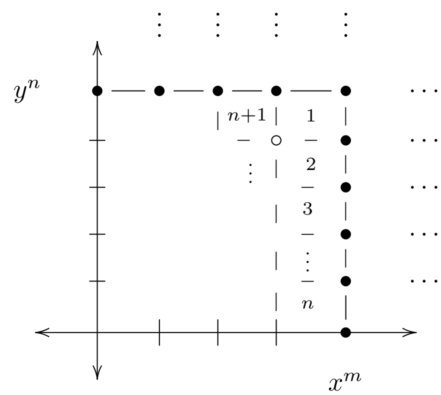
such that $I_j/I_{j-1} \cong R/(x, y)R$ for all $j \in [A]$ where $A = \dim_k(R/I)$. Considering that $I \subset I_j$ for all j we also have the chain

$$0 \subset I_1/I \subset I_2/I \subset \cdots \subset I_A/I = R/I$$

and by the Third Isomorphism Theorem

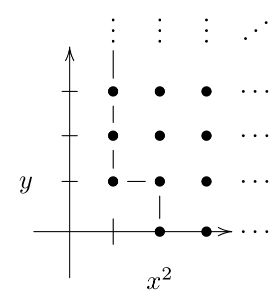
$$\frac{I_j/I}{I_{j-1}/I} \cong \frac{R}{(x, y)R}.$$

Since $(x, y)R \in \mathrm{Spec}(R)$, this chain is a prime filtration for R/I. In the proof of the lemma, we saw that the original chain from I to R is built by "throwing in" successive corner elements of I. In practice we can proceed in a methodical fashion as depicted in the lattice diagram below



where the element $x^{m-1}y^{n-1}$, marked with '$\circ$', represents a generator of I_1/I. We have thus demonstrated the existence of the prime filtration of R/I guaranteed by Theorem II.C.3.1.

ex100217h.b (b) Next consider the ideal $L = (x^2, xy)R$ with the following lattice diagram.



For elements $x + L,\ y + L \in R/L$, we already know $(x, y)R = \mathrm{Ann}_R(x + L)$ and $(x)R = \mathrm{Ann}_R(y + L)$, so

$$(x, y)R,\ (y)R \in \mathrm{Ass}(R/L) \qquad \text{(II.C.3.10.1)}$$ eqn012018f

So we want to show R/L has a prime filtration

$$0 \subseteq \langle x + L\rangle \subseteq R/L$$

To check the subsequent quotients, we first point out that by the argument in part (a) of this example

$$\frac{\langle x + L\rangle}{0} \cong \langle x + L\rangle \cong \frac{R}{(x, y)R}$$

so the condition for a prime filtration is satisfied for the first containment. To check the condition on the second containment we define the surjection

$$\begin{aligned} R/L &\longrightarrow R/(x)R \\ r+L &\longmapsto \overline{r} \end{aligned}$$

with kernel $\langle x+L\rangle$, so the condition is verified by the First Isomorphism Theorem and because $(x)R \in \operatorname{Spec}(R)$. (Well-definedness of the above map holds since $L \subseteq (x)R$.) Thus we have a prime filtration with $\mathfrak{p}_1 = (x,y)R$ and $\mathfrak{p}_2 = (x)R$, so $\operatorname{Ass}(R/L) \subseteq \{(x,y)R,\ (x)R\}$ by Theorem II.C.3.3 and in fact $\operatorname{Ass}(R/L) = \{(x,y)R,\ (x)R\}$ by (II.C.3.10.1).

prop100217i

Proposition II.C.3.11. *Let R be a non-zero commutative ring with identity, let M be an R-module, and let $U \subseteq R$ be a multiplicatively closed subset.*

prop100217i.a (a) $\operatorname{supp}_{U^{-1}R}(U^{-1}M) = \{U^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{supp}(M)$ *and* $\mathfrak{p} \cap U = \emptyset\}$

prop100217i.b (b) $\operatorname{Ass}_{U^{-1}R}(U^{-1}M) \supseteq \{U^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}(M)$ *and* $\mathfrak{p} \cap U = \emptyset\}$

prop100217i.c (c) *If R is also noetherian, then we have equality in (b).*

Proof. (a) The prime ideals of $U^{-1}R$ are described as follows.

$$\operatorname{Spec}(U^{-1}R) = \{U^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec}(R) \text{ and } \mathfrak{p} \cap U = \emptyset\}$$

For any ideal $U^{-1}\mathfrak{p}$, we know $M_{\mathfrak{p}} \cong (U^{-1}M)_{U^{-1}\mathfrak{p}}$ by the map

$$\begin{aligned} \Phi \colon M_{\mathfrak{p}} &\longrightarrow (U^{-1}M)_{U^{-1}\mathfrak{p}} \\ \frac{m}{v} &\longmapsto \frac{m/1}{v/1} \end{aligned}$$

and thus

$$(U^{-1}M)_{U^{-1}\mathfrak{p}} \neq 0 \iff M_{\mathfrak{p}} \neq 0$$

completing the proof of (a).

(b) Let $\mathfrak{p} \in \operatorname{Ass}(M)$ such that $U \cap \mathfrak{p} = \emptyset$. By Remark II.C.2.18 there exists a monomorphism

$$R/\mathfrak{p} \xhookrightarrow{\varphi} M$$

and therefore we also have the following monomorphism.

$$U^{-1}(R/\mathfrak{p}) \xhookrightarrow{U^{-1}\varphi} U^{-1}M$$

Hence $U^{-1}\mathfrak{p} \in \operatorname{Ass}_{U^{-1}R}(U^{-1}M)$ by the isomorphism

$$\frac{U^{-1}R}{U^{-1}\mathfrak{p}} \cong U^{-1}(R/\mathfrak{p}).$$

(c) Assume R is noetherian and let $U^{-1}\mathfrak{p} \in \operatorname{Ass}_{U^{-1}R}(U^{-1}M)$, where we know the form of such elements by the first line of the proof of part (a). We need to show $\mathfrak{p} \in \operatorname{Ass}(M)$. Since R is noetherian let $x_1,\ldots,x_n \in R$ such that $\mathfrak{p} = (x_1,\ldots,x_n)R$ and by virtue of being an associated prime we also know $U^{-1}\mathfrak{p} = \operatorname{Ann}_{U^{-1}R}(m/u)$ for some $m/u \in U^{-1}M$. For any $i \in [n]$, $x_i/1 \in U^{-1}\mathfrak{p}$ so

$$\frac{x_i}{1} \cdot \frac{m}{v} = 0$$

and thus there exist $u_1,\ldots,u_n \in U$ such that $u_i x_i m = 0$ for each $i \in [n]$. We define $u' = u_1 \cdots u_n \in U$ for which we have

$$x_i u' m = x_i u_1 \cdots u_i \cdots u_n m = 0$$

for every $i \in [n]$, implying

$$\mathfrak{p} = (x_1,\ldots,x_n)R \subseteq \operatorname{Ann}_R(u'm).$$

Now recall to prove $\mathfrak{p} \in \operatorname{Ass}(M)$ it suffices to find a monomorphism mapping from $R/\mathfrak{p}$ to M. Define the R-module homomorphism

$$\begin{aligned} \phi \colon R &\longrightarrow M \\ r &\longmapsto ru'm \end{aligned}$$

and notice that $\mathfrak{p} \subseteq \ker(\phi)$. Then by the Universal Mapping Property for quotients, the map

$$\begin{aligned} \alpha\colon R/\mathfrak{p} &\longrightarrow M \\ \overline{r} &\longmapsto ru'm \end{aligned}$$

is a well-defined R-module homomorphism as well. To show α is one-to-one, we consider a commutative diagram

$$\begin{tikzcd} R/\mathfrak{p} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & M \arrow[d, "\gamma"] \\ U^{-1}(R/\mathfrak{p}) \arrow[r, "U^{-1}\alpha"] & U^{-1}(M) \end{tikzcd}$$

and we will show both β and $U^{-1}\alpha$ are injective. First we verify the commutivity of the diagram. Indeed for any $r+\mathfrak{p} \in R/\mathfrak{p}$ we have

$$\begin{aligned} (U^{-1}\alpha \circ \beta)(r+\mathfrak{p}) = U^{-1}\alpha\left(\frac{r+\mathfrak{p}}{1}\right) &= \frac{\alpha(r+\mathfrak{p})}{1} \\ (\gamma \circ \alpha)(r+\mathfrak{p}) = \gamma(\alpha(r+\mathfrak{p})) &= \frac{\alpha(r+\mathfrak{p})}{1}. \end{aligned}$$

To show β is one-to-one, let $r+\mathfrak{p} \in R/\mathfrak{p}$ such that $\beta(r+\mathfrak{p}) = 0$. This implies

$$\frac{r+\mathfrak{p}}{1} = \frac{0+\mathfrak{p}}{v}$$

for any $v \in U$. By our definition of equality in the ring of fractions, there is some $w \in U$ such that

$$0+\mathfrak{p} = (w \cdot 1 \cdot 0) + \mathfrak{p} = (wrv) + \mathfrak{p}$$

so $wrv \in \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, from the conditions $w, v \in U$ and $U \cap \mathfrak{p} = \emptyset$, we conclude $r \in \mathfrak{p}$ and $r+\mathfrak{p} = 0+\mathfrak{p}$. Hence β is one-to-one.

Turning our attention to $U^{-1}\alpha$, we first note that since u/m and $u'm/1$ differ only by the unit u'/u we have

$$U^{-1}\mathfrak{p} = \operatorname{Ann}_{U^{-1}R}\left(\frac{u'm}{1}\right).$$

Next consider an element in the kernel of $U^{-1}\alpha$:

$$\begin{aligned} (U^{-1}\alpha)\left(\frac{r+\mathfrak{p}}{v}\right) = 0 &\iff \frac{ru'm}{v} = \frac{0}{v} \\ &\iff \frac{r}{v} \in U^{-1}\mathfrak{p} \\ &\iff r \in \mathfrak{p} \\ &\iff \frac{r+\mathfrak{p}}{v} = 0 \in U^{-1}(R/\mathfrak{p}). \end{aligned}$$

Hence $U^{-1}\alpha$ is one-to-one and therefore so is the composition $(U^{-1}\alpha)\circ\beta$. By the commutivity of the diagram the composition $\gamma \circ \alpha$ must also be one-to-one, so we conclude α is injective. $\square$

cor100717a

Corollary II.C.3.12. *Let $Q \in \operatorname{Spec}(R)$.*

(a) $\operatorname{supp}(M_Q) = \{\mathfrak{p}_Q \mid \mathfrak{p} \in \operatorname{supp}(M),\ \mathfrak{p} \subseteq Q\}$
(b) $\operatorname{Ass}(M_Q) \supseteq \{\mathfrak{p}_Q \mid \mathfrak{p} \in \operatorname{Ass}(M),\ \mathfrak{p} \subseteq Q\}$
(c) *If R is noetherian, then we have equality in (b).*

Proof. Set $U = R \setminus Q$ in the context of Proposition II.C.3.11. Then $\mathfrak{p} \cap U = \emptyset$ if and only if $\mathfrak{p} \subseteq Q$. $\square$

prop100717b

Proposition II.C.3.13. *Assume R is noetherian and $M \neq 0$ is a finitely generated R-module with prime filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for every $i \in [n]$.*

prop100717b.a (a) $\operatorname{Min}(\operatorname{Ass}(M)) = \operatorname{Min}\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \operatorname{Min}(\operatorname{supp}(M))$
prop100717b.b (b) $|\operatorname{Min}(\operatorname{supp}(M))| < \infty$ *and for all $\mathfrak{p} \in \operatorname{supp}(M)$, there exists some $\mathfrak{p}' \in \operatorname{Min}(\operatorname{supp}(M))$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$.*
prop100717b.c (c) $|\operatorname{Min}(\operatorname{Spec}(R))| < \infty$ *and for all $\mathfrak{p} \in \operatorname{Spec}(R)$, there exists some $\mathfrak{p}' \in \operatorname{Min}(\operatorname{Spec}(R))$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$.*

Proof. (a) Hypothetically, let $\mathfrak{p} \in \mathrm{Min}(\mathrm{supp}(M))$ and thus $M_{\mathfrak{p}} \neq 0$. Since R is noetherian we know $R_{\mathfrak{p}}$ is noetherian as well, so $\mathrm{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq \emptyset$ by Corollary II.C.2.21. Therefore by Corollary II.C.3.12 we let $\mathfrak{q} \in \mathrm{Ass}(M)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q}_{\mathfrak{p}} \in \mathrm{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Since $\mathrm{Ass}(M) \subseteq \mathrm{supp}(M)$ by Theorem II.C.3.3, the minimality of $\mathfrak{p}$ in $\mathrm{supp}(M)$ combined with the fact that $\mathfrak{q} \subseteq \mathfrak{p}$ implies $\mathfrak{p} = \mathfrak{q} \in \mathrm{Ass}(M)$. We claim $\mathfrak{p} \in \mathrm{Min}(\mathrm{Ass}(M))$. Since $\mathrm{Ass}(M)$ is finite also by Theorem II.C.3.3, there exists an ideal $\mathfrak{p}' \in \mathrm{Min}(\mathrm{Ass}(M))$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$. Again using the minimality of $\mathfrak{p} \in \mathrm{supp}(M)$ we conclude $\mathfrak{p} = \mathfrak{p}' \in \mathrm{Min}(\mathrm{Ass}(M))$ and we have thus shown

$$\mathrm{Min}(\mathrm{supp}(M)) \subseteq \mathrm{Min}(\mathrm{Ass}(M)). \qquad \text{(II.C.3.13.1)}$$ eqn012018d

Next let

$$\mathfrak{p}_i \in \mathrm{Min}\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \overset{(1)}{\subseteq} \mathrm{supp}(M)$$

the existence of which is guaranteed, because we are taking a minimal element of a finite set, and (1) is given by Theorem II.C.3.3. We want to show $\mathfrak{p}_i$ is a minimal element of $\mathrm{supp}(M)$. Suppose there is an ideal $\mathfrak{p}' \in \mathrm{supp}(M)$ such that $\mathfrak{p}' \subseteq \mathfrak{p}_i$. By Theorem II.C.3.3 there exists an ideal $\mathfrak{p}_j \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ such that

$$\mathfrak{p}_j \subseteq \mathfrak{p}' \subseteq \mathfrak{p}_i$$

and the minimality of $\mathfrak{p}_i$ yields

$$\mathfrak{p}_i \subseteq \mathfrak{p}_j \subseteq \mathfrak{p}' \subseteq \mathfrak{p}_i.$$

Hence $\mathfrak{p}' = \mathfrak{p}_i$, so $\mathfrak{p}_i$ is minimal and we have shown

$$\mathrm{Min}\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subseteq \mathrm{Min}(\mathrm{supp}(M)). \qquad \text{(II.C.3.13.2)}$$ eqn012018e

Note this containment verifies the legitimacy of the hypothetical element we took from $\mathrm{Min}(\mathrm{supp}(M))$ in the beginning of the proof. Finally let

$$\mathfrak{p} \in \mathrm{Min}(\mathrm{Ass}(M)) \subseteq \mathrm{Ass}(M) \overset{II.C.3.3}{\subseteq} \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}.$$

This implies $\mathfrak{p} = \mathfrak{p}_i$ for some $i \in [n]$ and since $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is finite, there exists some $\mathfrak{p}_j \in \mathrm{Min}\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ such that

$$\mathfrak{p}_j \subseteq \mathfrak{p}_i = \mathfrak{p}.$$

Using the results from Equations (II.C.3.13.1) and (II.C.3.13.2) we have shown

$$\mathfrak{p}_j \in \mathrm{Min}\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subseteq \mathrm{Min}(\mathrm{supp}(M)) \subseteq \mathrm{Min}(\mathrm{Ass}(M))$$

whereby we conclude $\mathfrak{p} = \mathfrak{p}_j$ by the minimality of $\mathfrak{p}$. This shows

$$\mathrm{Min}(\mathrm{Ass}(M)) \subseteq \mathrm{Min}\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}.$$

(b) Let $\mathfrak{p} \in \mathrm{supp}(M)$ and we have $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some $i \in [n]$ by Theorem II.C.3.3. Therefore there exists some

$$\mathfrak{p}_j \in \mathrm{Min}\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \mathrm{Min}(\mathrm{supp}(M))$$

such that

$$\mathfrak{p} \supseteq \mathfrak{p}_i \supseteq \mathfrak{p}_j$$

so taking $\mathfrak{p}_j$ as our $\mathfrak{p}'$ this proves the second part. The first part is verified simply as follows.

$$|\mathrm{Min}(\mathrm{supp}(M))| = |\mathrm{Min}\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}| \leq n < \infty$$

(c) This is more or less a corollary. Set $M = R$, note $\mathrm{supp}(R) = \mathrm{Spec}(R)$, and apply (b). □

We provide here two examples demonstrating that minimal associated primes may or may not be unique.

ex100717c Example II.C.3.14. Consider the ring of polynomials with complex coefficients $R = \mathbb{C}[x]$ and define the R-module

$$M = \frac{\mathbb{C}[x]}{x(x-1)\cdots(x-9)}.$$

From our work in Example II.C.3.6 we know

$$\mathrm{Ass}(M) = \{xR, (x-1)R, \ldots, (x-9)R\}.$$

Since $(x-i)R \not\subseteq (x-j)R$ for any $i \neq j$, we conclude

$$\mathrm{Min}(\mathrm{Ass}(M)) = \mathrm{Ass}(M).$$

ex100717d EXAMPLE II.C.3.15. Consider the ring of polynomials in two variables with coefficients in an arbitrary field, $R = \mathbb{K}[x, y]$, and define the R-module

$$M = \frac{\mathbb{K}[x, y]}{(x^2, xy)R}.$$

We already know from Example II.C.3.10 that

$$\mathrm{Ass}(M) = \{(x)R, (x, y)R\}$$

and since $(x)R \subset (x, y)R$ we have

$$\mathrm{Min}(\mathrm{Ass}(M)) = \{(x)R\}.$$

def100717e DEFINITION II.C.3.16. With R and M as in Proposition II.C.3.13, define

$$\mathrm{Min}_R(M) = \mathrm{Min}(\mathrm{Ass}(M)).$$

Any ideal $\mathfrak{p} \in \mathrm{Min}_R(M)$ is a minimal prime of M or a minimal associated prime of M. If $\mathfrak{q} \in \mathrm{Ass}(M) \setminus \mathrm{Min}_R(M)$, then $\mathfrak{q}$ is an embedded prime of M.

note100717f NOTE II.C.3.17. Let R be a noetherian ring. For any ideals $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$, we have $V(\mathfrak{p}) \subseteq V(\mathfrak{q})$. Applying Theorem II.C.3.3 and Proposition II.C.3.13, it follows that

$$\mathrm{supp}(M) = \bigcup_{i=1}^{n} V(\mathfrak{p}_i) = \bigcup_{\mathfrak{p}_i \in \mathrm{Min}(\mathrm{supp}(M))} V(\mathfrak{p}_i)$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are the prime ideals from some prime filtration of M.

II.C.4. Prime Avoidance and Nakayama's Lemma

ection062921n

In this section we prove two major results from abstract algebra. Prime Avoidance gives us more insight into the associated primes of finitely generated R-modules, especially when R is noetherian, as we will see in Corollaries II.C.4.2 and II.C.4.3. Nakayama's Lemma has a number of corollaries, some of which we will produce here. Perhaps most importantly, Nakayama's Lemma gives us Lemma II.C.4.19 which we will use directly in the proof of Theorem II.C.5.16.

lem100717g LEMMA II.C.4.1 (Prime Avoidance). *Let R be a non-zero commutative ring with identity and let $I_1, \ldots, I_n, J \leq R$ be ideals. Assume one of the following:*

(a) *R contains an infinite field as a subring, or*

(b) *the ideals $I_1, \ldots, I_{n-2}$ are prime.*

Then whenever $J \subseteq \bigcup_{i=1}^n I_i$, we have $J \subseteq I_i$ for some $i \in [n]$. Equivalently, if $J \not\subseteq I_i$ for all $i \in [n]$, then $J \not\subseteq \bigcup_{i=1}^n I_i$.

PROOF. Assume $\mathbb{K} \subseteq R$ as a subring with $\mathbb{K}$ an infinite field. For an arbitrary vector space V over $\mathbb{K}$, we have this fact:

$$V_1, \ldots, V_n \subsetneq V \text{ proper subspaces} \implies \bigcup_{i=1}^{n} V_i \subsetneq V. \qquad \text{(II.C.4.1.1)}$$ eqn012018g

Assume $J \not\subseteq I_j$ for any $j \in [n]$, which implies $J \cap I_j \subsetneq J$. Since ideals of R are $\mathbb{K}$-vector spaces, (II.C.4.1.1) gives

$$J \cap \left(\bigcup_{j=1}^{n} I_j \right) = \bigcup_{j=1}^{n} (J \cap I_j) \subsetneq J$$

and hence

$$J \not\subseteq \bigcup_{j=1}^{n} I_j.$$

Now assume $I_1, \ldots, I_{n-2}$ are prime and we will argue by induction on n. For the base case $n = 1$, the hypothesis is vacuous and the conclusion holds trivially. For the base case $n = 2$, suppose $J \not\subseteq I_1, I_2$ and suppose for the sake of contradiction that $J \subseteq (I_1 \cup I_2)$. Then there exist elements $x_1, x_2 \in J$ such that $x_1 \notin I_1$ and $x_2 \notin I_2$. We observe

$$x_1 \in J \subseteq (I_1 \cup I_2) \implies x_1 \in I_2$$
$$x_2 \in J \subseteq (I_1 \cup I_2) \implies x_2 \in I_1.$$

We know also that $x_1+x_2 \in J$. If $x_1+x_2 \in I_1$, then $x_1 = (x_1+x_2)-x_2 \in I_1$, a contradiction. An identical contradiction lets us conclude $x_1+x_2 \notin I_2$, giving us

$$x_1+x_2 \in J \setminus (I_1 \cup I_2)$$

which is a contradiction, proving the second base case.

For the induction step, assume $n \geq 3$ and assume the result holds for lists of length $n-1$. If $J \subseteq \bigcup_{i \neq l} I_i$ for some $l \in [n]$, then by the induction hypothesis $J \subseteq I_i$ for some $i \neq l$ and we are done. Therefore assume without loss of generality

$$J \not\subseteq \bigcup_{i \neq l} I_i$$

for all $l \in [n]$. For each $l \in [n]$, fix an element

$$x_l \in J \setminus \bigcup_{i \neq l} I_i \qquad \text{(II.C.4.1.2)}$$ eqn032518a

and consider the element $x' = x_1 + (x_2 \cdots x_n) \in J$. Suppose $J \subseteq \bigcup_{i=1}^n I_i$, implying $x' \in I_i$ for some i. If we first suppose $x' \in I_1$, it follows that $x_2 \cdots x_n = x' - x_1 \in I_1$. Since I_1 is a prime ideal, this implies $x_j \in I_1$ for some $j \geq 2$, contradicting our choices in (II.C.4.1.2). Second, if we suppose $x' \in I_i$ for some $i \geq 2$, then we have $x_2 \cdots x_n \in I_i$ and therefore $x_1 = x' - x_2 \cdots x_n \in I_i$, again contradicting (II.C.4.1.2). □

cor100717h COROLLARY II.C.4.2. *Let R be a noetherian ring and let M be a non-zero, finitely generated R-module. If $J \leq R$ consists entirely of zero-divisors on M, then there exists some ideal $\mathfrak{p} \in \operatorname{Ass}(M)$ such that $J \subseteq \mathfrak{p}$.*

PROOF. Since R is noetherian and M is finitely generated, Theorem II.C.3.3 implies $\operatorname{Ass}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ for some $n \geq 1$. By Proposition II.C.2.19 we have

$$J \subseteq \{\text{zero-divisors on } M\} = \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}.$$

Therefore $J \subseteq \mathfrak{p}_i$ for some $i \in [n]$ by Lemma II.C.4.1. □

cor100717i COROLLARY II.C.4.3. *Assume R is noetherian, let M be a non-zero finitely generated R-module, and let $\mathfrak{m} \lneq R$ be a maximal ideal. Then*

$$\mathfrak{m} \textit{ contains a non-zero-divisor on } M \iff \mathfrak{m} \notin \operatorname{Ass}(M)$$

or equivalently

$$\mathfrak{m} \textit{ consists entirely of zero-divisors on } M \iff \mathfrak{m} \in \operatorname{Ass}(M).$$

PROOF. If $\mathfrak{m}$ consists entirely of zero-divisors on M, then $\mathfrak{m} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}(M)$ by Corollary II.C.4.2. Since we have a maximal ideal inside a proper ideal, $\mathfrak{m} = \mathfrak{p} \in \operatorname{Ass}(M)$, proving one direction. On the other hand, if we suppose $\mathfrak{m} = \operatorname{Ann}_R(m)$ for some $0 \neq m \in M$, then $\mathfrak{m}$ consists entirely of zero-divisors on M. □

def100717j DEFINITION II.C.4.4. Let $I \leq R$ and M an R-module. IM is the submodule of M defined

$$IM = \langle im \in M \mid i \in I,\ m \in M \rangle = \left\{ \sum_{j=1}^n i_j m_j \mid i_j \in I,\ m_j \in M,\ \forall j \in [n];\ n \in \mathbb{N} \right\}.$$

fact100717k FACT II.C.4.5. *We can make M/IM into an R/I-module by the operation*

$$(r+I)(m+IM) = (rm) + IM.$$

PROOF. Checking the module axioms is straight-forward. For instance, we verify two questions of well-definedness here. Let $r_1, r_2 \in R$ such that $r_1 + I = r_2 + I$ and let $m \in M$. Therefore $r_1 - r_2 \in I$ and it follows that $(r_1 m) - (r_2 m) = (r_1 - r_2)m \in IM$. Thus

$$(r_1+I)(m+IM) = (r_1 m) + IM = (r_2 m) + IM = (r_2+I)(m+IM)$$

verifying one question. Now let $r \in R$ and let $m_1, m_2 \in M$ such that $m_1 + IM = m_2 + IM$. Then there exist $i \in I$ and $n \in M$ such that $m_1 - m_2 = in$. Therefore $(rm_1) - (rm_2) = r(m_1 - m_2) = r(in) = (ri)n \in IM$ and it follows that

$$(r+I)(m_1+IM) = (rm_1) + IM = (rm_2) + IM = (r+I)(m_2+IM)$$

verifying the second question, so the operation is well-defined. □

def100717l DEFINITION II.C.4.6. R is a local ring if it has a unique maximal ideal. If the unique maximal ideal is $\mathfrak{m} \lneq R$, then we say $(R, \mathfrak{m})$ is a local ring. Some texts would say this is a quasi-local ring, because we are not assuming R is noetherian.

fact100717m FACT II.C.4.7. *Let R be a commutative ring with identity.*

act100717m.a (a) *If $\mathfrak{m} \lneq R$ is maximal such that $1 + \mathfrak{m} \subseteq R^\times$, then $(R, \mathfrak{m})$ is local.*
act100717m.b (b) *The following are equivalent.*
ct100717m.b.i (i) *$(R, \mathfrak{m})$ is local.*
100717m.b.ii (ii) *$R \setminus R^\times$ is a proper ideal of R.*
00717m.b.iii (iii) *There exists a proper ideal $\mathfrak{a} \lneq R$ such that $R \setminus \mathfrak{a} \subseteq R^\times$.*
act100717m.c (c) *When the conditions of* (b) *are satisfied, we have $\mathfrak{m} = \mathfrak{a} = R \setminus R^\times$.*

PROOF. (b) Part (c) will follow from this argument as well. First assume $(R, \mathfrak{m})$ is local and we claim $\mathfrak{m} = R \setminus R^\times$, for which it suffices to show that $R \setminus \mathfrak{m} = R^\times$. For any $u \in R^\times$, $R = \langle u \rangle$, so $u \notin \mathfrak{m}$ since $\mathfrak{m}$ is a proper ideal. Thus $R^\times \subseteq R \setminus \mathfrak{m}$. For any $x \in R \setminus R^\times$, we have $\langle x \rangle \lneq R$. Since every proper ideal is contained in a maximal ideal, $\langle x \rangle \subseteq \mathfrak{m}$ and therefore $x \in \mathfrak{m}$. Having shown the contrapositive, we conclude $R^\times \supseteq R \setminus \mathfrak{m}$. Therefore (bi) implies (bii).

If we assume $R \setminus R^\times \lneq R$ and set $\mathfrak{a} = R \setminus R^\times$, then it is immediate that $R \setminus \mathfrak{a} = R^\times$. Hence (bii) implies (biii).

Finally, assume $\mathfrak{a} \lneq R$ is such that $R \setminus \mathfrak{a} \subseteq R^\times$. Taking the complement we have $\mathfrak{a} \supseteq R \setminus R^\times$. On the other hand, if we let $a \in \mathfrak{a}$, then $a \notin R^\times$, because $\mathfrak{a}$ is a proper ideal. Therefore $a \in R \setminus R^\times$ and thus $\mathfrak{a} \subseteq R \setminus R^\times$. Hence $\mathfrak{a} = R \setminus R^\times$. We claim for every maximal ideal $\eta \lneq R$ we have $\eta = \mathfrak{a}$. For any $y \in \eta$, since y does not generate the entire ring, $y \notin R^\times$. Therefore $y \in \mathbb{I}a$ and thus $\eta \subseteq \mathfrak{a} \subsetneq R$. Since η is maximal, $\eta = \mathfrak{a}$, completing the proof of (b) and (c).

(a) By part (b) it suffices to show $R \setminus \mathfrak{m} \subseteq R^\times$. Let $x \in R \setminus \mathfrak{m}$ and set $\langle x, \mathfrak{m} \rangle = \langle \{x\} \cup \mathfrak{m} \rangle$. It is straightforward to show

$$\langle x, \mathfrak{m} \rangle = \langle ax + m \mid a \in R, m \in \mathfrak{m} \rangle .$$

Then $\mathfrak{m} \subseteq \langle x, \mathfrak{m} \rangle$. Since $x \in \langle x, \mathfrak{m} \rangle$ and $x \notin \mathfrak{m}$ we have $\mathfrak{m} \subsetneq \langle x, \mathfrak{m} \rangle \subseteq R$. Therefore $\langle x, \mathfrak{m} \rangle = R$ by the maximality of $\mathfrak{m}$ and it follows that $1 \in \langle x, \mathfrak{m} \rangle$. Then we let $a \in R$ and $m \in \mathfrak{m}$ such that $1 = ax + m$. Therefore

$$ax = 1 - m \in 1 + \mathfrak{m} \subseteq R^\times$$

and we conclude $a, x \in R^\times$. □

NakLem LEMMA II.C.4.8 (Nakayama's Lemma). *Assume $(R, \mathfrak{m})$ is a local ring and M is a finitely generated R-module. The following conditions are equivalent.*

NakLem.i (i) $M = 0$
NakLem.ii (ii) $M = \mathfrak{m}M$
NakLem.iii (iii) $M/\mathfrak{m}M = 0$

PROOF. It is clear that (i) implies (iii) and it is also clear that conditions (ii) and (iii) are equivalent, so we need only show that (ii) implies (i). Assume $M = \mathfrak{m}M$, let $m_1, \ldots, m_n \in M$ be a generating sequence for M, and assume no proper subsequence of $m_1, \ldots, m_n$ generates M. Suppose for the sake of contradiction that $n \geq 1$. Then $m_1 \in M = \mathfrak{m}M$ can be written $m_1 = \sum_{i=1}^{n} r_i m_i$ for some $r_1, \ldots, r_n \in \mathfrak{m}$. Therefore

$$(1 - r_1) m_1 = m_1 - r_1 m_1 = \sum_{i=2}^{n} r_i m_i .$$

Since $r_1 \in \mathfrak{m}$, Fact II.C.4.7 above implies $1 - r_1 \in R^\times$ so $m_1 \in \langle m_2, \ldots, m_n \rangle$. In other words

$$M = \langle m_1, \ldots, m_n \rangle \subseteq \langle m_2, \ldots, m_n \rangle \subseteq M$$

giving equality at every step, which contradicts the minimality of our generating sequence. Therefore $n = 0$ and $M = \langle \emptyset \rangle = 0$. □

ex100817b EXAMPLE II.C.4.9. Let k be a field and consider the ring $R = k \times k$. We can define the projection

$$\begin{aligned} P_1 : R &\twoheadrightarrow k \\ (a,b) &\longmapsto a \end{aligned}$$

for which the kernel $\operatorname{Ker} P_1 = 0 \times k$ is a maximal ideal and we denote it $\mathfrak{m} = \operatorname{Ker} P_1$. Notice this maximal ideal is not unique. Consider the cyclic R-module $M = 0 \times k = \langle (0,1) \rangle$. In this case we have

$$\mathfrak{m}M = (0 \times k)(0 \times k) = (0 \times k) = M$$

but $M \neq 0$. The point here is that in order to use a maximal ideal in Nakayama's Lemma, we really do need that ideal to be unique.

ex100817c EXAMPLE II.C.4.10. Let $(R, \mathfrak{m})$ be a local integral domain, but not a field. Such rings could be

$$\mathbb{Z}_{\langle p \rangle} \qquad \text{or} \qquad \mathbb{K}[X]_{\langle X \rangle}$$

where $p \in \mathbb{N}$ a prime. Let $M = Q(R) \neq 0$ be the field of fractions of R. Since R is not a field, $\mathfrak{m} \neq 0$ and one can check that $\mathfrak{m} \cdot Q(R) = Q(R)$. So M must be finitely generated in Nakayama's Lemma.

cor100817d COROLLARY II.C.4.11. *If $(R, \mathfrak{m})$ is local, noetherian, and not a field, then $\mathfrak{m}^2 \subsetneq \mathfrak{m}$.*

PROOF. Since R is noetherian, we know $\mathfrak{m}$ is finitely generated, so by Nakayama's Lemma, if $\mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}^2 = \mathfrak{m}$, then $\mathfrak{m} = 0$, which is a contradiction, since R is not a field. □

cor100817e COROLLARY II.C.4.12. *Assume R is noetherian and M is a non-zero, finitely generated R-module.*

cor100817e.a (a) *If $(R, \mathfrak{m})$ is local and not a field with $\mathfrak{m} \notin \operatorname{Ass}(M)$, then $\mathfrak{m} \setminus \mathfrak{m}^2$ contains a non-zero-divisor on M.*

cor100817e.b (b) *If $\mathfrak{m} \lneq R$ a maximal ideal such that $\mathfrak{m}^2 \neq \mathfrak{m}$ and $\mathfrak{m} \notin \operatorname{Ass}(M)$, then $\mathfrak{m} \setminus \mathfrak{m}^2$ contains a non-zero-divisor on M.*

PROOF. We will prove part (b), then part (a) will follow by Corollary II.C.4.11. By Proposition II.C.2.19, the set of associated primes is nonempty and by Theorem II.C.3.3 we write $\operatorname{Ass}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ for some $n \geq 1$. We will apply Prime Avoidance (Lemma II.C.4.1) to the list $\mathfrak{p}_1, \ldots, \mathfrak{p}_n, \mathfrak{m}^2, \mathfrak{m}$ where $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are all prime. Since we are assuming $\mathfrak{m} \neq \mathfrak{m}^2$, it follows that $\mathfrak{m} \not\subseteq \mathfrak{m}^2$ (maximality of $\mathfrak{m}$ would force equality in this case). We are also assuming $\mathfrak{m} \notin \operatorname{Ass}(M)$, so by prime avoidance

$$\mathfrak{m} \not\subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n \cup \mathfrak{m}^2.$$

Therefore there exists an element $x \in \mathfrak{m}$ such that $x \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n = \operatorname{ZD}_R(M)$ and $x \notin \mathfrak{m}^2$ (cf. Proposition II.C.2.19). Hence we have found an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ that is a non-zero-divisor on M. □

cor101117b COROLLARY II.C.4.13. *Let $(R, \mathfrak{m})$ be local, M an R-module, and $N \subseteq M$ a submodule such that M/N is finitely generated over R. If $M = N + \mathfrak{m}M$, then $M = N$. (Note the stronger assumption that M is finitely generated would be sufficient to conclude M/N finitely generated.)*

PROOF. If $M = N + \mathfrak{m}M$, then we have

$$\mathfrak{m}\left(\frac{M}{N}\right) = \frac{\mathfrak{m}M + N}{N} = \frac{M}{N}.$$

Since M/N is finitely generated, we may apply Nakayama's Lemma to conclude $M = N$. □

def100917b DEFINITION II.C.4.14. Let M be a finitely generated R-module. A <u>minimal generating sequence</u> for M is a generating sequence $m_1, \ldots, m_n \in M$ such that no proper subsequence generates M.

ex100917c EXAMPLE II.C.4.15. For the ring $R = \mathbb{K}[x, y]$, the elements $x, y \in R$ form a minimal generating sequence for $\langle x, y \rangle$.

rmk101117d REMARK II.C.4.16. Note in our definition above, we make no claim on our ability to find a shorter generating sequence, but rather we cannot shorten this <u>particular</u> sequence without disrupting its generating property for M.

cor101117e COROLLARY II.C.4.17. *Let $(R, \mathfrak{m}, \mathfrak{K})$ be local and M a finitely generated R-module. Let $m_1, \ldots, m_n \in M$.*

cor101117e.a (a) *$M/\mathfrak{m}M$ is a finite-dimensional vector space over $\mathfrak{K}$ via the scalar multiplication*

$$\overline{r} \cdot \overline{m} = \overline{rm}.$$

cor101117e.b (b) *The sequence* $m_1, \dots, m_n$ *generates* M *as an* R*-module if and only if* $\overline{m}_1, \dots, \overline{m}_n \in M/\mathfrak{m}M$ *spans* $M/\mathfrak{m}M$ *as a* $\mathfrak{K}$*-vector space.*

cor101117e.c (c) $m_1, \dots, m_n \in M$ *is a minimal generating sequence for* M *if and only if* $\overline{m}_1, \dots, \overline{m}_n \in M/\mathfrak{m}M$ *is a basis for* $M/\mathfrak{m}M$ *over* $\mathfrak{K}$*. In particular, every minimal generating sequence for* M *has the same number of elements, namely*

$$\dim_{\mathfrak{K}}(M/\mathfrak{m}M).$$

PROOF. $M/\mathfrak{m}M$ is a $\mathfrak{K}$-module by Fact II.C.4.5. The $\mathfrak{K}$-vector space axioms follow directly from the R-module axioms. For example we have

$$\overline{r}(\overline{s} \cdot \overline{m}) = \overline{r} \cdot (\overline{sm}) = \overline{r(sm)} = \overline{(rs)m} = \overline{rs} \cdot \overline{m} = (\overline{r}\,\overline{s}) \cdot \overline{m}.$$

Since M is finitely generated we let $m_1, \dots, m_n$ generate M over R. For any $m \in M$, there exist $r_1, \dots, r_n \in R$ such that $m = \sum_{i=1}^n r_i m_i$. Therefore for any $\overline{m} \in M/\mathfrak{m}M$ there exist $\overline{r_1}, \dots, \overline{r_n} \in \mathfrak{K}$ such that

$$\overline{m} = \overline{\sum_{i=1}^n r_i m_i} = \sum_{i=1}^n \overline{r}_i \overline{m}_i$$

so $\overline{m}_1, \dots, \overline{m}_n$ spans $M/\mathfrak{m}M$ over $\mathfrak{K}$, completing the proof of part (a) as well as the forward implication of part (b).

Now assume $\overline{m}_1, \dots, \overline{m}_n$ spans $M/\mathfrak{m}M$ over $\mathfrak{K}$. We claim $M = \langle m_1, \dots, m_n \rangle + \mathfrak{m}M$. Certainly the reverse containment holds, because $\langle m_1, \dots, m_n \rangle, \mathfrak{m}M \subseteq M$ as submodules. On the other hand, if we let $m \in M$ be arbitrary, then our spanning assumption implies there exist $\overline{r}_1, \dots, \overline{r}_n \in \mathfrak{K}$ such that

$$\overline{m} = \sum_{i=1}^n \overline{r}_i \overline{m}_i = \overline{\sum_{i=1}^n r_i m_i}.$$

This yields

$$\begin{aligned} & m - \sum_{i=1}^n r_i m_i \in \mathfrak{m}M \\ \Longrightarrow\ & m = \sum_{i=1}^n r_i m_i + y && \text{for some } y \in \mathfrak{m}M \\ \Longrightarrow\ & m \in \langle m_1, \dots, m_n \rangle + \mathfrak{m}M \end{aligned}$$

proving the claim. Therefore we apply Corollary II.C.4.13 to conclude $M = \langle m_1, \dots, m_n \rangle$, so M is finitely generated, proving part (b).

For the forward direction of part (c), assume $m_1, \dots, m_n$ is a minimal generating sequence for M. Part (b) then implies $\overline{m}_1, \dots, \overline{m}_n$ spans $M/\mathfrak{m}M$. If we suppose for the sake of contradiction that $\overline{m}_1, \dots, \overline{m}_n$ is not a basis, then we can rearrange them if necessary to assume $\overline{m}_2, \dots, \overline{m}_n$ spans $M/\mathfrak{m}M$. Another application of part (b) implies $m_2, \dots, m_n$ is a generating sequence for M over R, contradicting our minimality assumption. Therefore $\overline{m}_1, \dots, \overline{m}_n$ is a basis for $M/\mathfrak{m}M$ as a $\mathfrak{K}$-vector space.

On the other hand, let us assume that $\overline{m}_1, \dots, \overline{m}_n$ is a basis for $M/\mathfrak{m}M$ over $\mathfrak{K}$. Equivalently, we have $\overline{m}_1, \dots, \overline{m}_n$ is a minimally spanning set for $M/\mathfrak{m}M$ over $\mathfrak{K}$. Then part (b) implies $m_1, \dots, m_n$ is a generating sequence for M as an R-module. Suppose $m_1, \dots, m_n$ is not minimal. That is, assume $m_2, \dots, m_n$ is a generating sequence for M, after some rearrangement of the m_i's if necessary. Again applying part (b) we know $\overline{m}_2, \dots, \overline{m}_n$ spans $M/\mathfrak{m}M$ as a $\mathfrak{K}$-vector space, contradicting the fact that $\overline{m}_1, \dots, \overline{m}_n$ is a minimal spanning set. □

cor101217a COROLLARY II.C.4.18. *Assume* $(R, \mathfrak{m}, \mathfrak{K})$ *is local and* P *is a finitely generated projective* R*-module. Then* P *is free with* $P \cong R^n$ *where* $n = \dim_{\mathfrak{K}}(P/\mathfrak{m}P)$*.*

PROOF. By Corollary II.C.4.13 there exist $p_1, \dots, p_n \in P$ such that they form a minimal generating sequence for P, where $n = \dim_{\mathfrak{K}}(P/\mathfrak{m}P)$. Note this implies $P/\mathfrak{m}P$ is an n-dimensional $\mathfrak{K}$-vector space and therefore $P/\mathfrak{m}P \cong \mathfrak{K}^n$. We therefore have the following well-defined, surjective R-module homomorphism.

$$\begin{aligned} \tau : R^n &\twoheadrightarrow P \\ e_i &\longmapsto p_i \\ \textstyle\sum_{i=1}^n r_i e_i &\longmapsto \textstyle\sum_{i=1}^n r_i p_i \end{aligned}$$

We may therefore define a short exact sequence $0 \longrightarrow \ker(\tau) \xrightarrow{\subseteq} R^n \xrightarrow{\tau} P \longrightarrow 0$ and from here it suffices to show that $\ker(\tau) = \{0\}$, thereby proving P is isomorphic to a free module. Let $K = \ker(\tau)$. Since P is projective, the short exact sequence splits and we write

$$\begin{aligned} R^n \cong K \oplus P &\xrightarrow{\pi} K \\ (x,p) &\longmapsto x \end{aligned}$$

where π is a well-defined, surjective R-module homomorphism, so K is also finitely generated. We will use this to apply Nakayama's Lemma. We have a string of isomorphisms.

$$\mathfrak{K}^n \cong \left(\frac{R}{\mathfrak{m}R}\right)^n \cong \frac{R^n}{\mathfrak{m}R^n} \cong \frac{K \oplus P}{\mathfrak{m}(K \oplus P)} \cong \frac{K}{\mathfrak{m}K} \oplus \frac{P}{\mathfrak{m}P} \cong \frac{K}{\mathfrak{m}K} \oplus \mathfrak{K}^n$$

Since isomorphic vector spaces have the same dimension we have $n = \dim_{\mathfrak{K}}(K/\mathfrak{m}K)+n$, implying $\dim_{\mathfrak{K}}(K/\mathfrak{m}K) = 0$ and therefore $K/\mathfrak{m}K = 0$. It follow from Nakayama's Lemma that $\ker(\tau) = K = 0$. $\square$

lem101217b

Lemma II.C.4.19. *Assume R is noetherian, let Γ and Δ be non-zero, finitely generated R-modules, and let $I \leq R$ such that* $\operatorname{supp}(\Delta) = V(I)$. *If $I \subseteq \operatorname{ZD}_R(\Gamma)$, then* $\operatorname{Hom}_R(\Delta, \Gamma) \neq 0$.

Proof. By Corollary II.C.4.2 there exists a prime ideal $\mathfrak{p} \in \operatorname{Ass}(\Gamma)$ such that $I \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is an associated prime, by Remark II.C.2.18 there exists an injective R-module homomorphism $R/\mathfrak{p} \hookrightarrow \Gamma$ and the exactness of localization gives the existence of an injection $(R/\mathfrak{p})_{\mathfrak{p}} \hookrightarrow \Gamma_{\mathfrak{p}}$. Moreover, by Theorem II.C.1.4 we have $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \cong (R/\mathfrak{p})_{\mathfrak{p}}$ and therefore we have an injection $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \hookrightarrow \Gamma_{\mathfrak{p}}$.

Since $I \subseteq \mathfrak{p}$, we know $\mathfrak{p}$ is an element of the variety $V(I) = \operatorname{supp}(\Delta)$. Therefore $\Delta_{\mathfrak{p}} \neq 0$. This is finitely generated over $R_{\mathfrak{p}}$ (since Δ finitely generated over R), so by Nakayama's Lemma we have

$$\frac{\Delta_{\mathfrak{p}}}{\mathfrak{p}_{\mathfrak{p}}\Delta_{\mathfrak{p}}} \neq 0 \tag{II.C.4.19.1}$$ eqn112218a

Similar to the context of Corollary II.C.4.17, the module in Equation (II.C.4.19.1) gives a vector space over $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$. Since one can always surject from a non-zero vector space onto the underlying field, we have the following commutative diagram.

$$\begin{tikzcd} \Delta_{\mathfrak{p}} \arrow[r, two heads] \arrow[drr, dashed, "\therefore\exists"'] & \frac{\Delta_{\mathfrak{p}}}{\mathfrak{p}_{\mathfrak{p}}\Delta_{\mathfrak{p}}} \arrow[r] & \frac{R_{\mathfrak{p}}}{\mathfrak{p}_{\mathfrak{p}}} \arrow[d, hook] \arrow[dl, phantom, "\circlearrowright"] \\ & & \Gamma_{\mathfrak{p}} \end{tikzcd}$$

Since $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ is non-zero, we have exhibited a non-zero R-module homomorphism in $\operatorname{Hom}_{R_{\mathfrak{p}}}(\Delta_{\mathfrak{p}}, \Gamma_{\mathfrak{p}})$ which is therefore non-zero. Therefore $\operatorname{Hom}_R(\Delta, \Gamma) \neq 0$ since $\operatorname{Hom}_{R_{\mathfrak{p}}}(\Delta_{\mathfrak{p}}, \Gamma_{\mathfrak{p}}) \cong \operatorname{Hom}_R(\Delta, \Gamma)$. $\square$

cor101217c

Corollary II.C.4.20. *Let R be noetherian, let M be a non-zero, finitely generated R-module, and let $I \lneq R$ be a proper ideal. If* $\operatorname{depth}(I, M) = 0$, *then* $\operatorname{Hom}_R(R/I, M) \neq 0$.

Proof. Since $\operatorname{depth}(I, M) = 0$, it follows that I is composed entirely of zero-divisors on M. Since $I \neq R$, the module R/I is non-zero and is also finitely generated over R. Therefore by Remark II.C.2.11 we have $\operatorname{supp}(R/I) = V(I)$ and hence $\operatorname{Hom}_R(R/I, M)$ is non-zero by Lemma II.C.4.19. $\square$

lem101217d

Lemma II.C.4.21. *Let R be noetherian and let M and N be R-modules such that M is finitely generated. Then*

$$\operatorname{Ass}(\operatorname{Hom}_R(M, N)) = \operatorname{supp}(M) \cap \operatorname{Ass}(N).$$

Proof. If $M = 0$, then $\operatorname{Hom}_R(M, N) = 0$, so $\operatorname{Ass}(\operatorname{Hom}_R(M, N)) = \operatorname{Ass}(0) = \emptyset$, $\operatorname{supp}(M) = \emptyset$, and therefore $\operatorname{supp}(M) \cap \operatorname{Ass}(N) = \emptyset$. Hence the conclusion holds in this case. The conclusion holds by the same reasoning if $N = 0$, so assume without loss of generality that $M, N \neq 0$.

Let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p} \notin \operatorname{supp}(M)$ and we will argue that $\mathfrak{p} \notin \operatorname{Ass}(\operatorname{Hom}_R(M, N))$. Since $M_{\mathfrak{p}} = 0$ we have $\operatorname{Hom}_R(M, N)_{\mathfrak{p}} \cong \operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ by Proposition II.C.1.8. Therefore

$$\mathfrak{p} \notin \operatorname{supp}(\operatorname{Hom}_R(M, N)) \supseteq \operatorname{Ass}(\operatorname{Hom}_R(M, N)).$$

We've shown $\mathfrak{p}$ not in the support of M implies $\mathfrak{p}$ not in the associated primes of the homomorphism module. This is the contrapositive of

$$\mathfrak{p} \in \operatorname{Ass}(\operatorname{Hom}_R(M, N)) \implies \mathfrak{p} \in \operatorname{supp}(M)$$

so $\operatorname{Ass}(\operatorname{Hom}_R(M,N)) \subseteq \operatorname{supp}(M)$.

Now notice M finitely generated implies there exists some $t \geq 1$ such that we can map surjectively from R^t onto M. The left-exactness of $\operatorname{Hom}_R(-,N)$ along with Hom-cancellation and Example II.C.1.2 gives

$$\operatorname{Hom}_R(M,N) \hookrightarrow \operatorname{Hom}_R(R^t,N) \cong N^t.$$

Therefore by Remark II.C.2.26 and Lemma II.C.2.28 we have

$$\operatorname{Ass}(\operatorname{Hom}_R(M,N)) \subseteq \operatorname{Ass}(N^t) = \operatorname{Ass}(N)$$

which proves $\operatorname{Ass}(\operatorname{Hom}_R(M,N)) \subseteq \operatorname{supp}(M) \cap \operatorname{Ass}(N)$.

For the reverse containment, first consider that in the degenerate case when $\operatorname{supp}(M) \cap \operatorname{Ass}(N)$ is empty, we have nothing to show, since every set contains the empty set. Now assume without loss of generality that $\mathfrak{p}$ lies in the intersection. We claim

$$\operatorname{Hom}_R(M, R/\mathfrak{p}) \neq 0. \tag{II.C.4.21.1}$$ eqn032818a

To put things in the notation of Lemma II.C.4.19, set $I = \operatorname{Ann}_R(M)$, $\Gamma = R/\mathfrak{p}$, and $\Delta = M$. By Remark II.C.2.11 we have

$$\operatorname{supp}(\Delta) = V(\operatorname{Ann}_R(M)) = \operatorname{supp}(M)$$

which implies $I \subseteq \mathfrak{p}$, since $\mathfrak{p} \in \operatorname{supp}(M)$. Therefore $I \cdot R/\mathfrak{p} = 0$, so $I \subseteq \operatorname{ZD}(R/\mathfrak{p})$, and thus Equation (II.C.4.21.1) holds by Lemma II.C.4.19.

Next we claim if $\alpha \in \operatorname{Hom}_R(M, R/\mathfrak{p})$ is non-zero, then $\operatorname{Ann}_R(\alpha) = \mathfrak{p}$. If $x \in \mathfrak{p}$, then $x \cdot R/\mathfrak{p} = 0$ and in particular $x \cdot \alpha(m) = 0$ for all $m \in M$. Equivalently, this means $(x\alpha)(m) = 0$ for all $m \in M$, so $x \in \operatorname{Ann}_R(\alpha)$ and we have shown $\mathfrak{p} \subseteq \operatorname{Ann}_R(\alpha)$. On the other hand, since $\alpha \neq 0$, let $m \in M$ such that $\alpha(m) \neq 0$. Notice also that since $\mathfrak{p}$ is prime and $0 \neq \alpha(m) \in R/\mathfrak{p}$, then $\operatorname{Ann}_R(\alpha(m)) = \mathfrak{p}$. Now for any $y \in \operatorname{Ann}_R(\alpha)$, we have $y \cdot \alpha(m) = (y\alpha)(m) = 0$, implying $y \in \operatorname{Ann}_R(\alpha(m)) = \mathfrak{p}$ and we conclude $\operatorname{Ann}_R(\alpha) = \mathfrak{p}$ by mutual containment.

It now suffices to show $\mathfrak{p} \in \operatorname{Ass}(\operatorname{Hom}_R(M,N))$. By Equation II.C.4.21.1, there exists some element $\alpha \in \operatorname{Hom}_R(M, R/\mathfrak{p}) \setminus \{0\}$ and by our second claim $\operatorname{Ann}_R(\alpha) = \mathfrak{p}$. Define the R-module homomorphism

$$\begin{aligned} \phi : R &\longrightarrow \operatorname{Hom}_R(M, R/\mathfrak{p}) \\ r &\longmapsto r\alpha \end{aligned}$$

and note $\ker(\phi) = \operatorname{Ann}_R(\alpha) = \mathfrak{p}$. Therefore by the First Isomorphism Theorem we have the injective R-module homomorphism

$$\begin{aligned} \overline{\phi} \colon R/\mathfrak{p} &\longrightarrow \operatorname{Hom}_R(M, R/\mathfrak{p}) \\ \overline{r} &\longmapsto r\alpha. \end{aligned}$$

Moreover since $\mathfrak{p} \in \operatorname{Ass}(N)$, there also exists an injective R-module homomorphism $R/\mathfrak{p} \hookrightarrow N$ by Remark II.C.2.18. Since $\operatorname{Hom}_R(M,-)$ is right exact we have the horizontal injection in the following commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_R(M, R/\mathfrak{p}) \arrow[r, hook] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{Hom}_R(M,N) \\ R/\mathfrak{p} \arrow[u, hook] \arrow[ur, hook, "\therefore\exists"'] & \end{tikzcd}$$

and we conclude $\mathfrak{p} \in \operatorname{Ass}(\operatorname{Hom}_R(M,N))$ by Remark II.C.2.18. $\square$

ex101317a EXAMPLE II.C.4.22. Recall the running example we began in Note II.C.2.16. We can see our new lemma in action.

	supp	Ass
$R = \mathbb{K}[x,y]$	$\operatorname{Spec}(R)$	$\{0\}$
$A = R/((x,y)R)^n$	$\{(x,y)R\}$	$\{(x,y)R\}$
$B = R/(x^n, y^m)R$	$\{(x,y)R\}$	$\{(x,y)R\}$
$C = R/(x^2, xy)R$	$V(xR)$	$\{xR, (x,y)R\}$

Notice that

$$\operatorname{Ass}(\operatorname{Hom}_R(A,R)) = \operatorname{supp}(A) \cap \operatorname{Ass}(R) = \{(x,y)\} \cap \{0\} = \emptyset,$$

which implies that $\operatorname{Hom}_R(A,R) = 0$ by Corollary II.C.2.21. Similarly,

$$\operatorname{Ass}(\operatorname{Hom}_R(B,C)) = \operatorname{supp}(B) \cap \operatorname{Ass}(C) = \{(x,y)R\} \cap \{xR, (x,y)R\} = \{(x,y)R\} \neq \emptyset,$$

which also implies that $\mathrm{Hom}_R(B,C) \neq 0$ by Corollary II.C.2.21.

II.C.5. Regular Sequences and Ext

ection062921o

We begin the section with a re-characterization of regular sequences and a characterization of the radical ideal in terms of the intersection of prime ideals (see Lemma II.C.5.7). After stating four facts, we use them to prove Lemma II.C.5.13 before finally achieving our goal of the chapter by proving Theorem II.C.5.16.

disc101317b DISCUSSION II.C.5.1. We have already defined M-regular elements and sequences (Definition II.B.2.1). We can also give a different characterization of M-regular sequences.

Assume $M \neq 0$ is a finitely generated R-module. We claim a sequence $a_1, \ldots, a_n \in R$ is M-regular if and only if $a_1 \notin \mathrm{ZD}_R(M)$, $a_i \notin \mathrm{ZD}_R(M/(a_1, \ldots, a_{i-1})M)$ for all $i = 2, \ldots, n$, and $(a_1, \ldots, a_n)M \neq M$.

PROOF. One implication is trivial from Definition II.B.2.1, so assume $a_1 \notin \mathrm{ZD}_R(M)$, furthermore that $a_i \notin \mathrm{ZD}_R(M/(a_1, \ldots, a_{i-1})M)$ for all $i = 2, \ldots, n$, and $(a_1, \ldots, a_n)M \neq M$. Suppose for the sake of contradiction that $a_1 M = M$. This implies

$$(a_1, \ldots, a_n)M \subseteq M = a_1 M \subseteq (a_1, \ldots, a_n)M.$$

This contradicts the assumption that $(a_1, \ldots, a_n)M \neq M$. Thus $a_1 M \neq M$.

Now suppose $a_i(M/(a_1, \ldots, a_{i-1})M) = M/(a_1, \ldots, a_{i-1})M$ for some $i \geq 2$. Then we have

$$\frac{M}{(a_1, \ldots, a_{i-1})M} = a_i \cdot \frac{M}{(a_1, \ldots, a_{i-1})M} = \frac{(a_1, \ldots, a_i)M}{(a_1, \ldots, a_{i-1})M}$$

which implies

$$(a_1, \ldots, a_n)M \subseteq M = (a_1, \ldots, a_i)M \subseteq (a_1, \ldots, a_n)M.$$

However, this again contradicts the assumption $(a_1, \ldots, a_n)M \neq M$ and we conclude

$$a_i(M/(a_1, \ldots, a_{i-1})M) \neq M/(a_1, \ldots, a_{i-1})M$$

for all $i \in [n]$. Hence, $a_1, \ldots, a_n$ is M-regular. □

disc110218b DISCUSSION II.C.5.2. If $(R, \mathfrak{m})$ is a local ring and $M \neq 0$ is a finitely generated R-module, then Nakayama's Lemma implies $\mathfrak{a}M \neq M$ for all $\mathfrak{a} \subsetneq R$. In particular, a sequence $a_1, \ldots, a_n \in \mathfrak{a} \subsetneq R$ is M-regular if and only if $a_1 \notin \mathrm{ZD}_R(M)$ and $a_i \notin \mathrm{ZD}_R(M/(a_1, \ldots, a_{n-1})M)$ for all $i = 2, \ldots, n$, by Discussion II.C.5.1.

ex101917a EXAMPLE II.C.5.3. Let $\mathbb{K}$ be a field.

(a) Let $R = \mathbb{K}[X_1, \ldots, X_d]$ for some $d \geq 1$. We claim for any $n \leq d$, the sequence $X_1, \ldots, X_n \in R$ is R-regular. In the case when $n = 1$, note that $X_1 \notin \mathrm{ZD}_R(R)$ and $R/X_1R \cong \mathbb{K}[X_2, \ldots, X_n] \neq 0$, so $X_1 R \neq R$ and thus X_1 is R-regular.

Now assume $d \geq n \geq 2$ and the sequence $X_1, \ldots, X_{n-1}$ is R-regular. To show $X_1, \ldots, X_n$ to be regular we need only point out that $X_n \notin \mathrm{ZD}_R(R/(X_1, \ldots, X_{n-1})R)$ and that

$$\frac{R}{(X_1, \ldots, X_n)R} \cong \mathbb{K}[X_{n+1}, \ldots, X_d] \neq 0$$

implying by Nakayama's Lemma that

$$(X_1, \ldots, X_n)R \neq R.$$

Therefore $X_1, \ldots, X_n$ is R-regular by Discussion II.C.5.1.

(b) Consider the ring $\mathbb{Z}$. For any $n \in \mathbb{Z}$ with $n \geq 2$, n is $\mathbb{Z}$-regular, because n is a non-zero, non-unit element of an integral domain. However, we can show that $\mathbb{Z}$ has no regular sequences of length two.

Suppose $m, n \in \mathbb{Z}$ is $\mathbb{Z}$-regular. Then $m \neq 0$ is a non-unit and to build a regular sequence n must be $\mathbb{Z}/m\mathbb{Z}$-regular. Two things can go wrong:

(1) If $(m, n) = 1$, then

$$n \cdot \mathbb{Z}/m\mathbb{Z} = (m, n)\mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/m\mathbb{Z}$$

so n is not $\mathbb{Z}/m\mathbb{Z}$-regular.

(2) If $(m, n) = d \geq 2$, then $d|m$, $d|n$, and we have cases to check. For instance if $m|n$, then $n \cdot \mathbb{Z}/m\mathbb{Z} = 0$ so m, n is not regular. If $m \nmid n$, then $\overline{d} \in \mathbb{Z}/m\mathbb{Z}$ is non-zero and $n \cdot \overline{d} = \overline{0} \in \mathbb{Z}/m\mathbb{Z}$. Therefore n is a zero-divisor and thus m, n fails to be regular.

(c) Any field $\mathbb{K}$ has no regular sequences, because $\mathbb{K} \setminus \{0\} = \mathbb{K}^\times$, so $k\mathbb{K} = \mathbb{K}$ for all $k \neq 0$.

(d) The quotient ring $R = \mathbb{K}[x]/(x^2)$ has no regular sequences, because the non-units of R are of the form $a\overline{x}$ for some $a \in \mathbb{K}$ and $a\overline{x} \cdot \overline{x} = 0$ implies $a\overline{x} \in \mathrm{ZD}_R(R)$.

def102917c DEFINITION II.C.5.4. An M-regular sequence $x_1, \ldots, x_n \in I \leq R$ is a maximal M-regular sequence in I if for any $x_{n+1} \in I$, the sequence $x_1, \ldots, x_{n+1}$ is not M-regular.

rmk102917b REMARK II.C.5.5. If R is noetherian and M an R-module, then for any $I \lneq R$, M has a maximal regular sequence in I. Moreover every M-regular sequence in $I \leq R$ extends to a maximal M-regular sequence in I, which we prove here.

PROOF. Let $\underline{x} = x_1, \ldots, x_n \in I$ be an M-regular sequence and suppose for any $N > n$ and any $x_{n+1}, \ldots, x_N \in I$ such that the sequence $\underline{\underline{x}} = x_1, \ldots, x_N$ is regular, $\underline{\underline{x}}$ is not maximal. Then for any $N > n$ there exists some $x_{N+1} \in I$ such that $\underline{\underline{x}}, x_{N+1}$ is regular. Define $I_k = \langle x_1, \ldots, x_k \rangle$ and consider the chain

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

which we claim is made of proper containments. We need to show $x_{k+1} \notin I_k$ for any k. If we suppose otherwise, then $x_{k+1} \cdot M/I_kM = 0$, which violates the regularity of the sequence in a big way. Therefore we have exhibited a chain

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots \subseteq R$$

that does not stabilize, contradicting the fact that R is noetherian. Therefore $\underline{x}$ must extend to a maximal regular sequence. □

rmk102917d REMARK II.C.5.6. If $(R, \mathfrak{m})$ is a local, noetherian, ring and $M \neq 0$ is a finitely generated R-module, then we have an algorithm for finding maximal M-regular sequences.

Step 1: If $\mathfrak{m} \in \mathrm{Ass}(M)$, then $\mathfrak{m} \subseteq \mathrm{ZD}_R(M)$ by Corollary II.C.4.3, so the empty set is a maximal M-regular sequence and we can therefore stop.

Step 2: Assume $\mathfrak{m} \notin \mathrm{Ass}(M)$, i.e., $\mathfrak{m} \not\subseteq \mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Ass}(M)$. Then by prime avoidance

$$\mathfrak{m} \not\subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p} \quad \Longrightarrow \quad \mathrm{ZD}_R(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p} \subsetneq \mathfrak{m}.$$

Hence there exits an element $x_1 \in \mathfrak{m} \setminus \mathrm{ZD}_R(M)$ (i.e., $x_1 \in \mathfrak{m} \setminus \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$).

Step 3: Repeat Steps 1 and 2 with the module M/x_1M in place of M. If $\mathfrak{m} \in \mathrm{Ass}(M/x_1M)$, then x_1 is a maximal M-regular sequence, so we stop. Otherwise there exists some $x_2 \in \mathfrak{m}$ such that $x_2 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(M/x_1M)$.

Step 4: Repeat Steps 1 and 2 with $M/(x_1, x_2)M$. And so on.

By the proof of Remark II.C.5.5, this process must terminate after finitely many steps.

Associated primes are indispensable for the proof of Theorem II.C.5.16, though their use is a bit hidden. The point of the following lemma is to give in part (b) a context in which they are a bit easier to write down.

lem102917f LEMMA II.C.5.7. *Let R be a non-zero, noetherian, commutative ring with identity and let $I \lneq R$ be a proper ideal.*

lem102917f.a (a) $\mathrm{rad}(I) = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(R/I)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \mathrm{Min}_R(R/I)} \mathfrak{p}$

lem102917f.b (b) *If I is the intersection of a finite number of prime ideals, then*

$$\mathrm{Ass}(R/I) = \mathrm{Min}_R(R/I) = \{\textit{minimum elements in the intersection defining } I\}.$$

lem102917f.c (c) *If I is an intersection of prime ideals, then it is the intersection of a finite number of prime ideals.*

PROOF. (a) This is justified by the following string of containments.

$$\begin{aligned}
\operatorname{rad}(I) &= \bigcap_{\mathfrak{p}\in V(I)} \mathfrak{p} && II.C.2.6\\
&= \bigcap_{\mathfrak{p}\in \operatorname{supp}(R/I)} \mathfrak{p} && II.C.2.11\\
&\subseteq \bigcap_{\mathfrak{p}\in \operatorname{Ass}(R/I)} \mathfrak{p} && II.C.2.19\\
&\subseteq \bigcap_{\mathfrak{p}\in \operatorname{Min}_R(R/I)} \mathfrak{p} && II.C.3.16\\
&\subseteq \bigcap_{\mathfrak{p}\in \operatorname{supp}(R/I)} \mathfrak{p} && II.C.3.13
\end{aligned}$$

Hence we have equality at every step.

(b) Assume $I = \bigcap_{i=1}^n \mathfrak{p}_i$ and re-order if necessary to assume $\mathfrak{p}_1, \ldots, \mathfrak{p}_j$ are the minimal elements in $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ with respect to containment. Therefore $I = \bigcap_{i=1}^j \mathfrak{p}_i$ and we first claim $\mathfrak{p}_1, \ldots, \mathfrak{p}_j \in \operatorname{Min}_R(R/I)$. By 4 in Remark II.C.2.11 and Proposition II.C.3.13 we have

$$\operatorname{Min}_R(R/I) = \operatorname{Min}(\operatorname{supp}(R/I)) = \operatorname{Min}(V(I))$$

(see also Definition II.C.3.16) so it suffices to show $\mathfrak{p}_i$ is minimal in $V(I)$ for each $i = [j]$. We know $\mathfrak{p}_k \in V(I)$ for any $k \in [j]$ since each contains the intersection $\cap_{i=1}^j \mathfrak{p}_i$, which is precisely I. To show minimality suppose $\mathfrak{p} \in V(I)$ such that $\mathfrak{p} \subseteq \mathfrak{p}_i$. Observe the following with the product of prime ideals.

$$\mathfrak{p}_1 \cdots \mathfrak{p}_j \subseteq \bigcap_{k=1}^{j} \mathfrak{p}_k = I \subseteq \mathfrak{p} \subseteq \mathfrak{p}_i$$

The fact that $\mathfrak{p}$ is prime implies there exists some index $l \in [j]$ such that $\mathfrak{p}_l \subseteq \mathfrak{p} \subseteq \mathfrak{p}_i$, but since $\mathfrak{p}_i$ is minimal among the $\mathfrak{p}_k$'s, we know $\mathfrak{p}_i \subseteq \mathfrak{p}_l$. Therefore we have

$$\mathfrak{p}_l \subseteq \mathfrak{p} \subseteq \mathfrak{p}_i \subseteq \mathfrak{p}_l$$

forcing equality at every step and hence $\mathfrak{p} = \mathfrak{p}_i$. By this argument and by Definition II.C.3.16, we have shown

$$\begin{aligned}
\{\text{minimal elements in the intersection defining } I\} &= \{\mathfrak{p}_1, \ldots, \mathfrak{p}_j\}\\
&\subseteq \operatorname{Min}(V(I))\\
&= \operatorname{Min}_R(R/I)\\
&\subseteq \operatorname{Ass}(R/I).
\end{aligned}$$

So to complete the proof of this part it suffices to show $\operatorname{Ass}(R/I) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_j\}$. Let $\mathfrak{p} \in \operatorname{Ass}(R/I)$ and by definition of being an associated prime there exists some $\overline{0} \neq \overline{x} \in R/I$ such that $\mathfrak{p} = \operatorname{Ann}_R(\overline{x})$. By definition of what it means to be zero in the quotient module R/I we have

$$\mathfrak{p}x \subseteq I = \bigcap_{i=1}^{j} \mathfrak{p}_j.$$

Since $\overline{x}$ is non-zero, $x \notin I$, so there exists some $k \in [j]$ such that $x \notin \mathfrak{p}_k$. Rewriting our last line we have

$$\mathfrak{p}x \subseteq I = \bigcap_{i=1}^{j} \mathfrak{p}_j \subseteq \mathfrak{p}_k$$

where the fact that $\mathfrak{p}_k$ is prime implies $\mathfrak{p} \subseteq \mathfrak{p}_k$ (if x isn't in $\mathfrak{p}_k$, then $\mathfrak{p}$ must be). Since we have already shown the minimal elements of the intersection defining I ($\mathfrak{p}_k$ in particular) are also in $\operatorname{Min}_R(R/I)$, this implies $\mathfrak{p} = \mathfrak{p}_k$ and completes the proof of part (b).

(c) If I is the intersection of prime ideals, say $\mathfrak{p}_\lambda$ for $\lambda \in \Lambda$, then we have

$$\bigcap_{\lambda\in\Lambda} \mathfrak{p}_\lambda = I \subseteq \operatorname{rad}(I) = \bigcap_{\mathfrak{p}\in\operatorname{Min}_R(R/I)} \mathfrak{p} \subseteq \bigcap_{\lambda\in\Lambda} \mathfrak{p}_\lambda$$

and equality at every step follows. Since $\operatorname{Min}_R(R/I)$ is finite by Corollary II.C.3.4, part (c) holds. $\square$

ex102917e EXAMPLE II.C.5.8. Let $\mathbb{K}$ be a field, let $R = \mathbb{K}[x,y]_{(x,y)}$, which is a local ring with unique maximal ideal $\mathfrak{m} = (x,y)R$, and define the R-module $M = R/(xy)R$. We will use the steps in Remark II.C.5.6 to find a maximal M-regular sequence. We know $(x)R = \operatorname{Ann}_R(\overline{y})$ and $(y)R = \operatorname{Ann}_R(\overline{x})$, for $\overline{x}, \overline{y} \in M$, so $(x)R$, $(y)R \in \operatorname{Ass}(M)$. It is straight forward to show $(xy)R = (x)R \cap (y)R$, so we actually have $\operatorname{Ass}(M) = \{(x)R, (y)R\}$ by Lemma II.C.5.7(b). Since $\mathfrak{m} \notin \operatorname{Ass}(M)$, we need to find an element

$$a_1 \in (x,y)R \setminus [(x)R \cup (y)R].$$

That is, we want to find $a_1 = fx + gy$ such that $x \nmid a_1$ and $y \nmid a_1$, for some $f, g \in R$. In particular we can take $a_1 = x - y$.

For Step 3, we first repeat Step 1 with the module M/a_1M, so we need to determine if $\mathfrak{m} \in \operatorname{Ass}(M/a_1M)$. Observe

$$\frac{M}{a_1M} = \frac{R/(xy)R}{(x-y)\cdot R/(xy)R} \cong \frac{R}{(x-y,xy)R} \cong \frac{R/(x-y)R}{(xy)\cdot R/(x-y)R}$$

where

$$\frac{R}{(x-y)R} = \frac{\mathbb{K}[x,y]_{(x,y)}}{(x-y)\mathbb{K}[x,y]_{(x,y)}} \cong \mathbb{K}[x]_{(x)}.$$

Colloquially, the last isomorphism above holds because setting $x - y = 0$ is the same as setting $x = y$. This gives

$$\frac{M}{a_1M} \cong \frac{\mathbb{K}[x]_{(x)}}{x^2\cdot\mathbb{K}[x]_{(x)}}. \qquad \text{(II.C.5.8.1)}$$ eqn041018a

We will now argue

$$\operatorname{Ann}_R\left(\overline{x} \in \frac{M}{a_1M}\right) = (x,y)R$$

and will thereby have showed $\mathfrak{m} \in \operatorname{Ass}(M/a_1M)$. Since $x = y$ we have

$$y \cdot \overline{x} = x \cdot \overline{x} = \overline{x^2} = 0 \in M/a_1M$$

by Equation (II.C.5.8.1) and therefore $(x,y)R \subseteq \operatorname{Ann}_R(\overline{x})$. Moreover we know $\overline{x} \neq \overline{0} \in M/a_1M$ by Equation (II.C.5.8.1), implying $\operatorname{Ann}_R(\overline{x}) \neq R$ and in fact the maximality of $(x,y)R$ implies $\mathfrak{m} = (x,y)R = \operatorname{Ann}_R(\overline{x})$. Hence $\mathfrak{m} \in \operatorname{Ass}(M/a_1M)$ and we can stop. That is, $x - y$ is a maximal M-regular sequence in M of length 1.

We shall assume the following fact without proof (for the moment) and use it to prove the subsequent fact. (See Discussion II.E.2.3 for existence of the induced maps.)

fact110117a FACT II.C.5.9. *Let $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ R-module homomorphisms. For any $i \geq 0$, there exist R-module homomorphisms*

$$\operatorname{Ext}^i_R(A,g) : \operatorname{Ext}^i_R(A,B) \longrightarrow \operatorname{Ext}^i_R(A,B')$$

$$\operatorname{Ext}^i_R(f,B) : \operatorname{Ext}^i_R(A',B) \longrightarrow \operatorname{Ext}^i_R(A,B).$$

If $f' : A' \longrightarrow A''$ and $g' : B' \longrightarrow B''$ are also two R-module homomorphisms, then the following diagrams commute.

$$\begin{tikzcd} \operatorname{Ext}^i_R(A,B) \arrow[r, "\operatorname{Ext}^i_R(A,g)"] \arrow[dr, "\operatorname{Ext}^i_R(A,g'\circ g)"'] & \operatorname{Ext}^i_R(A,B') \arrow[d, "\operatorname{Ext}^i_R(A,g')"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \operatorname{Ext}^i_R(A,B'') \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Ext}^i_R(A'',B) \arrow[r, "\operatorname{Ext}^i_R(f',B)"] \arrow[dr, "\operatorname{Ext}^i_R(f'\circ f,B)"'] & \operatorname{Ext}^i_R(A',B) \arrow[d, "\operatorname{Ext}^i_R(f,B)"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \operatorname{Ext}^i_R(A,B) \end{tikzcd}$$

Colloquially, we are saying that $\operatorname{Ext}^i_R(A,-)$ and $\operatorname{Ext}^i_R(-,B)$ each respect compositions.

Next, we use this fact to establish the following.

`fact110117b`

FACT II.C.5.10. *If A, A', B, and B' are all R-modules, then*

$$\mathrm{Ext}_R^i(A, 0_{B'}^B) = 0 = \mathrm{Ext}_R^i(0_{A'}^A, B)$$

where $0_{B'}^B$ denotes the zero map from B into B' and $0_{A'}^A$ denotes the zero map from A into A'.

PROOF. As silly as it looks to write down, we begin with the following commutative diagram.

$$\begin{tikzcd} A \arrow[r, "0_0^A"] \arrow[dr, "0_{A'}^A"'] \arrow[dr, phantom, bend left=20, "\circlearrowleft"] & 0 \arrow[d, "0_{A'}^0"] \\ & A' \end{tikzcd}$$

From Fact II.C.5.9, the following diagram also commutes.

$$\begin{tikzcd} \mathrm{Ext}_R^i(A', B) \arrow[r, "\mathrm{Ext}_R^i(0_{A'}^0, B)"] \arrow[dr, "\mathrm{Ext}_R^i(0_{A'}^A, B)"'] \arrow[dr, phantom, bend left=20, "\circlearrowleft"] & \mathrm{Ext}_R^i(0, B) \arrow[d, "\mathrm{Ext}_R^i(0_0^A, B)"] \\ & \mathrm{Ext}_R^i(A, B) \end{tikzcd}$$

Since $\mathrm{Ext}_R^i(0, B) = 0$ by Proposition II.B.1.12, the commutivity of the diagram forces $\mathrm{Ext}_R^i(0_{A'}^A, B) = 0$ as well.

In a similar fashion, we have two more commutative diagrams.

$$\begin{tikzcd} B \arrow[r, "0_0^B"] \arrow[dr, "0_{B'}^B"'] \arrow[dr, phantom, bend left=20, "\circlearrowleft"] & 0 \arrow[d, "0_{B'}^0"] \\ & B' \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \mathrm{Ext}_R^i(A, B) \arrow[r, "\mathrm{Ext}_R^i(A,0)"] \arrow[dr, "\mathrm{Ext}_R^i(A, 0_{B'}^B)"'] \arrow[dr, phantom, bend left=20, "\circlearrowleft"] & \mathrm{Ext}_R^i(A, 0) \arrow[d, "\mathrm{Ext}_R^i(A,0)"] \\ & \mathrm{Ext}_R^i(A, B') \end{tikzcd}$$

From the second diagram we conclude $\mathrm{Ext}_R^i(A, 0_{B'}^B) = 0$ as desired, using similar reasoning as before. $\square$

Similarly, we assume the next fact without proof and use it to prove the subsequent one. See Discussion II.E.2.4 for some justification of Fact II.C.5.11.

`fact110117c`

FACT II.C.5.11. *Let $r \in R$ and let A, B be R-modules. The multiplication map*

$$\mu_r^B \colon B \longrightarrow B$$

$$b \longmapsto rb$$

is a well-defined R-module homomorphism (cf. Notation II.B.1.8). The induced maps

$$\mathrm{Ext}_R^i(A, \mu_r^B) : \mathrm{Ext}_R^i(A, B) \xrightarrow{r\cdot} \mathrm{Ext}_R^i(A, B)$$

$$\mathrm{Ext}_R^i(\mu_r^A, B) : \mathrm{Ext}_R^i(A, B) \xrightarrow{r\cdot} \mathrm{Ext}_R^i(A, B)$$

from Fact II.C.5.9 are the multiplication maps $\mu_r^{\mathrm{Ext}_R^i(A,B)}$ and $\mu_r^{\mathrm{Ext}_R^i(A,B)}$. That is, the map on Ext induced by a multiplication map is itself a multiplication map.

`fact110117d`

FACT II.C.5.12. *Given the two identity maps $\mathrm{id}_A : A \longrightarrow A$ and $\mathrm{id}_B : B \longrightarrow B$, we have*

$$\mathrm{Ext}_R^i(\mathrm{id}_A, B) = \mathrm{id}_{\mathrm{Ext}_R^i(A,B)} = \mathrm{Ext}_R^i(A, \mathrm{id}_B).$$

PROOF. This is essentially a corollary of Fact II.C.5.11.

$$\mathrm{Ext}_R^i(\mathrm{id}_A, B) = \mathrm{Ext}_R^i(\mu_1^A, B) = \mu_1^{\mathrm{Ext}_R^i(A,B)} = \mathrm{id}_{\mathrm{Ext}_R^i(A,B)}$$

$$\mathrm{Ext}_R^i(A, \mathrm{id}_B) = \mathrm{Ext}_R^i(A, \mu_1^B) = \mu_1^{\mathrm{Ext}_R^i(A,B)} = \mathrm{id}_{\mathrm{Ext}_R^i(A,B)}$$

$\square$

We will see all four of the above facts again in II.E.2, where we will justify Facts II.C.5.9 and II.C.5.11, and where we will give alternative proofs of Facts II.C.5.10 and II.C.5.12. For now, we use them to prove a lemma.

lem110217a

LEMMA II.C.5.13. *If M and N are R-modules, then for all $i \geq 0$ we have*

$$(\mathrm{Ann}_R(M) \cup \mathrm{Ann}_R(N)) \subseteq \mathrm{Ann}_R\left(\mathrm{Ext}^i_R(M,N)\right).$$

That is, if $x \in R$ such that $xM = 0$ or $xN = 0$, then $x \cdot \mathrm{Ext}^i_R(M,N) = 0$.

PROOF. Let $x \in R$ and assume $xM = 0$. Therefore $\mu^M_x = 0^M_M$ and applying Facts II.C.5.10 and II.C.5.11 we have

$$\mu_x^{\mathrm{Ext}^i_R(M,N)} = \mathrm{Ext}^i_R(\mu^M_x, N) = \mathrm{Ext}^i_R(0^M_M, N) = 0.$$

The proof is done similarly if $y \in R$ such that $yN = 0$. $\square$

ex110217b

EXAMPLE II.C.5.14. If $m, n \in \mathbb{Z}$ (not both 0) and $g = \gcd(m,n)$, then we first claim that

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/g\mathbb{Z}.$$

We take a projective resolution of $\mathbb{Z}/m\mathbb{Z}$, truncate it, and apply the functor $\mathrm{Hom}_R(-, \mathbb{Z}/n\mathbb{Z})$.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m\cdot} \mathbb{Z} \xrightarrow{\tau} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m\cdot} \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{(m\cdot)^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \longrightarrow 0$$

By Hom-cancellation the final sequence is isomorphic to

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{m\cdot} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \longrightarrow \cdots.$$

Thus we compute

$$\begin{aligned}\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) &= \frac{\mathrm{Ker}\ \ \mathbb{Z}/n\mathbb{Z} \longrightarrow 0}{\mathrm{Im}\ \ \mathbb{Z}/n\mathbb{Z} \xrightarrow{m\cdot} \mathbb{Z}/m\mathbb{Z}} \\ &= \frac{\mathbb{Z}/n\mathbb{Z}}{m \cdot (\mathbb{Z}/n\mathbb{Z})} = \frac{\mathbb{Z}/n\mathbb{Z}}{(m,n)\mathbb{Z}/n\mathbb{Z}} = \frac{\mathbb{Z}/n\mathbb{Z}}{g\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/g\mathbb{Z}.\end{aligned}$$

Note since $\mathbb{Z}/g\mathbb{Z}$ is annihilated by both m and n, it is also annihilated by $m\mathbb{Z} \cup n\mathbb{Z}$, so as separate verification of the conclusion of Lemma II.C.5.13 in this special case, we note

$$\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}) \cup \mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) = m\mathbb{Z} \cup n\mathbb{Z} \subseteq \mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/g\mathbb{Z}) \subseteq \mathrm{Ann}_{\mathbb{Z}}(\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})).$$

rmk110217c

REMARK II.C.5.15. In general, if M and N are R-modules, then we at least have

$$\mathrm{Ann}_R(M) \cup \mathrm{Ann}_R(N) \subseteq \mathrm{Ann}_R(\mathrm{Ext}^i_R(M,N))$$

but we cannot assume equality here. However, since this is true for all $i \in \mathbb{Z}$, we can strengthen the conclusion of the lemma to write

$$\mathrm{Ann}_R(M) \cup \mathrm{Ann}_R(N) \subseteq \bigcap_{i=0}^{\infty} \mathrm{Ann}_R(\mathrm{Ext}^i_R(M,N)).$$

For instance, in Example II.C.5.14 we have

$$\mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$$

and therefore

$$\mathrm{Ann}_{\mathbb{Z}}\left(\mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})\right) = \mathbb{Z}.$$

Yet notice

$$\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}) + \mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) = m\mathbb{Z} + n\mathbb{Z} = g\mathbb{Z}.$$

So if m and n are relatively prime, then $g = 1$ and we achieve the equality

$$\mathrm{Ann}_R(\mathbb{Z}/m\mathbb{Z}) \cup \mathrm{Ann}_R(\mathbb{Z}/n\mathbb{Z}) = g\mathbb{Z} = \mathbb{Z} = \mathrm{Ann}_R(\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})).$$

In general, however, $g\mathbb{Z} \neq \mathbb{Z}$. Thus equality in Lemma II.C.5.13 is achievable, but does not hold in general.

Here we finally achieve the goal of the chapter by characterizing depth in terms of vanishing Ext modules.

DepthExt **Theorem II.C.5.16.** *Assume R is noetherian, let $I \leq R$ be an ideal, and assume M is a finitely generated R-module such that $IM \neq M$. Let $n \geq 0$. The following are equivalent.*

DepthExt.i (i) $\operatorname{Ext}^i_R(N, M) = 0$ *for all $i < n$ and all finitely generated R-modules N satisfying* $\operatorname{supp}(N) \subseteq V(I)$.

DepthExt.ii (ii) $\operatorname{Ext}^i_R(R/I, M) = 0$ *for all $i < n$.*

DepthExt.iii (iii) $\operatorname{Ext}^i_R(N, M) = 0$ *for all $i < n$ and for* <u>*some*</u> *finitely generated R-module N satisfying* $\operatorname{supp}(N) = V(I)$.

DepthExt.iv (iv) *Every M-regular sequence in I of length no greater than n can be extended to an M-regular sequence in I of length equal to n.*

DepthExt.v (v) *M has a regular sequence in I of length n.*

Proof. Since $\operatorname{supp}(R/I) = V(I)$ by Remark II.C.2.11, (i) implies (ii) and (ii) implies (iii) are already done (consider $N = R/I$). To show (iii) implies (iv), assume N is a finitely generated R-module with $\operatorname{supp}(N) = V(I)$ such that $\operatorname{Ext}^i_R(N, M) = 0$ for all $i < n$. Since the case $n = 0$ is trivial we assume $n \geq 1$. Therefore by assumption $\operatorname{Hom}_R(N, M) \cong \operatorname{Ext}^0_R(N, M) = 0$ and it follows from Lemma II.C.4.19 that $I \not\subseteq \operatorname{ZD}_R(M)$, i.e., there exists $a_1 \notin \operatorname{ZD}_R(M) \cap I$. Now we induct on n. If $n = 1$, then we're done since if we start with a sequence of length 0, we can extend to a_1 and if we start with a sequence of length 1, we needn't extend at all.

Assume $n \geq 2$ and the result holds for all finitely generated R-modules M' satisfying $\operatorname{Ext}^i_R(N, M') = 0$ for all $i < n - 1$. Let $a_1, \ldots, a_k \in I$ be an M-regular sequence with $k \leq n$. If $k = n$, then we're done. If $k = 0$, then we already know there exists an M-regular element $a_1 \in I$ from which we would start our sequence, so assume $1 \leq k \leq n - 1$. The sequence

$$0 \longrightarrow M \xrightarrow{a_1 \cdot} M \longrightarrow M/a_1M \longrightarrow 0$$

is exact and yields the long exact sequence (II.C.5.16.1) from which it follows

$$\operatorname{Ext}^i_R(N, M/a_1M) = 0$$

for all $i < n - 1$ (i.e., whenever $i + 1 < n$). We assumed $a_1, \ldots, a_k$ is M-regular, so $a_2, \ldots, a_k$ is a M/a_1M-regular sequence of length $k - 1 < n - 1$. Therefore under our induction hypothesis we may extend to a M/a_1M-regular sequence $a_2, \ldots, a_k, \ldots, a_n$ of length $n - 1$. Hence to conclude $a_1, \ldots, a_n \in I$ is M-regular of length n and thus complete the proof of this implication, it suffices to show $I \cdot M/a_1M \neq M/a_1M$. Indeed since $IM \neq M$ we have

$$I \cdot \frac{M}{a_1M} = \frac{IM}{a_1M} \neq \frac{M}{a_1M}.$$

$$\begin{array}{l} 0 \longrightarrow \operatorname{Hom}_R(N, M) \xrightarrow{a_1 \cdot} \operatorname{Hom}_R(N, M) \longrightarrow \operatorname{Hom}_R(N, M/a_1M) \longrightarrow \\ \longrightarrow \operatorname{Ext}^1_R(N, M) \xrightarrow{a_1 \cdot} \cdots \end{array} \qquad \text{(II.C.5.16.1)}$$ eqn041018c

$$\begin{array}{l} \cdots \xrightarrow{a_1 \cdot} \underbrace{\operatorname{Ext}^i_R(N, M)}_{=0} \longrightarrow \underbrace{\operatorname{Ext}^i_R(N, M/a_1M)}_{\therefore = 0} \longrightarrow \\ \longrightarrow \underbrace{\operatorname{Ext}^{i+1}_R(N, M)}_{=0} \xrightarrow{a_1 \cdot} \cdots \end{array}$$

$$\begin{array}{l} \cdots \xrightarrow{a_1 \cdot} \operatorname{Ext}^{n-1}_R(N, M) \longrightarrow \operatorname{Ext}^{n-1}_R(N, M/a_1M) \longrightarrow \\ \longrightarrow \operatorname{Ext}^n_R(N, M) \longrightarrow \cdots \end{array}$$

To show (iv) implies (v), we simply point out that if we assume (iv), then the empty sequence can be extended to an M-regular sequence of length n.

Proving (v) implies (i) is again by induction. Assume M has a regular sequence $a_1, \ldots, a_n \in I$ and let N be a finitely generated R-module such that $\operatorname{supp}(N) \subseteq V(I)$. We want to show $\operatorname{Ext}^i_R(N, M) = 0$ for all $i < n$. In the case when $n = 1$, our sequence is merely the element $a_1 \in I$. By Remark II.C.2.11 and our assumption we have

$$V(\operatorname{Ann}_R(N)) = \operatorname{supp}(N) \subseteq V(I)$$

so by Lemma II.C.2.7 we have $a_1^t \in I^t \subseteq \operatorname{Ann}_R(N)$ (i.e., $a_1^t N = 0$) for all t sufficiently large. By construction the sequence

$$0 \longrightarrow M \xrightarrow{a_1 \cdot} M \longrightarrow M/a_1M \longrightarrow 0$$

is exact. By the left exactness of $\operatorname{Hom}_R(N, -)$, this implies the sequence

$$0 \longrightarrow \operatorname{Hom}_R(N, M) \xhookrightarrow{a_1 \cdot} \operatorname{Hom}_R(N, M)$$

is also exact, as is the sequence

$$0 \longrightarrow \operatorname{Hom}_R(N, M) \xhookrightarrow{a_1^t \cdot} \operatorname{Hom}_R(N, M)$$

for all $t \geq 1$, since the composition of injective functions is still injective. Since $a_1^t \cdot$ is the (injective) zero map for all sufficiently large t by Lemma II.C.5.13, we conclude $\operatorname{Hom}_R(N, M) = 0$ as desired and we let this serve as the base case.

Assume $n \geq 2$ and the result holds for all $i < n - 1$. We want to show $\operatorname{Ext}^{n-1}_R(N, M) = 0$. Let $a_1, \ldots, a_{n-1} \in I$ be an M-regular sequence guaranteed by (v). By our induction hypothesis $\operatorname{Ext}^i_R(N, M) = 0$ for all $i < n-1$ and we also know I contains an M/a_1M-regular sequence of length $n-1$, namely $a_2, \ldots, a_n$. Our induction hypothesis again implies

$$\operatorname{Ext}^i_R(N, M/a_1M) = 0$$

for all $i < n - 1$. In particular, $\operatorname{Ext}^{n-2}_R(N, M/a_1M) = 0$ and therefore we have

$$\cdots \longrightarrow 0 \longrightarrow \operatorname{Ext}^{n-1}_R(N, M) \xhookrightarrow{a_1 \cdot} \operatorname{Ext}^{n-1}_R(N, M) \longrightarrow \cdots$$

by the exactness of the sequence (see (II.C.5.16.1)). Since the composition of injective functions yields an injective function, the sequence

$$0 \longrightarrow \operatorname{Ext}^{n-1}_R(N, M) \xhookrightarrow{a_1^t \cdot} \operatorname{Ext}^{n-1}_R(N, M)$$

is also exact for any $t > 0$. As in the base case we can take $t \gg 0$ such that $a_1^t \cdot N = 0$ and by Lemma II.C.5.13, $a_1^t \cdot \operatorname{Ext}^{n-1}_R(N, M) = 0$ as well. Thus we have an injective zero map, implying the domain must be zero, i.e., $\operatorname{Ext}^{n-1}_R(N, M) = 0$ as desired. □

Exercises

exer170919a

Exercise II.C.5.17. Let R be a non-zero commutative ring with identity. Let M, M', N, N' be R-modules, and let $U \subseteq R$ be a multiplicatively closed subset. Let $f \colon M \to M'$ and $g \colon N \to N'$ be R-modules. Recall the $U^{-1}R$-module homomorphism $\Theta_{U,M,N} \colon U^{-1}\operatorname{Hom}_R(M, N) \to \operatorname{Hom}_{U^{-1}R}(U^{-1}M, U^{-1}N)$ from Proposition II.C.1.8. Prove that the following diagrams commute.

$$\begin{tikzcd}
U^{-1}\operatorname{Hom}_R(M, N) \arrow[r, "U^{-1}\operatorname{Hom}_R(M,g)"] \arrow[d, "\Theta_{U,M,N}"'] & U^{-1}\operatorname{Hom}_R(M, N') \arrow[d, "\Theta_{U,M,N'}"'] \\
\operatorname{Hom}_{U^{-1}R}(U^{-1}M, U^{-1}N) \arrow[r, "\operatorname{Hom}_{U^{-1}R} U^{-1}M{,}U^{-1}g)"] & \operatorname{Hom}_{U^{-1}R}(U^{-1}M, U^{-1}N')
\end{tikzcd}$$

$$\begin{tikzcd}
U^{-1}\operatorname{Hom}_R(M', N) \arrow[r, "U^{-1}\operatorname{Hom}_R(f,N)"] \arrow[d, "\Theta_{U,M',N}"'] & U^{-1}\operatorname{Hom}_R(M, N) \arrow[d, "\Theta_{U,M,N}"'] \\
\operatorname{Hom}_{U^{-1}R}(U^{-1}M', U^{-1}N) \arrow[r, "\operatorname{Hom}_{U^{-1}R}(U^{-1}f{,}U^{-1}N)"] & \operatorname{Hom}_{U^{-1}R}(U^{-1}M, U^{-1}N)
\end{tikzcd}$$

exer170919b EXERCISE II.C.5.18. Let A be an integral domain, and consider the polynomial ring $R = k[X,Y]$ and the ideal $I = (X^2, XY)R$.

exer170919b1 (a) Prove that $\mathrm{rad}(I) = XR$. Conclude that $\mathrm{Supp}_R(R/I) = V(XR)$.

exer170919b2 (b) Prove that the prime ideals XR and $(X,Y)R$ are in $\mathrm{Ass}\,R(R/I)$.

exer170919c EXERCISE II.C.5.19. Let R be a commutative ring, and let $r \in R$ be an R-regular element, that is, a non-unit that is not a zero-divisor on R. Prove that $\mathrm{Ass}\,R(R/r^nR) = \mathrm{Ass}\,R(R/rR)$ for all $n \geq 1$. [Hint: Verify that the following sequence is exact:

$$0 \to R/rR \xrightarrow{r^{n-1}} R/r^nR \to R/r^{n-1}R \to 0$$

and use induction on n.]

exer171004a EXERCISE II.C.5.20. Let R be a commutative ring, and let M and N be R-modules.

exer171004a3 (a) Prove that if M and N are non-zero free R-modules, then $M \otimes_R N \neq 0$. [Hint: There are surjections $M \to R$ and $N \to R$. Combine these and use the right-exactness of tensor product to obtain a surjection $M \otimes_R N \to R \otimes_R R \cong R \neq 0$.] In particular, if k is a field and V and W are non-zero vector spaces over k, then $V \otimes_k W \neq 0$.

exer171004a1 (b) [Reading exercise] Let $I \subseteq R$ be an ideal. Recall that there is an R/I-module isomorphism $(R/I) \otimes_R M \xrightarrow{\cong} M/IM$ such that $(r + I) \otimes m \mapsto (rm) + IM$. Thus, we have isomorphisms

$$\begin{aligned}
(M/IM) \otimes_{R/I} (N/IN) &\cong ((R/I) \otimes_R M) \otimes_{R/I} ((R/I) \otimes_R N) & \\
&\cong (((R/I) \otimes_R M) \otimes_{R/I} (R/I)) \otimes_R N & \text{associativity} \\
&\cong ((R/I) \otimes_R M) \otimes_R N & \text{cancellation} \\
&\cong (R/I) \otimes_R (M \otimes_R N) & \text{associativity} \\
&\cong (M \otimes_R N)/I(M \otimes_R N) &
\end{aligned}$$

exer171004a2 (c) Let $U \subseteq R$ be a multiplicatively closed subset. Recall that there is a $U^{-1}R$-module isomorphism $(U^{-1}R) \otimes_R M \xrightarrow{\cong} U^{-1}M$ such that $(r/u) \otimes m \mapsto (rm)/u$. Using this, argue as in part (b) to show that there is a $U^{-1}R$-module isomorphism $(U^{-1}M) \otimes_{U^{-1}R} (U^{-1}N) \cong U^{-1}(M \otimes_R N)$. In particular, for each prime ideal $P \in \mathrm{Spec}(R)$, there exists an R_P-module isomorphism $M_P \otimes_{R_P} N_P \cong (M \otimes_R N)_P$.

exer030206 EXERCISE II.C.5.21. Let R be a commutative local ring. Let M and N be non-zero finitely generated R-modules. Prove that, $M \otimes_R N \neq 0$. [Hint: Exercise II.C.5.20(a)–(b) and Nakayama's Lemma.]

exer030297' EXERCISE II.C.5.22. Let R be a commutative ring, and let M and N be finitely generated R-modules.

exer030297'z (a) Prove that for each multiplicatively closed subset $U \subseteq R$, the modules $U^{-1}M$ and $U^{-1}N$ are finitely generated over $U^{-1}R$. In particular, for each prime ideal $P \in \mathrm{Spec}(R)$, the modules M_P and N_P are finitely generated over R_P.

(b) Prove that $\mathrm{Supp}_R(M \otimes_R N) = \mathrm{Supp}_R(M) \cap \mathrm{Supp}_R(N)$. [Hint: Part (a) and Exercises II.C.5.20(c) and II.C.5.21.]

exer030505 EXERCISE II.C.5.23. Let R be a non-zero commutative noetherian ring with identity, and let M be a finitely generated R-module. Let $I = (a_1, \ldots, a_n)R$ be an ideal of R such that $IM \neq M$. Prove that if $a_1, \ldots, a_n$ is M-regular, then $a_1, \ldots, a_n$ is a maximal M-regular sequence in I.

exer030505z EXERCISE II.C.5.24. Let R be a non-zero commutative ring with identity, and let X be an R-module.

(a) Prove that $X = 0$ if and only if the sequence $0 \to X \to 0$ is exact.

(b) Let $0 \to X' \to X \to X'' \to 0$ be an exact sequence of R-modules. Prove that $X = 0$ if and only if $X' = 0 = X''$.

exer030506 EXERCISE II.C.5.25 (Depth Lemma). Let R be a non-zero commutative noetherian ring with identity, and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated R-modules. Let $I \subseteq R$ be an ideal, and prove the following inequalities:

$$\begin{aligned}
\mathrm{depth}_R(I, M) &\geq \inf\{\mathrm{depth}_R(I, M'), \mathrm{depth}_R(I, M'')\} \\
\mathrm{depth}_R(I, M') &\geq \inf\{\mathrm{depth}_R(I, M), \mathrm{depth}_R(I, M'') + 1\} \\
\mathrm{depth}_R(I, M'') &\geq \inf\{\mathrm{depth}_R(I, M') - 1, \mathrm{depth}_R(I, M)\}.
\end{aligned}$$

(Hint: Long exact sequence in Ext with Exercise II.C.5.24.)

CHAPTER II.D

Homology

apter062921c

The ultimate goal of the next three chapters is to prove the well-definedness of Ext and the existence of long exact sequences as described in Theorem II.B.1.1. We begin by introducing some of the basics of homology.

II.D.1. Chain Complexes and Homology

ction062921p

In this section we define chain complexes and homology modules. We present some of their basic characteristics and show in Theorem II.D.1.9 that they play well with Hom modules. We also see in Example II.D.1.7 is that Ext modules are specific homology modules.

def110417a

Definition II.D.1.1. A chain complex of R-modules and R-module homomorphisms, also known as an R-complex, is a sequence

$$M_\bullet = \qquad \cdots \xrightarrow{\partial^M_{i+2}} M_{i+1} \xrightarrow{\partial^M_{i+1}} M_i \xrightarrow{\partial^M_i} M_{i-1} \xrightarrow{\partial^M_{i-1}} \cdots$$

of R-modules and R-module homomorphisms such that $\partial^M_n \circ \partial^M_{n+1} = 0$ for all $n \in \mathbb{Z}$. The map ∂^M_i is the i^{th} differential of the complex and

$$H_i(M_\bullet) = \frac{\operatorname{Ker} \partial^M_i}{\operatorname{Im} \partial^M_{i+1}}$$

is the i^{th} homology module of $M_\bullet$. Note this quotient makes sense to write down since $\operatorname{Im} \partial^M_{i+1} \subseteq \operatorname{Ker} \partial^M_i$ if and only if $\partial^M_i \circ \partial^M_{i+1} = 0$.

rmk110417b

Remark II.D.1.2. An R-complex $M_\bullet$ is exact if and only if $H_i(M_\bullet) = 0$ for all $i \in \mathbb{Z}$. Colloquially, $H_i(M_\bullet)$ measures how far $M_\bullet$ is from being exact at the i^{th} position, M_i.

ex110417c

Example II.D.1.3. Let M be an R-module and let $P^+_\bullet$ be an augmented projective resolution of M.

$$P^+_\bullet = \qquad \cdots \xrightarrow{\partial^P_3} P_2 \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \xrightarrow{\tau} M \longrightarrow 0$$

$$P_\bullet = \qquad \cdots \xrightarrow{\partial^P_3} P_2 \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \longrightarrow 0$$

Since $P^+_\bullet$ is exact, it is also an R-complex with $H_i(P^+_\bullet) = 0$ for all $i \in \mathbb{Z}$. On the other hand, while $P_\bullet$ is not exact, it is still an R-complex since $\partial^P_i \circ \partial^P_{i+1} = 0$ for all positive i and for all negative i we have $\partial^P_i \circ \partial^P_{i+1} = 0 \circ \partial^P_{i+1} = 0$. Therefore $H_i(P_\bullet) = 0$ for all $i \neq 0$ and leveraging the exactness of $P^+_\bullet$ we have

$$H_0(P_\bullet) = \frac{\operatorname{Ker} P_0 \longrightarrow 0}{\operatorname{Im} \partial^P_1} = \frac{P_0}{\operatorname{Ker} \tau} \cong \operatorname{Im} \tau = M.$$

Conversely, if the sequence

$$Q_\bullet = \qquad \cdots \xrightarrow{\partial^Q_3} Q_2 \xrightarrow{\partial^Q_2} Q_1 \xrightarrow{\partial^Q_1} Q_0 \longrightarrow 0$$

is an R-complex such that each Q_i is projective and $H_i(Q_\bullet) = 0$ for all $i \neq 0$, then $Q_\bullet$ is a projective resolution of the homology module $H_0(Q_\bullet)$.

Proof. Note that

$$H_0(Q_\bullet) = \frac{\operatorname{Ker} Q_0 \longrightarrow 0}{\operatorname{Im} \partial^Q_1} = \frac{Q_0}{\operatorname{Im} \partial^Q_1}.$$

Denote the natural epimorphism

$$\pi : Q_0 \longrightarrow \frac{Q_0}{\operatorname{Im} \partial^Q_1}$$

which once adjoined to $Q_\bullet$ gives the exact sequence

$$Q_\bullet^+ = \qquad \cdots \xrightarrow{\partial_3^Q} Q_2 \xrightarrow{\partial_2^Q} Q_1 \xrightarrow{\partial_1^Q} Q_0 \xrightarrow{\pi} \frac{Q_0}{\operatorname{Im}\partial_1^Q} \longrightarrow 0$$

completing the proof. □

def110417d

Definition II.D.1.4. Let $M_\bullet$ be an R-complex and let N be an R-module. We define lower star and upper star on R-complexes as we did with exact sequences. Define

$$M_{\bullet *} = \operatorname{Hom}_R(N, M_\bullet) = \qquad \cdots \xrightarrow{(\partial_{i+2}^M)_*} \underset{=(M_{i+1})_*}{\operatorname{Hom}_R(N, M_{i+1})} \xrightarrow[\partial_{i+1}^M \circ (-)]{(\partial_{i+1}^M)_*} \underset{=(M_i)_*}{\operatorname{Hom}_R(N, M_i)} \xrightarrow{(\partial_i^M)_*} \cdots$$

$$M_\bullet^* = \operatorname{Hom}_R(M_\bullet, N) = \qquad \cdots \xrightarrow{(\partial_{i-1}^M)^*} \underset{=(M_{i-1})^*}{\operatorname{Hom}_R(M_{i-1}, N)} \xrightarrow[(-)\circ\partial_i^M]{(\partial_i^M)^*} \underset{=(M_i)^*}{\operatorname{Hom}_R(M_i, N)} \xrightarrow{(\partial_{i+1}^M)^*} \cdots$$

where

$$\left(\partial_i^M\right)_* = \operatorname{Hom}_R(N, \partial_i^M) \qquad\qquad \left(\partial_i^M\right)^* = \operatorname{Hom}_R(\partial_i^M, N).$$

prop110417e

Proposition II.D.1.5. *Both $M_{\bullet *}$ and $M_\bullet^*$ are R-complexes.*

Proof. The argument is written succinctly as follows.

$$(\partial_i^M)^* \circ (\partial_{i-1}^M)^* = (\partial_{i-1}^M \circ \partial_i^M)^* = 0^* = 0$$
$$(\partial_i^M)_* \circ (\partial_{i+1}^M)_* = (\partial_{i+1}^M \circ \partial_i^M)_* = 0_* = 0$$

□

notn110417f

Notation II.D.1.6. We add some more short-hand.

$$(M_*)_i = M_{i*} \qquad\qquad \partial_i^{M_*} = (\partial_i^M)_*$$
$$(M^*)_j = (M_{-j})^* \qquad\qquad \partial_j^{M^*} = (\partial_{-j+1}^M)^*$$

ex110417g

Example II.D.1.7. Let M and N be R-modules and let $P_\bullet$ be a projective resolution of M. Observe that the indices for the projective resolution are decreasing, whereas after applying $\operatorname{Hom}_R(P_\bullet, N)$ to get $P_\bullet^*$ the indices are increasing.

$$P_\bullet = \qquad \cdots \xrightarrow{\partial_3^P} P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \longrightarrow 0$$
$$P_\bullet^* = \qquad 0 \longrightarrow P_0^* \xrightarrow{(\partial_1^P)^*} P_1^* \xrightarrow{(\partial_2^P)^*} P_2^* \xrightarrow{(\partial_3^P)^*} \cdots$$

From Definition II.D.1.1, when calculating homology modules we require decreasing indices, so in the case when we want $H_j(P_\bullet)$, the indices line up nicely.

$$H_j(P_\bullet) = \frac{\operatorname{Ker}\partial_j^P}{\operatorname{Im}\partial_{j+1}^P}$$

To put $P_\bullet^*$ in the form of having decreasing indices, we can define $k = -i$ and set $M_k = P_{-k}^*$ in order to write the following.

$$\begin{array}{ccccccccc} P_\bullet^* = & 0 & \longrightarrow & P_0^* & \xrightarrow{(\partial_1^P)^*} & P_1^* & \xrightarrow{(\partial_2^P)^*} & P_2^* & \xrightarrow{(\partial_3^P)^*} \cdots \\ & \| & & \| & & \| & & \| & \\ & M_1 & & M_0 & & M_{-1} & & M_{-2} & \end{array}$$

Hence building the i^{th} homology module from these M_i amounts to finding $\operatorname{Ext}_R^{-i}(M, N)$, i.e.,

$$H_i(P_\bullet^*) = H_i(\operatorname{Hom}_R(P_\bullet, N)) = \operatorname{Ext}_R^{-i}(M, N).$$

We can align the homology modules with their respective projective modules in $P_\bullet^*$ to make this even clearer.

$$\begin{array}{cccccccccccc} P_\bullet^* = & 0 & \longrightarrow & P_0^* & \longrightarrow & P_1^* & \longrightarrow & P_2^* & \longrightarrow \cdots \longrightarrow & P_i^* & \longrightarrow \cdots \\ & & & H_0(P_\bullet^*) & & H_{-1}(P_\bullet^*) & & H_{-2}(P_\bullet^*) & & H_{-i}(P_\bullet^*) & \end{array}$$

lem042818a

Lemma II.D.1.8. *Assume we have the following commutative diagram of R-modules and R-module homomorphisms where the upper and lower horizontal sequences are exact, as are the diagonal sequences.*

$$\begin{tikzcd}
0 & & & & & & & & 0 \arrow[dl] & & \\
& A \arrow[ul] & & & & 0 & & H \arrow[ll] \arrow[dl, "\nu"'] & & I \arrow[ll, "\mu"'] \arrow[ul] \arrow[dl, phantom, "\circlearrowright"] & \\
& & B \arrow[ul, "\alpha"] & & & & C \arrow[llll, "\beta"'] \arrow[dl, "\sigma"] \arrow[dlll, phantom, "\circlearrowright"] & & & & D \arrow[llll, "\gamma"] \arrow[ul, "\theta"'] \\
& 0 & & E \arrow[ll] \arrow[ul, "\varepsilon"] & & F \arrow[ll, "\zeta"] \arrow[dl] & & G \arrow[ll, "\rho"] & & 0 \arrow[ll] & \\
& & & & 0 \arrow[ul] & & & & & &
\end{tikzcd}$$

If $\beta \circ \gamma = 0$, then $G \cong \dfrac{\ker(\beta)}{\operatorname{Im} \gamma}$.

Proof. First, we claim that σ with domain restricted to $\ker(\beta)$ surjects onto $\ker(\zeta)$. We denote σ with such a restriction as $\hat{\sigma} \colon \ker(\beta) \longrightarrow F$ and we want to show by mutual containment that

$$\operatorname{Im} \hat{\sigma} = \ker(\zeta). \tag{II.D.1.8.1}$$ eqn042818b

Let $c \in \ker(\beta)$ and since $\beta(c) = 0$, by the commutivity of the diagram we also have $\varepsilon(\zeta(\hat{\sigma}(c))) = 0$. Since ε is injective we have $\zeta(\hat{\sigma}(c)) = 0$ and therefore $\hat{\sigma}(c) \in \ker(\zeta)$. This takes care of the forward containment of Equation (II.D.1.8.1).

Let $f \in \ker(\zeta)$ and it follows $\varepsilon(\zeta(f)) = \varepsilon(0) = 0 \in B$. Since σ is surjective, there exists some $c \in C$ such that $\sigma(c) = f$. By the commutivity of the diagram $\beta(c) = 0$, so $\hat{\sigma}$ is defined at c and moreover $f = \widehat{\sigma(c)} \in \operatorname{Im} \hat{\sigma}$. This handles the reverse containment and thus Equation (II.D.1.8.1) is proven.

For our second claim, we will show

$$\ker(\hat{\sigma}) = \operatorname{Im} \gamma \tag{II.D.1.8.2}$$ eqn042818c

again by mutual containment. Let $c \in \ker(\hat{\sigma}) \subseteq \ker(\sigma) = \operatorname{Im} \nu$ and let $h \in H$ such that $\nu(h) = c$. Since μ and θ are each surjective, choose $i \in I$ such that $\mu(i) = h$ and choose $d \in D$ such that $\theta(d) = i$. By the commutivity of the diagram we have $\gamma(d) = \nu(\mu(\theta(d))) = c$ and therefore $c \in \operatorname{Im} \gamma$, justifying the forward containment of Equation (II.D.1.8.2).

Now let $d \in D$ and set $c = \gamma(d)$. By the commutivity of the diagram $\nu(\mu(\theta(d))) = c$, so $c \in \operatorname{Im} \nu = \ker(\sigma)$ and $\sigma(c) = 0$. Moreover, $\beta(c) = \varepsilon(\zeta(\sigma(c))) = 0$ by the commutivity of the diagram as well. Thus $c \in \ker(\beta)$ so $\hat{\sigma}$ is defined at c with $\hat{\sigma}(c) = 0$ and therefore $c \in \ker(\hat{\sigma})$. This completes our justification of Equation (II.D.1.8.2).

By the First Isomorphism Theorem, and Equations (II.D.1.8.1) and (II.D.1.8.2), we have

$$\frac{\ker(\beta)}{\operatorname{Im} \gamma} = \frac{\ker(\beta)}{\ker(\hat{\sigma})} \cong \operatorname{Im} \hat{\sigma} = \ker(\zeta).$$

By the exactness of the bottom row we also have

$$\frac{\ker(\beta)}{\operatorname{Im} \gamma} \cong \operatorname{Im} \rho$$

and since ρ is injective the apply the First Isomorphism Theorem again to conclude

$$\frac{\ker(\beta)}{\operatorname{Im} \gamma} \cong G.$$

$\square$

thm110417h

Theorem II.D.1.9. *Let $M_\bullet$ be an R-complex and let N be an R-module.*

thm110417h.a (a) *If N is projective, then $H_i(\operatorname{Hom}_R(N, M_\bullet)) \cong \operatorname{Hom}_R(N, H_i(M_\bullet))$.*

thm110417h.b (b) *If N is injective, then $H_i(\operatorname{Hom}_R(M_\bullet, N)) \cong \operatorname{Hom}_R(H_{-i}(M_\bullet), N)$.*

We also write each of these colloquially.

(a) *'Homming' with a projective module in the first slot commutes with taking homology.*
(b) *'Homming' with an injective module in the second slot commutes with taking homology, as long as we are careful about indices.*

Proof. (a) Consider the following diagram.

(II.D.1.9.1) eqn210725a

$$
\begin{tikzcd}
0 \arrow[d] & & & & & & & 0 \arrow[dr] & & \\
\operatorname{Ker}\partial^{i+1}_M \arrow[dr, "\subseteq", "\epsilon_{i+1}"'] & & & & & 0 \arrow[r] & \operatorname{Im}\partial^M_i \arrow[rr, "\subseteq", "\alpha_i"'] \arrow[ur] \arrow[drr, phantom, "\circlearrowleft"] & & \operatorname{Ker}\partial^M_{i-1} \arrow[dr, "\epsilon_{i-1}", "\subseteq"'] & \\
 & M_{i+1} \arrow[rrrr, "\partial^M_{i+1}"] \arrow[dr, "\delta^M_{i+1}"'] \arrow[drrr, phantom, "\circlearrowleft"] & & & & M_i \arrow[rrrr, "\partial^M_i"'] \arrow[ur, "\delta^M_i"] & & & & M_{i-1} \\
0 \arrow[rr] & & \operatorname{Im}\partial^M_{i+1} \arrow[rr, "\alpha_{i+1}", "\subseteq"'] \arrow[dr] & & \operatorname{Ker}\partial^M_i \arrow[ur, "\subseteq", "\epsilon_i"'] \arrow[rr, "\tau_i"] & & H_i(M_\bullet) \arrow[r] & 0 & & \\
 & & & 0 \arrow[ur] & & & & & &
\end{tikzcd}
$$

where δ^M_i is the map induced by ∂^M_i and τ_i is the natural surjection. The diagonals are all exact, as is the lower horizontal sequence. Moreover, $\operatorname{Hom}_R(N,-)=(-)_*$ is exact, because N is projective. Therefore we have the commutative diagram given below.

(II.D.1.9.2) eqn210725b

$$
\begin{tikzcd}
0 \arrow[d] & & & & & & & 0 \arrow[dr] & \\
\operatorname{Ker}\partial^{i+1}_M{}_* \arrow[dr, "(\epsilon_{i+1})_*"'] & & & & & & \operatorname{Im}\partial^M_i{}_* \arrow[rr, "(\alpha_i)_*"'] \arrow[ur] \arrow[drr, phantom, "\circlearrowleft"] & & \operatorname{Ker}\partial^M_{i-1}{}_* \arrow[d, "(\epsilon_{i-1})_*"] \\
 & (M_{i+1})_* \arrow[rrrr, "(\partial^M_{i+1})_*"] \arrow[dr, "(\delta^M_{i+1})_*"'] \arrow[drrr, phantom, "\circlearrowleft"] & & & & (M_i)_* \arrow[rrr, "(\partial^M_i)_*"'] \arrow[ur, "(\delta^M_i)_*"] & & & (M_{i-1})_* \\
0 \arrow[rr] & & \operatorname{Im}\partial^M_{i+1}{}_* \arrow[rr, "(\alpha_{i+1})_*"] \arrow[dr] & & \operatorname{Ker}\partial^M_i{}_* \arrow[ur, "(\epsilon_i)_*"'] \arrow[rr, "(\tau_i)_*"] & & H_i(M_\bullet)_* \arrow[r] & 0 & \\
 & & & 0 \arrow[ur] & & & & &
\end{tikzcd}
$$

Since the exactness of the lower horizontal sequence is preserved, we claim the following is an isomorphism of short exact sequences, where ε and π are the natural injection and surjection, respectively.

(II.D.1.9.3) eqn091218b

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{Im}\partial^M_{i+1}{}_* \arrow[r, "(\alpha_{i+1})_*"] \arrow[d, dashed, "\beta", "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{Ker}\partial^M_i{}_* \arrow[r, "(\tau)_*"] \arrow[d, dashed, "\gamma", "\cong"'] \arrow[dr, phantom, "\text{s.t. } \circlearrowleft"] & H_i(M_\bullet)_* \arrow[r] \arrow[d, dashed, "\therefore \exists\theta", "\cong"'] & 0 \\
0 \arrow[r] & \operatorname{Im}(\partial^M_{i+1})_* \arrow[r, "\varepsilon"'] & \operatorname{Ker}(\partial^M_i)_* \arrow[r, "\pi"'] & H_i(M_{\bullet *}) \arrow[r] & 0
\end{tikzcd}
$$

The map γ is induced by $(\epsilon_i)_*$ and is well-defined, because of diagram (II.D.1.9.2) above. It is also a monomorphism, because $(\epsilon_i)_*$ is a monomorphism. Moreover it is onto, which one can see from a standard diagram chase. The map β is induced by $(\epsilon_i)_* \circ (\alpha_{i+1})_*$ and is an isomorphism for similar reasons as γ. Also, it is straightforward to show that the left-hand square of (II.D.1.9.3) commutes. It follows that there exists some θ making the right-hand square commute and θ is an isomorphism by the Short-Five Lemma. Since $(M_\bullet)_* = \operatorname{Hom}_R(N, M_\bullet)$, we have

$$H_i(\operatorname{Hom}_R(N, M_\bullet)) = H_i(M_{\bullet *}) \cong H_i(M_\bullet)_* = \operatorname{Hom}_R(N, H_i(M_\bullet)).$$

(b) Let $i \in \mathbb{Z}$ be given and we apply $\operatorname{Hom}_R(-, N)$ to commutative diagram (II.D.1.9.1), which preserves the exactness and flips everything.

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & & & 0 & \\
 & & \operatorname{Ker}\partial_{i-1}^{M}{}^{*} \arrow[ur] \arrow[r, two heads, "(\alpha_{i-1})^*"'] & \operatorname{Im}\partial_i^{M*} \arrow[r] \arrow[d, "(\delta_i^M)^*"] & 0 & & \operatorname{Ker}\partial_{i+1}^{M}{}^{*} \arrow[u] & \\
\cdots \arrow[r] & (M_{i-1})^* \arrow[ur, "(\varepsilon_{i-1})^*"] \arrow[rr, "(\partial_i^M)^*"'] & & (M_i)^* \arrow[rr, "(\partial_{i+1}^M)^*"] \arrow[d, "(\varepsilon_i)^*"'] & & (M_{i+1})^* \arrow[ur, "(\varepsilon_{i+1})^*"] \arrow[r] & \cdots & \\
 & 0 \arrow[r] & (H_i(M_\bullet))^* \arrow[r, "(\tau_i)^*"'] & \operatorname{Ker}\partial_i^{M*} \arrow[r, two heads, "(\alpha_{i+1})^*"'] \arrow[d] & \operatorname{Im}\partial_{i+1}^{M}{}^{*} \arrow[ur, "(\delta_{i+1}^M)^*"'] \arrow[r] & 0 & & \\
 & & & 0 \arrow[ur] & & & &
\end{tikzcd}
$$

Therefore we have

$$
\begin{aligned}
(H_i(M_\bullet))^* = \operatorname{Hom}_R(H_i(M_\bullet), N) &\cong \frac{\operatorname{Ker}(\partial_{i+1}^M)^*}{\operatorname{Im}(\partial_i^M)^*} && \text{Lemma } II.D.1.8 \\
&= \frac{\operatorname{Ker}\partial_{-i}^{M^*}}{\operatorname{Im}\partial_{-i+1}^{M^*}} && \text{Notation } II.D.1.6 \\
&= H_{-i}(M_\bullet^*) \\
&= H_{-i}(\operatorname{Hom}_R(M_\bullet, N)) && \text{Definition } II.D.1.4
\end{aligned}
$$

completing the proof of part (b). □

section062921q

II.D.2. Ext Modules

There are two main propositions in this section. We state formally in Proposition II.D.2.3 why one says Ext detects whether a given module is projective. In Proposition II.D.2.8 we give conditions under which we know Ext modules are finitely generated.

disc110417i

Discussion II.D.2.1. We have already put a fair amount of time into describing Ext_R^i, so in this section we add only a few more things. Let M be an R-module and $P_\bullet$ a projective resolution of M. That is

$$P_\bullet = \quad \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \longrightarrow 0$$

$$P_\bullet^+ = \quad \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \longrightarrow 0$$

where all P_i are projective and $P_\bullet^+$ is exact. We saw in Example II.D.1.7 that

$$\operatorname{Ext}_R^i(M, N) = H_{-i}(\operatorname{Hom}_R(P_\bullet, N))$$

for all $i \in \mathbb{Z}$, where

$$
\begin{array}{rcccccccc}
\operatorname{Hom}_R(P_\bullet, N) = P_\bullet^* = \cdots & \longrightarrow & 0 & \longrightarrow & P_0^* & \longrightarrow & P_1^* & \longrightarrow & P_2^* \longrightarrow \cdots \\
\text{position:} & & -1 & & 0 & & 1 & & 2
\end{array}
$$

and the $-i^{th}$ module is built from the i^{th} position of $P_\bullet^*$, i.e.,

$$(P_\bullet^*)_i = P_{-i}^*.$$

The following theorem was previously stated as Fact II.B.1.13 and will be proven in Theorem II.F.5.2.

thm110417j

Theorem II.D.2.2. *If $P_\bullet$ and $Q_\bullet$ are two projective resolutions of M, then*

$$H_{-i}(\operatorname{Hom}_R(P_\bullet, N)) \cong H_{-i}(\operatorname{Hom}_R(Q_\bullet, N))$$

for all $i \in \mathbb{Z}$. The slogan here is '$\operatorname{Ext}_R^i(M, N)$ is independent of our choice of projective resolution.'

prop110517a **Proposition II.D.2.3.** *Let M and N be R-modules.*

prop110517a.a (a) *If M is projective, then $\operatorname{Ext}^i_R(M,N)=0$ for all $i>0$.*

prop110517a.b (b) *If N is injective, then $\operatorname{Ext}^i_R(M,N)=0$ for all $i>0$.*

Proof. (a) Since M is projective, the augmented projective resolution and projective resolution are as follows.

$$P_\bullet^+ = \quad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M \xrightarrow{\mathrm{id}} M \longrightarrow 0$$

$$P_\bullet = \quad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M \longrightarrow 0$$

In practice, we stop writing terms for the projective resolution, but in reality we may write more completely

$$\begin{array}{rccccccccc} P_\bullet = & \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow \cdots. \\ \text{position:} & & 2 & & 1 & & 0 & & -1 & \end{array}$$

Since $\operatorname{Hom}_R(-,N)$ is arrow-reversing, this gives

$$\begin{array}{rccccccccc} P_\bullet^* = & \cdots \longrightarrow & 0 & \longrightarrow & M^* & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots. \\ \text{position:} & & -1 & & 0 & & 1 & & 2 & \end{array}$$

Therefore for all $i>0$ we have as desired, namely

$$\operatorname{Ext}^i_R(M,N) = H_{-i}(0) = 0.$$

(b) Let $P_\bullet$ be a projective resolution of M and we have the following.

$$\begin{aligned} \operatorname{Ext}^i_R(M,N) = H_{-i}(\operatorname{Hom}_R(P_\bullet,N)) &\overset{II.D.1.9}{\cong} \operatorname{Hom}_R(H_i(P_\bullet),N) \\ &\overset{II.D.1.3}{\cong} \operatorname{Hom}_R(0,N), \text{ for all } i>0 \\ &\overset{II.B.1.12}{=} 0 \end{aligned}$$

□

The next several results set up the proof of Proposition II.D.2.8. We begin with a definition.

def082118b **Definition II.D.2.4.** Let R be a non-zero commutative ring with identity and let M be an R module. M is a noetherian module if it satisfies the following equivalent conditions.

(i) Every submodule of M is finitely generated.
(ii) M satisfies the ascending chain condition for submodules.
(iii) Every nonempty set S of R-submodules of M has a maximal element. That is, there exists an element $N\in S$ such that for all $N'\in S$, if $N\subseteq N'$, then $N=N'$.

note082118a **Note II.D.2.5.** R is a noetherian ring if and only if R is noetherian as an R-module.

prop082018a **Proposition II.D.2.6.** *Let R be a non-zero commutative ring with identity and consider an exact sequence*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

of R-modules and R-module homomorphisms. In this setting M is a noetherian module over R if and only if M' and M'' are noetherian over R.

Proof. First assume that M is a noetherian R-module and let $N'\subset M'$ be a submodule. $f(N')\subset M$ is a finitely generated submodule, since M is noetherian. Therefore since f is injective, N' is finitely generated by the First Isomorphism Theorem. Since N' was arbitrarily taken, M' is noetherian.

Now consider an chain $N_1''\subset N_2''\subset\ldots$ of submodules of M''. Then there is a chain $N_1\subset N_2\subset\ldots$ of submodules of M with $N_i=g^{-1}(N_i'')$. Since M is noetherian, there is some $k\in\mathbb{N}$ such that $N_j=N_k$ for all $j\geq k$. Since g is surjective, $g(N_j)=N_j''$ for all j and we have

$$N_j'' = g(N_j) = g(N_k) = N_k''$$

for all $j\geq k$. Hence the chain $N_1''\subset N_2''\subset\ldots$ stabilizes and we conclude M'' is noetherian.

Second, we instead assume both M' and M'' are noetherian. We want to show an arbitrary submodule $N\subseteq M$ is finitely generated. Since $g(N)\subseteq M''$ as a submodule and M'' is noetherian, it is finitely generated.

We let $n_1, \ldots, n_p \in N$ such that $g(N) = \langle g(n_1), \ldots, g(n_p)\rangle$. Similarly, $f^{-1}(N) \subseteq M'$ as a submodule and we let $m'_{p+1}, \ldots, m'_q \in M'$ such that $f^{-1}(N) = \langle m'_{p+1}, \ldots, m'_q\rangle$. We claim $N = \langle n_1, \ldots, n_q\rangle$ where $n_i = f(m'_i)$ for every $i = p+1, \ldots, q$. Since one containment is by choice of n_i, it suffices to show $N \subseteq \langle n_1, \ldots, n_q\rangle$. Let $n \in N$ be given. Then there exist $r_1, \ldots, r_p \in R$ such that

$$g(n) = \sum_{i=1}^{p} r_i g(n_i) = g\left(\sum_{i=1}^{p} r_i n_i\right).$$

Since g is an R-module homomorphism it follows that

$$n - \sum_{i=1}^{p} r_i n_i \in \ker(g) = \operatorname{Im} f.$$

Moreover, since $n - \sum_{i=1}^{p} r_i n_i \in N$ as well, we have an element $x \in f^{-1}(N) = \langle m'_{p+1}, \ldots, m'_q\rangle$ such that $f(x) = n - \sum_{i=1}^{p} r_i n_i$. So there are $r_{p+1}, \ldots, r_q \in R$ such that $x = \sum_{i=p+1}^{q} r_i m'_i$. It follows that

$$n - \sum_{i=1}^{p} r_i n_i = f\left(\sum_{i=p+1}^{q} r_i m'_i\right) = \sum_{i=p+1}^{q} r_i f(m'_i) = \sum_{i=p+1}^{q} r_i n_i$$

and therefore $n = \sum_{i=1}^{q} r_i n_i$. $\square$

prop082018b

Proposition II.D.2.7. *Let R be a non-zero commutative ring with identity and let M be an R-module.*

rop082018b.a (a) *The following are equivalent.*
p082018b.a.i (i) *M is noetherian over R.*
082018b.a.ii (ii) *M^n is noetherian over R for all $n \in \mathbb{N}$.*
82018b.a.iii (iii) *M^n is noetherian over R for some $n \in \mathbb{N}$.*
rop082018b.b (b) *The following are equivalent.*
p082018b.b.i (i) *R is a noetherian ring.*
082018b.b.ii (ii) *R^n is noetherian over R for all $n \in \mathbb{N}$.*
82018b.b.iii (iii) *R^n is noetherian over R for some $n \in \mathbb{N}$.*
rop082018b.c (c) *In the case when R is a noetherian ring, the following are equivalent.*
p082018b.c.i (i) *M is finitely generated over R.*
082018b.c.ii (ii) *M is noetherian over R.*
82018b.c.iii (iii) *M has a degree-wise finite free resolution, that is, there is an exact sequence*

$$\cdots \longrightarrow R^{\beta_2} \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow M \longrightarrow 0$$

with each $\beta_i \in \mathbb{N}_0$.

Proof. (a) Consider the following short exact sequence for any $n > 1$.

$$0 \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M \longrightarrow 0$$

$$\begin{pmatrix} m_1 \\ \vdots \\ m_{n-1} \end{pmatrix} \longmapsto \begin{pmatrix} m_1 \\ \vdots \\ m_{n-1} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \longmapsto m_n$$

If M is noetherian, then we apply Proposition II.D.2.6 to the short exact sequence above to conclude by induction on n that M^n is noetherian for any $n \geq 1$. Therefore (ai) implies (aii). The implication (aii) implies (aiii) is trivial. If we assume M^n is noetherian for some $n \in \mathbb{N}$, then applying the same exercise to the same short exact sequence we conclude M is noetherian, so (aiii) implies (ai).
(b) By Note II.D.2.5, this is a corollary of part (a).

(c) M is noetherian over R if and only if every submodule of M is finitely generated over R. In particular, M is finitely generated since it is a submodule of itself, so (cii) implies (ci). From the exact sequence

$$\cdots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \xrightarrow{\tau} M \longrightarrow 0$$

we can build a short exact sequence

$$0 \longrightarrow \ker(\tau) \xrightarrow{\subset} R^{\beta_0} \xrightarrow{\tau} M \longrightarrow 0\,.$$

Since R^{β_0} is noetherian by part (b), M is noetherian as well by Proposition II.D.2.6. Thus (ciii) implies (cii).

Now we assume M is finitely generated over R and we want to build a degree-wise finite free resolution of M. Let $m_1, \ldots, m_{\beta_0} \in M$ be a set of generators for M and define the surjection

$$\begin{aligned} \tau_0 : R^{\beta_0} &\longrightarrow M \\ \textstyle\sum_{i=1}^{\beta_0} r_i e_i &\longmapsto \textstyle\sum_{i=1}^{\beta_0} r_i m_i \end{aligned}$$

where $e_1, \ldots, e_{\beta_0}$ is the standard basis of the free module R^{β_0}, which is noetherian by part (b). Therefore the submodule $\ker(\tau_0) \subset R^{\beta_0}$ is finitely generated and we write $\ker(\tau_0) = (f_1, \ldots, f_{\beta_1})R^{\beta_0}$ for some $f_1, \ldots, f_{\beta_1} \in R^{\beta_0}$. We may then approximate $\ker(\tau_0)$ by the free module R^{β_1} using the surjection

$$\begin{aligned} \tau_1 : R^{\beta_1} &\longrightarrow \ker(\tau_0) \\ \textstyle\sum_{i=1}^{\beta_1} r_i e'_i &\longmapsto \textstyle\sum_{i=1}^{\beta_1} r_i f_i \end{aligned}$$

where $e'_1, \ldots, e'_{\beta_1}$ is the standard basis. Since R^{β_1} is again noetherian, $\ker(\tau_1)$ is again finitely generated and this process may continue.

For any $j \geq 1$ define $\partial_j = \tau_j \circ I_{j-1}$ where for any $k \geq 0$ we define I_k to be the containment map from $\ker(\tau_k)$ into R^{β_k}. Then we can build the following commutative diagram where the row is exact, because the diagonals are exact by construction.

$$\begin{tikzcd}
 & & & 0 \arrow[dr] & & 0 & & & \\
 & & & & \ker(\tau_0) \arrow[ur] \arrow[dr, "I_0"] \arrow[d, phantom, "\circlearrowleft"] & & & & \\
\cdots \arrow[r] & R^{\beta_2} \arrow[rr, "\partial_2"] \arrow[dr, "\tau_2"'] & & R^{\beta_1} \arrow[rr, "\partial_1"'] \arrow[ur, "\tau_1"] & & R^{\beta_0} \arrow[rr, "\tau_0"] \arrow[dr, "\tau_0"'] & & M \arrow[r] & 0 \\
 & & \ker(\tau_1) \arrow[ur, "I_1"'] \arrow[dr] \arrow[u, phantom, "\circlearrowleft"] & & & & M \arrow[ur] \arrow[dr] \arrow[u, phantom, "\circlearrowleft"] & & \\
 & 0 \arrow[ur] & & 0 & & & & 0 &
\end{tikzcd}$$

$\square$

prop110517b

Proposition II.D.2.8. *Let R be noetherian. If M and N are finitely generated R-modules, then $\operatorname{Ext}^i_R(M,N)$ is finitely generated for all $i \in \mathbb{Z}$.*

Proof. Since R is noetherian and M is finitely generated, by Proposition II.D.2.7 M has a projective resolution of the form

$$P_\bullet = \qquad \cdots \longrightarrow R^{\beta_2} \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow 0$$

where $\beta_i \in \mathbb{N}_0$ for all $i \in \mathbb{N}$. Therefore from Fact II.C.1.1 and from Hom-cancellation we have

$$\operatorname{Hom}_R(R^{\beta_i}, N) \cong \operatorname{Hom}_R(R, N)^{\beta_i} \cong N^{\beta_i}.$$

Since N is finitely generated and R is noetherian, N^{β_i} is also finitely generated and noetherian. Therefore the submodule $\operatorname{Ker} \partial_{-i}^{P_\bullet^*}$ is finitely generated and hence so is the following.

$$\frac{\operatorname{Ker} \partial_{-i}^{P_\bullet^*}}{\operatorname{Im} \partial_{-i+1}^{P_\bullet^*}} = H_{-i}(P_\bullet^*) = \operatorname{Ext}_R^i(M, N)$$

□

Exercises

exer020212

Exercise II.D.2.9. Let R be a commutative noetherian ring with identity, and let $M_\bullet$ be an R-complex. Prove that, if $i \in \mathbb{Z}$ is such that M_i is finitely generated over R, then $\mathrm{H}_i(M_\bullet)$ is finitely generated over R.

exer020153s

Exercise II.D.2.10. Prove Theorem II.D.1.9(b): Let R be a commutative ring with identity, and let $M_\bullet$ be an R-complex. If N is an injective R-module, then for each $i \in \mathbb{Z}$ there is an R-module isomorphism $\mathrm{H}_i(\operatorname{Hom}_R(M_\bullet, N)) \cong \operatorname{Hom}_R(\mathrm{H}_{-i}(M_\bullet), N)$.

exer020502

Exercise II.D.2.11. Let G be a finitely generated $\mathbb{Z}$-module, and let H be a $\mathbb{Z}$-module. Prove that $\operatorname{Ext}_{\mathbb{Z}}^i(G, H) = 0$ for all $i > 1$.

CHAPTER II.E

Chain Maps and Induced Maps on Ext

apter062921d

In this chapter we continue to build the technology needed to prove that Ext is well-defined and to establish long exact sequences.

II.E.1. Chain Maps

ection062921r

In this section we introduce chain maps and show in Proposition II.E.1.3 that these induce maps on homology modules. We will use this fact heavily when we prove the existence of the mother of all long exact sequences (Theorem II.F.1.2).

def110517c DEFINITION II.E.1.1. Let $M_\bullet$ and $N_\bullet$ be R-complexes. A chain map from $M_\bullet$ into $N_\bullet$ is a sequence of R-module homomorphisms

$$F_\bullet = \{F_i : M_i \longrightarrow N_i \mid i \in \mathbb{Z}\}$$

such that the following diagram commutes.

$$\begin{tikzcd}
M_\bullet = \arrow[d, "F_\bullet"'] & \cdots \arrow[r] & M_i \arrow[r, "\partial_i^M"] \arrow[d, "F_i"'] \arrow[dr, phantom, "\circlearrowleft"] & M_{i-1} \arrow[r] \arrow[d, "F_{i-1}"] & \cdots \\
N_\bullet = & \cdots \arrow[r] & N_i \arrow[r, "\partial_i^N"'] & N_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

We denote such a sequence as

$$F_\bullet \colon M_\bullet \longrightarrow N_\bullet.$$

Chain maps are also known as commutative ladder diagrams. An isomorphism from $M_\bullet$ to $N_\bullet$ is a chain map such that each F_i is an isomorphism.

ex110517d EXAMPLE II.E.1.2. Consider the ring $R = \mathbb{Z}_{12} = \mathbb{Z}/12\mathbb{Z}$ and let $M_\bullet$ and $N_\bullet$ each be the constant sequence of copies of R with the R-module homomorphisms defined below. Defining various multiplication maps from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{12}$ (vertically) we have a chain map from $M_\bullet$ to $N_\bullet$.

$$\begin{tikzcd}
M_\bullet = & \cdots \arrow[r, "6\cdot"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z}_{12} \arrow[r, "4\cdot"] \arrow[d, "2\cdot"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z}_{12} \arrow[r, "6\cdot"] \arrow[d, "3\cdot"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z}_{12} \arrow[r, "4\cdot"] \arrow[d, "2\cdot"'] \arrow[dr, phantom, "\circlearrowleft"] & \cdots \\
N_\bullet = & \cdots \arrow[r, "4\cdot"'] & \mathbb{Z}_{12} \arrow[r, "6\cdot"'] & \mathbb{Z}_{12} \arrow[r, "4\cdot"'] & \mathbb{Z}_{12} \arrow[r, "6\cdot"'] & \cdots
\end{tikzcd}$$

apsOnHomology PROPOSITION II.E.1.3. *Let* $F_\bullet \colon M_\bullet \longrightarrow N_\bullet$ *be a chain map.*

sOnHomology.a (a) $F_i\left(\operatorname{Ker}\partial_i^M\right) \subseteq \operatorname{Ker}\partial_i^N$

sOnHomology.b (b) $F_i\left(\operatorname{Im}\partial_{i+1}^M\right) \subseteq \operatorname{Im}\partial_{i+1}^N$

sOnHomology.c (c) F_i *induces a well-defined* R*-module homomorphism from* $H_i(M_\bullet)$ *to* $H_i(N_\bullet)$ *given by*

$$\begin{aligned}
H_i(F_\bullet) : \qquad H_i(M_\bullet) &\longrightarrow H_i(N_\bullet) \\
\overline{m} &\longmapsto \overline{F_i(m)} \\
m + \operatorname{Im}\partial_{i+1}^M &\longmapsto F_i(m) + \operatorname{Im}\partial_{i+1}^N.
\end{aligned}$$

To put it yet another way

$$H_i(F_\bullet)(\overline{m}) = \overline{F_i(m)}.$$

PROOF. (a) For any $\alpha \in \ker(\partial_i^M)$ we have

$$\partial_i^N(F_i(\alpha)) = F_{i-1}(\partial_i^M(\alpha)) = F_{i-1}(0) = 0$$

because $F_\bullet$ is a chain map, completing this part.

(b) For any $\beta \in \operatorname{Im}\partial_{i+1}^M$ we can lift to some $\gamma \in M_{i+1}$ such that $\partial_{i+1}^M(\gamma) = \beta$. Then since $F_\bullet$ is a chain map we have

$$\partial_{i+1}^N(F_i(\gamma)) = F_i(\partial_{i+1}^M(\gamma)) = F_i(\beta).$$

(c) This is a corollary. Part (a) ensures that $H_i(F_\bullet)$ lands well, part (b) ensures that $H_i(F_\bullet)$ preserves equality, and the R-linearity of F_i gives the R-linearity of $H_i(F_\bullet)$. □

rmk091618a REMARK II.E.1.4. The construction of $H_i(F_\bullet)$ is summarized in the following commutative diagram with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Im}\partial_{i+1}^M \arrow[r, "\subseteq"] \arrow[d, "\alpha_i"] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{Ker}\partial_i^M \arrow[r] \arrow[d, "\beta_i"] \arrow[dr, phantom, "\circlearrowleft"] & H_i(M_\bullet) \arrow[r] \arrow[d, dashed, "\therefore\exists! H_i(F_\bullet)"] & 0 \\ 0 \arrow[r] & \operatorname{Im}\partial_{i+1}^N \arrow[r, "\subseteq"'] & \operatorname{Ker}\partial_i^N \arrow[r] & H_i(N_\bullet) \arrow[r] & 0 \end{tikzcd}$$

Here α_i and β_i are each induced by F_i (by parts (b) and (a) of Proposition II.E.1.3, respectively).

ex110517f EXAMPLE II.E.1.5. Recall $F_\bullet$, $M_\bullet$, and $N_\bullet$ from Example II.E.1.2. We have the homology modules

$$H_0(M_\bullet) = \frac{\operatorname{Ker} 6\cdot}{\operatorname{Im} 4\cdot} = \frac{2\cdot\mathbb{Z}_{12}}{4\cdot\mathbb{Z}_{12}} \cong \frac{2\mathbb{Z}}{4\mathbb{Z}} \cong \mathbb{Z}_2$$

$$H_0(N_\bullet) = \frac{\operatorname{Ker} 4\cdot}{\operatorname{Im} 6\cdot} = \frac{3\cdot\mathbb{Z}_{12}}{6\cdot\mathbb{Z}_{12}} \cong \mathbb{Z}_2$$

and the following map induced by $F_0 = 3\cdot$.

$$\begin{array}{rccc} H_0(F_\bullet): & H_0(M_\bullet) & \longrightarrow & H_0(N_\bullet) \\ & \frac{2\mathbb{Z}_{12}}{4\mathbb{Z}_{12}} & \xrightarrow{\quad 3\cdot \quad} & \frac{3\mathbb{Z}_{12}}{6\mathbb{Z}_{12}} \\ & \overline{2n} & \longmapsto & \overline{3\cdot 2n} = \overline{0} \end{array}$$

Note this implies $H_0(F_\bullet)$ is actually the zero map. The point is one might suspect this induced map to be multiplication by 3 from $\mathbb{Z}_2$ into $\mathbb{Z}_2$, but it can't be, because that would be an isomorphism and what we have found clearly is not.

In a similar fashion, we can study the induced map $H_1(F_\bullet)$.

$$\begin{array}{rccc} H_1(F_\bullet): & H_1(M_\bullet) & \longrightarrow & H_1(N_\bullet) \\ & \frac{3\mathbb{Z}_{12}}{6\mathbb{Z}_{12}} & \xrightarrow{\quad 2\cdot \quad} & \frac{2\mathbb{Z}_{12}}{4\mathbb{Z}_{12}} \\ & \overline{3k} & \longmapsto & \overline{2\cdot 3k} = \overline{6k} \end{array}$$

Note this is an isomorphism since it sends 0 to 0 and sends $\overline{3}$ to $\overline{6} = \overline{2}$. That is, it sends the generator of an order-2 cyclic group to the generator of another order-2 cyclic group.

II.E.2. Liftings and Resolutions

ection210623a

In this section we show that an R-module homomorphism can be extended to produce a chain map on projective resolutions. Then we give some justification for Facts II.C.5.9 and II.C.5.11, as promised.

lem111017a LEMMA II.E.2.1. *Consider the following diagram of R-modules and R-module homomorphisms with exact rows.*

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "\alpha"] & P \arrow[r, "\gamma"] & M \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "\delta"'] & Q \arrow[r, "\sigma"'] & N \arrow[r] & 0 \end{tikzcd}$$

If P is projective, then there exist R-module homomorphisms f' and F making the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\alpha"] \arrow[d, dashed, "f'"] \arrow[dr, phantom, "\circlearrowright"] & P \arrow[r, "\gamma"] \arrow[d, dashed, "F"] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & N' \arrow[r, "\delta"'] & Q \arrow[r, "\sigma"'] & N \arrow[r] & 0
\end{tikzcd}$$

Before proving this lemma, we give the following application.

prop111017b

Proposition II.E.2.2. *Let $P_\bullet^+$ be an augmented projective resolution of M and let $Q_\bullet^+$ be a "left resolution of N", i.e., an exact sequence*

$$Q_\bullet^+ = \qquad \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$$

where the modules $Q_0, Q_1, \dots$ may not be projective. In this case, for every R-module homomorphism $f\colon M \longrightarrow N$, there exists a commutative diagram

$$\begin{tikzcd}
P_\bullet^+ = \arrow[d, "F_\bullet^+"'] & \cdots \arrow[r] \arrow[dr, phantom, "\circlearrowright"] & P_1 \arrow[r] \arrow[d, dashed, "F_1"] \arrow[dr, phantom, "\circlearrowright"] & P_0 \arrow[r] \arrow[d, dashed, "F_0"] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r] \arrow[d, "f"] & 0 \\
Q_\bullet^+ = & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0.
\end{tikzcd}$$

Proof. We give a convincing diagram (II.E.2.2.1) and the general idea. (One may also want to revisit the construction in Discussion II.B.1.2.) The maps F_0 and f' come from Lemma II.E.2.1. Then the maps F_1 and f'' come from the same lemma, and so on, inductively. A diagram chase shows the larger rectangles commute as well. $\square$

$$
\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & 0 & & & \\
 & & & M'' \arrow[ur] \arrow[dr] \arrow[dddddr, dashed, "f''"] & & & & M \arrow[ur] \arrow[dr, "="] \arrow[dddddr, "f"] & & & & \\
\cdots \arrow[rr, "\partial_3^P"] & & P_2 \arrow[rr, "\partial_2^P"] \arrow[ur] \arrow[dddddr, dashed, "F_2"] & & P_1 \arrow[rr, "\partial_1^P"] \arrow[dr] \arrow[dddddr, dashed, "F_1"] & & P_0 \arrow[rr, "\tau"] \arrow[ur] \arrow[dddddr, dashed, "F_0"] & & M \arrow[rr] \arrow[dddddr, "f"] & & 0 & \\
 & M''' \arrow[ur] \arrow[dddddr, dashed, "f'''"] & & & & M' \arrow[ur] \arrow[dr] \arrow[dddddr, dashed, "f'"] & & & & & & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & & & & & \\
 & & & 0 \arrow[dr] & & 0 & & & & 0 & & \\
 & & & & N'' \arrow[ur] \arrow[dr] & & & & N \arrow[ur] \arrow[dr, "="] & & & \\
 & \cdots \arrow[rr, "\partial_3^Q"] & & Q_2 \arrow[rr, "\partial_2^Q"] \arrow[ur] & & Q_1 \arrow[rr, "\partial_1^Q"] \arrow[dr] & & Q_0 \arrow[rr, "\pi"] \arrow[ur] & & N \arrow[rr] & & 0 \\
 & & N''' \arrow[ur] & & & & N' \arrow[ur] \arrow[dr] & & & & & \\
 & 0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & & & &
\end{tikzcd}
\tag{II.E.2.2.1}
$$

eqn091018a

We now prove Lemma II.E.2.1.

Proof of Lemma II.E.2.1. Since P is projective, Definition II.A.1.14(b) gives the existence of a function F such that the following diagram commutes.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists F"'] \arrow[d, "f\circ\gamma"] \arrow[dl, phantom, shift left=3, "\circlearrowleft"] & \\ Q \arrow[r, "\sigma"'] & N \arrow[r] & 0 \end{tikzcd}$$

This is precisely one of the functions we seek. We can also prove this using another characterization of projective modules. Specifically $\mathrm{Hom}_R(P,-)$ is exact by Definition II.A.1.14(a) so applying it to the bottom row of the diagram we get the following short exact sequence.

$$0 \longrightarrow \mathrm{Hom}_R(P,N') \longrightarrow \mathrm{Hom}_R(P,Q) \xrightarrow{\sigma_*} \mathrm{Hom}_R(P,N) \longrightarrow 0$$

Noting the surjectivity of σ_*, there exists an R-module homomorphism $F \in \mathrm{Hom}_R(P,Q)$ such that

$$\sigma_*(F) = f\circ\gamma \in \mathrm{Hom}_R(P,N).$$

Since $\sigma_*(F) = \sigma\circ F$, this also yields the desired map.

Proving the existence of f' takes a bit more work. For any $m' \in M'$, the commutivity afforded by F and the exactness of the rows give

$$\sigma(F(\alpha(m'))) = f(\gamma(\alpha(m'))) = f(0) = 0.$$

Therefore $F(\alpha(m')) \in \mathrm{Ker}\,\sigma = \mathrm{Im}\,\delta$ and there exists some $n' \in N'$ such that $F(\alpha(m')) = \delta(n')$. In fact, since δ is injective, this n' is unique. Therefore we have the well-defined map

$$f' : M' \longrightarrow N'$$
$$m' \longmapsto n'$$

which we claim is an R-module homomorphism. To check R-linearity, first let $m' \in M'$ and let $r \in R$. Then there exists some $n' \in N'$ such that $F(\alpha(m')) = \delta(n')$ and we consider $rn' \in N'$ to find

$$\delta(rn') = r\cdot\delta(n') = r\cdot F(\alpha(m')) = F(\alpha(rm')).$$

Therefore

$$f'(rm') = rn' = r\cdot f'(m').$$

We prove the additivity of f' in a similar fashion. Let $m'_1, m'_2 \in M'$ and there exist n'_1, $n'_2 \in N'$ such that

$$F(\alpha(m'_1)) = \delta(n'_1) \qquad F(\alpha(m'_2)) = \delta(n'_2)$$

Therefore considering the element $n'_1 + n'_2 \in N'$ we have

$$\delta(n'_1+n'_2) = \delta(n'_1)+\delta(n'_2) = F(\alpha(m'_1)) + F(\alpha(m'_2)) = F(\alpha(m'_1+m'_2))$$

and hence

$$f'(m'_1+m'_2) = n'_1+n'_2 = f'(m'_1)+f'(m'_2).$$

□

Here we construct the induced maps on Ext from Fact II.C.5.9, but we will still put off some questions of well-definedness.

disc110817d

Discussion II.E.2.3. Consider R-module homomorphisms

$$f : M \longrightarrow M' \qquad\qquad g : N \longrightarrow N'$$

and we want to derive, though by no means completely at this point, the following induced maps.

$$\mathrm{Ext}^i_R(M,N) \xrightarrow{\mathrm{Ext}^i_R(M,g)} \mathrm{Ext}^i_R(M,N')$$
$$\mathrm{Ext}^i_R(M',N) \xrightarrow{\mathrm{Ext}^i_R(f,N)} \mathrm{Ext}^i_R(M,N)$$

The two maps we seek between Ext's come from chain maps

$$\mathrm{Hom}_R(P_\bullet, N) \xrightarrow{\mathrm{Hom}_R(P_\bullet, g)} \mathrm{Hom}_R(P_\bullet, N')$$
$$\mathrm{Hom}_R(P'_\bullet, N) \xrightarrow{\mathrm{Hom}_R(F_\bullet, N)} \mathrm{Hom}_R(P_\bullet, N)$$

where $P_\bullet$ is a projective resolution of M, $P'_\bullet$ is a projective resolution of M', and $F_\bullet \colon P_\bullet \longrightarrow P'_\bullet$ is a "lift" of f. That is, given the two augmented projective resolutions, because of the map f, there exist maps F_0, F_1, and so on that make each of the successive diagrams commute.

$$\begin{tikzcd}
P_\bullet^+ = & \cdots \arrow[r, "\partial_2^P"] \arrow[d, phantom, "\cdots"] & P_1 \arrow[r, "\partial_1^P"] \arrow[d, dashed, "\exists F_1"'] \arrow[dr, phantom, "\circlearrowleft"] & P_0 \arrow[r, "\tau"] \arrow[d, dashed, "\exists F_0"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[r] \arrow[d, "f"] & 0 \\
(P'_\bullet)^+ = & \cdots \arrow[r, "\partial_2^{P'}"'] & P'_1 \arrow[r, "\partial_1^{P'}"'] & P'_0 \arrow[r, "\tau'"'] & M' \arrow[r] & 0
\end{tikzcd}$$

Therefore restricting down to the projective resolutions we have

$$\begin{tikzcd}
P_\bullet = & \cdots \arrow[r, "\partial_2^P"] \arrow[d, phantom, "\cdots"] \arrow[dr, phantom, "\circlearrowleft"] & P_1 \arrow[r, "\partial_1^P"] \arrow[d, "F_1"'] \arrow[dr, phantom, "\circlearrowleft"] & P_0 \arrow[r] \arrow[d, "F_0"'] & 0 \\
P'_\bullet = & \cdots \arrow[r, "\partial_2^{P'}"'] & P'_1 \arrow[r, "\partial_1^{P'}"'] & P'_0 \arrow[r] & 0.
\end{tikzcd}$$

We say $F_\bullet = \{F_0, F_1, F_2, \dots\}$ <u>is a chain map compatible with</u> f. Or to put it another way, $F_\bullet$ is a chain map such that

$$\begin{tikzcd}
H_0(P_\bullet) \arrow[r, "H_0(F_\bullet)"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & H_0(P'_\bullet) \arrow[d, "\cong"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd}$$

using the induced map from Proposition II.E.1.3. We want to show that the ladder diagrams $\mathrm{Hom}_R(P_\bullet, g)$ and $\mathrm{Hom}_R(F_\bullet, N)$ commute. First we consider $\mathrm{Hom}_R(P_\bullet, g)$.

$$\begin{tikzcd}
\mathrm{Hom}_R(P_\bullet, N) = \arrow[d, "\mathrm{Hom}_R(P_\bullet, g)"'] & \cdots \arrow[r] & \mathrm{Hom}_R(P_i, N) \arrow[r, "\mathrm{Hom}_R(\partial_{i+1}^P, N)"] \arrow[d, "\mathrm{Hom}_R(P_i, g)"'] \arrow[dr, phantom, "\circlearrowleft?"] & \mathrm{Hom}_R(P_{i+1}, N) \arrow[r] \arrow[d, "\mathrm{Hom}_R(P_{i+1}, g)"] & \cdots \\
\mathrm{Hom}_R(P_\bullet, N') = & \cdots \arrow[r] & \mathrm{Hom}_R(P_i, N') \arrow[r, "\mathrm{Hom}_R(\partial_{i+1}^P, N')"'] & \mathrm{Hom}_R(P_{i+1}, N') \arrow[r] & \cdots
\end{tikzcd}$$

To check commutivity we track an arbitrary $\phi \in \mathrm{Hom}_R(P_i, N)$.

$$\begin{tikzcd}
\phi \arrow[r, mapsto, "\mathrm{Hom}_R(\partial_{i+1}^P, N)"] \arrow[d, mapsto, "\mathrm{Hom}_R(P_i, g)"'] & \phi \circ \partial_{i+1}^P \arrow[drr, mapsto, "\mathrm{Hom}_R(P_{i+1}, g)"] & & \\
g \circ \phi \arrow[r, mapsto, "\mathrm{Hom}_R(\partial_{i+1}^P, N')"'] & (g \circ \phi) \circ \partial_{i+1}^P & = & g \circ (\phi \circ \partial_{i+1}^P)
\end{tikzcd}$$

Therefore the diagram commutes by the associativity of function composition and we define the first of our two maps as

$$\mathrm{Ext}_R^i(M, g) = H_{-i}(\mathrm{Hom}_R(P_\bullet, g))$$

where

$$\mathrm{Ext}_R^i(M, g)\left(\overline{\phi}\right) = \overline{\mathrm{Hom}_R(P_\bullet, g)_{-i}(\phi)} = \overline{\mathrm{Hom}_R(P_i, g)(\phi)} = \overline{g \circ \phi}.$$

The chain map $\operatorname{Hom}_R(F_\bullet, N)$ also arises from maps between the chain complexes used to define the Ext's of the domain and codomain.

$$\operatorname{Hom}_R(F_\bullet, N) : \operatorname{Hom}_R(P'_\bullet, N) \longrightarrow \operatorname{Hom}_R(P_\bullet, N)$$

As with the first map, there is a question of commutivity in a particular diagram we need answered in order to verify we have a chain map.

$$\begin{tikzcd}
\operatorname{Hom}_R(P'_\bullet, N) = \arrow[d, "\operatorname{Hom}_R(F_\bullet,N)"'] & \cdots \arrow[r] & \operatorname{Hom}_R(P'_i, N) \arrow[r, "\operatorname{Hom}_R(\partial^{P'}_{i+1},N)"] \arrow[d, "\operatorname{Hom}_R(F_i,N)"'] \arrow[dr, phantom, "\circlearrowleft?"] & \operatorname{Hom}_R(P'_{i+1}, N) \arrow[r] \arrow[d, "\operatorname{Hom}_R(F_{i+1},N)"] & \cdots \\
\operatorname{Hom}_R(P_\bullet, N) = & \cdots \arrow[r] & \operatorname{Hom}_R(P_i, N) \arrow[r, "\operatorname{Hom}_R(\partial^{P}_{i+1},N)"'] & \operatorname{Hom}_R(P_{i+1}, N) \arrow[r] & \cdots
\end{tikzcd}$$

We again ignore well-definedness and check commutivity.

$$\begin{tikzcd}
\psi \arrow[r, mapsto, "\operatorname{Hom}_R(\partial^{P'}_{i+1},N)"] \arrow[d, mapsto, "\operatorname{Hom}_R(F_i,N)"'] & \psi \circ \partial^{P'}_{i+1} \arrow[drr, mapsto, "\operatorname{Hom}_R(F_{i+1},N)"] & & \\
\psi \circ F_i \arrow[r, mapsto, "\operatorname{Hom}_R(\partial^{P}_{i+1},N)"'] & (\psi \circ F_i) \circ \partial^P_{i+1} & = & (\psi \circ \partial^{P'}_{i+1}) \circ F_{i+1}
\end{tikzcd}$$

Where the equality holds since $F_\bullet$ is a chain map. Therefore we define the second map below.

$$\operatorname{Ext}^i_R(f, N)(\overline{\psi}) = \overline{\operatorname{Hom}_R(F_\bullet, N)_{-i}(\psi)} = \overline{\psi \circ F_i}$$

In all reality, one also needs to show this construction is independent of choice of $P_\bullet$, $P'_\bullet$, and $F_\bullet$, but we will end our discussion for now.

Here we give some justification for Fact II.C.5.11.

disc110917a Discussion II.E.2.4. Let $r \in R$, let $L_\bullet$ be an R-complex, and define the map

$$\begin{aligned} \mu^M_r : M &\longrightarrow M \\ m &\longmapsto rm \end{aligned}$$

where M is any R-module. Notice that we can build a chain map from $L_\bullet$ to itself out of such R-module homomorphisms.

$$\begin{tikzcd}
L_\bullet = & \cdots \arrow[r] & L_i \arrow[r, "\partial^L_i"] \arrow[d, "\mu^{L_i}_r"'] \arrow[dr, phantom, "\circlearrowleft"] & L_{i-1} \arrow[r] \arrow[d, "\mu^{L_{i-1}}_r"] & \cdots \\
L_\bullet = & \cdots \arrow[r] & L_i \arrow[r, "\partial^L_i"'] & L_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

We confirm the commutivity of the diagram by tracking an arbitrary element $\ell \in L_i$.

$$\begin{tikzcd}
\ell \arrow[r, mapsto] \arrow[d, mapsto] & \partial^L_i(\ell) \arrow[drr, mapsto] & & \\
r\ell \arrow[r, mapsto] & \partial^L_i(r\ell) & = & r \cdot \partial^L_i(\ell)
\end{tikzcd}$$

Hence we say

$$(\mu^{L_\bullet}_r)_\bullet : L_\bullet \longrightarrow L_\bullet.$$

Furthermore, the map induced on homologies is also a multiplication map. That is

$$H_i((\mu^{L_\bullet}_r)_\bullet) = \mu^{H_i(L_\bullet)}_r$$

because of the following.

$$H_i((\mu^{L_\bullet}_r)_\bullet)(\ell) = \overline{(\mu^{L_\bullet}_r)_i(\ell)} = \overline{r\ell} = r \cdot \overline{\ell} = \mu^{H_i(L_\bullet)}_r\left(\overline{\ell}\right)$$

We now claim

$$\operatorname{Ext}^i_R(\mu_r^M, N) = \mu_r^{\operatorname{Ext}^i_R(M,N)} = \operatorname{Ext}^i_R(M, \mu_r^N).$$

Indeed the second equality in our claim follows from

$$\operatorname{Ext}^i_R(M, \mu_r^N)(\overline{\phi}) = \overline{\mu_r^N \circ \phi} \underset{\ddagger}{=} \overline{r\phi} = r \cdot \overline{\phi} = \mu_r^{\operatorname{Ext}^i_R(M,n)}(\overline{\phi}) \tag{II.E.2.4.1}$$

`eqn092818a`

where ‡ holds since $(\mu_r^N \circ \phi)(x) = r \cdot \phi(x) = (r\phi)(x)$. For the first equality in our claim, we need $F_\bullet$.

$$\begin{tikzcd}
P_\bullet^+ = \arrow[d, "\left(\mu_r^{P_\bullet^+}\right)_\bullet"'] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, dashed, "\mu_r^{P_1}"] & P_0 \arrow[r] \arrow[d, dashed, "\mu_r^{P_0}"] & M \arrow[r] \arrow[d, "\mu_r^M"] & 0 \\
P_\bullet^+ = & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

It is straightforward to show that this diagram commutes, i.e., it satisfies the conclusion of Proposition II.E.2.2. Thus, we have the following.

$$\operatorname{Ext}^i_R(\mu_r^M, N)(\overline{\psi}) = \overline{\psi \circ \mu_r^{P_i}} = \overline{r \cdot \psi} = r \cdot \overline{\psi} = \mu_r^{\operatorname{Ext}^i_R(M,N)}$$

`ex111217a` Example II.E.2.5. Let $R = \mathbb{Z}_{12}$ and define R-modules $M = \mathbb{Z}_6$ and $N = \mathbb{Z}_3$. We then have the following chain map $F_\bullet$, where τ, π, and ρ are all natural surjections.

$$\begin{tikzcd}
P_\bullet^+ = & \cdots \arrow[r, "2\cdot"] & \mathbb{Z}_{12} \arrow[r, "6\cdot"] \arrow[d, dashed, "F_3"', "2\cdot"] & \mathbb{Z}_{12} \arrow[r, "2\cdot"] \arrow[d, dashed, "F_2"', "1\cdot"] & \mathbb{Z}_{12} \arrow[r, "6\cdot"] \arrow[d, dashed, "F_1"', "2\cdot"] & \mathbb{Z}_{12} \arrow[r, "\tau"] \arrow[d, dashed, "F_0"', "1\cdot"] & \mathbb{Z}_6 \arrow[r] \arrow[d, dashed, "\rho"] & 0 \\
Q_\bullet^+ = & \cdots \arrow[r, "4\cdot"] & \mathbb{Z}_{12} \arrow[r, "3\cdot"] & \mathbb{Z}_{12} \arrow[r, "4\cdot"] & \mathbb{Z}_{12} \arrow[r, "3\cdot"] & \mathbb{Z}_{12} \arrow[r, "\pi"] & \mathbb{Z}_3 \arrow[r] & 0
\end{tikzcd}$$

Reducing from the augmented resolutions, we lose our exactness on the right side, but we still have a chain map.

$$\begin{tikzcd}
P_\bullet = \arrow[d, "F_\bullet"'] & \cdots \arrow[r, "2\cdot"] & \mathbb{Z}_{12} \arrow[r, "6\cdot"] \arrow[d, dashed, "F_3"', "2\cdot"] & \mathbb{Z}_{12} \arrow[r, "2\cdot"] \arrow[d, dashed, "F_2"', "1\cdot"] & \mathbb{Z}_{12} \arrow[r, "6\cdot"] \arrow[d, dashed, "F_1"', "2\cdot"] & \mathbb{Z}_{12} \arrow[r] \arrow[d, dashed, "F_0"', "1\cdot"] & 0 \\
Q_\bullet = & \cdots \arrow[r, "4\cdot"] & \mathbb{Z}_{12} \arrow[r, "3\cdot"] & \mathbb{Z}_{12} \arrow[r, "4\cdot"] & \mathbb{Z}_{12} \arrow[r, "3\cdot"] & \mathbb{Z}_{12} \arrow[r] & 0
\end{tikzcd}$$

We want to compute maps on Ext induced by ρ. Specifically, we want to compute the maps

$$\begin{array}{ccccc}
\operatorname{Ext}^i_{\mathbb{Z}_{12}}(\rho, \mathbb{Z}_{12}) & : & \operatorname{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_{12}) & \longrightarrow & \operatorname{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_6, \mathbb{Z}_{12}) \\
\| & & \| & & \| \\
H_{-i}(\operatorname{Hom}_{\mathbb{Z}_{12}}(F_\bullet, \mathbb{Z}_{12})) & & H_{-i}(\operatorname{Hom}_{\mathbb{Z}_{12}}(Q_\bullet, \mathbb{Z}_{12})) & & H_{-i}(\operatorname{Hom}_{\mathbb{Z}_{12}}(P_\bullet, \mathbb{Z}_{12})).
\end{array}$$

From Discussion II.E.2.3, we know exactly how this map behaves for any given index i.

$$H_{-i}(\operatorname{Hom}_{\mathbb{Z}_{12}}(F_\bullet, \mathbb{Z}_{12}))(\overline{\phi}) = \overline{\phi \circ F_i}$$

In order to understand this better, we apply the functor $\operatorname{Hom}_{\mathbb{Z}_{12}}(-, \mathbb{Z}_{12})$ to the chain map above.

$$\begin{tikzcd}
P_\bullet^* = & 0 \arrow[r] & \operatorname{Hom}_{\mathbb{Z}_{12}}(\mathbb{Z}_{12}, \mathbb{Z}_{12}) \arrow[r, "6\cdot"] & \operatorname{Hom}_{\mathbb{Z}_{12}}(\mathbb{Z}_{12}, \mathbb{Z}_{12}) \arrow[r, "2\cdot"] & \operatorname{Hom}_{\mathbb{Z}_{12}}(\mathbb{Z}_{12}, \mathbb{Z}_{12}) \arrow[r, "6\cdot"] & \cdots \\
Q_\bullet^* = & 0 \arrow[r] & \operatorname{Hom}_{\mathbb{Z}_{12}}(\mathbb{Z}_{12}, \mathbb{Z}_{12}) \arrow[r, "3\cdot"'] \arrow[u, "1\cdot"'] & \operatorname{Hom}_{\mathbb{Z}_{12}}(\mathbb{Z}_{12}, \mathbb{Z}_{12}) \arrow[r, "4\cdot"'] \arrow[u, "2\cdot"'] & \operatorname{Hom}_{\mathbb{Z}_{12}}(\mathbb{Z}_{12}, \mathbb{Z}_{12}) \arrow[r, "3\cdot"'] \arrow[u, "1\cdot"'] & \cdots
\end{tikzcd}$$

Note we still have multiplication maps (see our justification for ‡ in Equation (II.E.2.4.1)). By Hom-cancellation we have the following.

$$\begin{tikzcd} P^*_\bullet \cong & 0 \arrow[r] & \mathbb{Z}_{12} \arrow[r, "6\cdot"] & \mathbb{Z}_{12} \arrow[r, "2\cdot"] & \mathbb{Z}_{12} \arrow[r, "6\cdot"] & \cdots \\ Q^*_\bullet \cong & 0 \arrow[r] & \mathbb{Z}_{12} \arrow[r, "3\cdot"'] \arrow[u, "1\cdot"] & \mathbb{Z}_{12} \arrow[r, "4\cdot"'] \arrow[u, "2\cdot"] & \mathbb{Z}_{12} \arrow[r, "3\cdot"'] \arrow[u, "1\cdot"] & \cdots \end{tikzcd}$$

Noticing this ladder diagram is merely the second diagram in this example with the arrows reversed, we know there is only one place where the rows are not exact, namely at the 0^{th} index. Therefore the Ext^i's vanish for all $i > 0$. So we write

$$\mathrm{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_6, \mathbb{Z}_{12}) = 0 = \mathrm{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_{12})$$

for all $i > 0$ and hence

$$\mathrm{Ext}^i_{Z_{12}}(\rho, \mathbb{Z}_{12}) : 0 \longrightarrow 0$$

is the zero map. At the $i = 0$ position we have

$$\mathrm{Ext}^0_R(\mathbb{Z}_6, \mathbb{Z}_{12}) \cong \mathrm{Ker}\ \ \mathbb{Z}_{12} \xrightarrow{6\cdot} \mathbb{Z}_{12} \ \cong \langle \overline{2} \rangle$$

$$\mathrm{Ext}^0_R(\mathbb{Z}_3, \mathbb{Z}_{12}) \cong \mathrm{Ker}\ \ \mathbb{Z}_{12} \xrightarrow{3\cdot} \mathbb{Z}_{12} \ \cong \langle \overline{4} \rangle$$

Therefore the map induced by $\rho = \left(\mathbb{Z}_6 \xrightarrow{1\cdot} \mathbb{Z}_6 \right)$,

$$\mathrm{Ext}^0_R(\rho, \mathbb{Z}_{12})\colon \ \mathrm{Ext}^0_R(\mathbb{Z}_3, \mathbb{Z}_{12}) \longrightarrow \mathrm{Ext}^0_R(\mathbb{Z}_6, \mathbb{Z}_{12})$$

is just the inclusion map $\langle \overline{4} \rangle \xhookrightarrow{\subseteq} \langle \overline{2} \rangle$.

ex112017a

Example II.E.2.6. Next, we generalize the previous example by computing $\mathrm{Ext}^i_{Z_{12}}(\rho, \mathbb{Z}_n)$ for several n satisfying $n|12$.

First we handle the $n = 2$ and $n = 4$ cases. Since $2 \cdot \mathbb{Z}_2 = 0$ and $3 \cdot \mathbb{Z}_3 = 0$, by Discussion II.E.2.4 we know

$$2 \cdot \mathrm{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_2) = 0 = 3 \cdot \mathrm{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_2)$$

and therefore

$$1 \cdot \mathrm{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_2) = (3 - 2) \cdot \mathrm{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_2) = 0.$$

Thus $\mathrm{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_2) = 0$ for all $i \in \mathbb{Z}$ and for almost identical reasons $\mathrm{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_4) = 0$ for all $i \in \mathbb{Z}$ as well, so we need not endeavor any further to study the induced maps on homologies in these cases (maps between zeros are boring).

For the case when $n = 6$, much of the derivation is a replication of Example II.E.2.5, so we will not reproduce it here, but the resulting ladder diagram is below.

$$\begin{tikzcd} P^*_\bullet \cong & 0 \arrow[r] & \mathbb{Z}_6 \arrow[r, "6\cdot"] & \mathbb{Z}_6 \arrow[r, "2\cdot"] & \mathbb{Z}_6 \arrow[r, "6\cdot"] & \cdots \\ Q^*_\bullet \cong & 0 \arrow[r] & \mathbb{Z}_6 \arrow[r, "3\cdot"'] \arrow[u, "1\cdot"] & \mathbb{Z}_6 \arrow[r, "4\cdot"'] \arrow[u, "2\cdot"] & \mathbb{Z}_6 \arrow[r, "3\cdot"'] \arrow[u, "1\cdot"] & \cdots \end{tikzcd}$$

At the $i = 0$ position we have the following homology modules.

$$H_0(P^*_\bullet) = \frac{\mathrm{Ker}\ \ \mathbb{Z}_6 \xrightarrow{0\cdot} \mathbb{Z}_6}{\mathrm{Im}\ \ 0 \longrightarrow \mathbb{Z}_6} = \frac{\mathbb{Z}_6}{0} \cong \mathbb{Z}_6$$

$$H_0(Q^*_\bullet) = \frac{\mathrm{Ker}\ \ \mathbb{Z}_6 \xrightarrow{3\cdot} \mathbb{Z}_6}{\mathrm{Im}\ \ 0 \longrightarrow \mathbb{Z}_6} \cong \frac{2 \cdot \mathbb{Z}_6}{0} \cong 2 \cdot \mathbb{Z}_6$$

Therefore the multiplication map $1\cdot$ is essentially a containment map.

$$H_0(F_\bullet)\ :\ H_0(Q^*_\bullet) \xrightarrow{1\cdot} H_0(P^*_\bullet)$$

$$2\cdot\mathbb{Z}_6 \xrightarrow[\subseteq]{1\cdot} \mathbb{Z}_6$$

That is, it is injective, and is neither onto nor the zero map. On the other hand, at the $i=-1$ position we have

$$H_{-1}(P^*_\bullet) = \mathrm{Ext}^1_{\mathbb{Z}_{12}}(\mathbb{Z}_6,\mathbb{Z}_6) = \frac{\mathrm{Ker}\ \mathbb{Z}_6 \xrightarrow{2\cdot} \mathbb{Z}_6}{\mathrm{Im}\ \mathbb{Z}_6 \xrightarrow{0\cdot} \mathbb{Z}_6} = \frac{3\cdot\mathbb{Z}_6}{0} \cong 3\mathbb{Z}_6$$

and

$$H_{-1}(Q^*_\bullet) = \mathrm{Ext}^1_{\mathbb{Z}_{12}}(\mathbb{Z}_3,\mathbb{Z}_6) = \frac{\mathrm{Ker}\ \mathbb{Z}_6 \xrightarrow[=-2\cdot]{4\cdot} \mathbb{Z}_6}{\mathrm{Im}\ \mathbb{Z}_6 \xrightarrow{3\cdot} \mathbb{Z}_6} \cong \frac{3\cdot\mathbb{Z}_6}{3\cdot\mathbb{Z}_6} = 0.$$

Therefore the induced map is the zero map and by the periodicity of our diagram, the same will hold for all odd i. Similarly

$$H_{-2}(Q^*_\bullet) = \mathrm{Ext}^2_{\mathbb{Z}_{12}}(\mathbb{Z}_3,\mathbb{Z}_6) = \frac{\mathrm{Ker}\ \mathbb{Z}_6 \xrightarrow{3\cdot} \mathbb{Z}_6}{\mathrm{Im}\ \mathbb{Z}_6 \xrightarrow[=-2\cdot]{4\cdot} \mathbb{Z}_6} = \frac{2\cdot\mathbb{Z}_6}{2\cdot\mathbb{Z}_6} = 0$$

so the periodicity of our ladder diagram lets us conclude $\mathrm{Ext}^i_{\mathbb{Z}_{12}}(\rho,\mathbb{Z}_6)=0$ for all $i>0$.

Exercises

`exer040102` Exercise II.E.2.7. Let R be a commutative ring with identity, and let $M'_\bullet, M''_\bullet$ be R-complexes. Define $M'_\bullet \oplus M''_\bullet := (M'\oplus M'')_\bullet$ where for each $i\in\mathbb{Z}$ we have $(M'\oplus M'')_i = M'_i\oplus M''_i$ and $\partial_i^{M'\oplus M''}\begin{pmatrix} m'_i \\ m''_i\end{pmatrix} = \begin{pmatrix}\partial^{M'}(m'_i)\\ \partial^{M''}(m''_i)\end{pmatrix}$. In other words, the map $\partial_i^{M'\oplus M''}$ is represented by the diagonal matrix $\begin{pmatrix}\partial^{M'} & 0\\ 0 & \partial^{M''}\end{pmatrix}$. For each $i\in\mathbb{Z}$, let $\epsilon_i\colon M'_i\to M'_i\oplus M''_i$ be the natural inclusion $\epsilon_i(m'_i)=\begin{pmatrix} m'_i\\ 0\end{pmatrix}$, and let $p_i\colon M'_i\oplus M''_i\to M''_i$ be the natural surjection $p_i\begin{pmatrix} m'_i\\ m''_i\end{pmatrix} = m''_i$. Note that each map $\partial_i^{M'\oplus M''}$, ϵ_i, and p_i is an R-module homomorphism, and each sequence $0\to M'_i\xrightarrow{\epsilon_i} M'_i\oplus M''_i \xrightarrow{p_i} M''_i\to 0$ is exact. (You do not need to prove this.)

(a) Prove that $M'_\bullet\oplus M''_\bullet$ is an R-complex.

(b) For each $i\in\mathbb{Z}$, prove that $\mathrm{H}_i(M'_\bullet\oplus M''_\bullet)\cong \mathrm{H}_i(M'_\bullet)\oplus\mathrm{H}_i(M''_\bullet)$.

`exer040102c` (c) Prove that $\epsilon_\bullet\colon M'_\bullet\to M'_\bullet\oplus M''_\bullet$ and $p_\bullet\colon M'_\bullet\oplus M''_\bullet\to M''_\bullet$ are chain maps. (It follows immediately that the sequence

$$0\to M'_\bullet \xrightarrow{\epsilon_\bullet} M'_\bullet\oplus M''_\bullet \xrightarrow{p_\bullet} M''_\bullet\to 0 \qquad \text{(II.E.2.7.1)}$$ `eq171116a`

is exact; you do not need to prove this.)

`exer040102d` (d) For each $i\in\mathbb{Z}$, prove that the sequence

$$0\to \mathrm{H}_i(M'_\bullet)\xrightarrow{\mathrm{H}_i(\epsilon_\bullet)} \mathrm{H}_i(M'_\bullet\oplus M''_\bullet)\xrightarrow{\mathrm{H}_i(p_\bullet)} \mathrm{H}_i(M''_\bullet)\to 0$$

is split exact.

`exer040102e` (e) For each $i\in\mathbb{Z}$, let $\eth_i\colon \mathrm{H}_i(M''_\bullet)\to\mathrm{H}_{i-i}(M'_\bullet)$ be the connecting homomorphism for the long exact sequence coming from (II.E.2.7.1). Use part (d) to prove that $\eth_i=0$.

(f) Use the definition/construction of $\eth_i$ to give another proof of the fact that $\eth_i=0$.

`exer040101` Exercise II.E.2.8. (Functoriality of long exact sequences) Let R be a commutative ring with identity, and consider the following diagram of chain maps:

$$\begin{tikzcd}
0 \arrow[r] & M'_\bullet \arrow[r, "F_\bullet"] \arrow[d, "f_\bullet"'] & M_\bullet \arrow[r, "G_\bullet"] \arrow[d, "g_\bullet"'] & M''_\bullet \arrow[r] \arrow[d, "h_\bullet"'] & 0 \\
0 \arrow[r] & N'_\bullet \arrow[r, "H_\bullet"] & N_\bullet \arrow[r, "K_\bullet"] & N''_\bullet \arrow[r] & 0.
\end{tikzcd}$$

Assume that, for each integer i, the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & M'_i \arrow[r, "F_i"] \arrow[d, "f_i"'] & M_i \arrow[r, "G_i"] \arrow[d, "g_i"'] & M''_i \arrow[r] \arrow[d, "h_i"'] & 0 \\ 0 \arrow[r] & N'_i \arrow[r, "H_i"] & N_i \arrow[r, "K_i"] & N''_i \arrow[r] & 0. \end{tikzcd}$$

Prove that the following diagram of long exact sequences commutes:

$$\begin{tikzcd} \cdots \arrow[r, "\eth^M_{i+1}"] & \mathrm{H}_i(M'_\bullet) \arrow[r, "\mathrm{H}_i(F_\bullet)"] \arrow[d, "\mathrm{H}_i(f_\bullet)"'] & \mathrm{H}_i(M_\bullet) \arrow[r, "\mathrm{H}_i(G_\bullet)"] \arrow[d, "\mathrm{H}_i(g_\bullet)"'] & \mathrm{H}_i(M''_\bullet) \arrow[r, "\eth^M_i"] \arrow[d, "\mathrm{H}_i(h_\bullet)"'] & \mathrm{H}_{i-1}(M'_\bullet) \arrow[r, "\mathrm{H}_{i-1}(F_\bullet)"] \arrow[d, "\mathrm{H}_{i-1}(f_\bullet)"'] & \cdots \\ \cdots \arrow[r, "\eth^N_{i+1}"] & \mathrm{H}_i(N'_\bullet) \arrow[r, "\mathrm{H}_i(H_\bullet)"] & \mathrm{H}_i(N_\bullet) \arrow[r, "\mathrm{H}_i(K_\bullet)"] & \mathrm{H}_i(N''_\bullet) \arrow[r, "\eth^N_i"] & \mathrm{H}_{i-1}(N'_\bullet) \arrow[r, "\mathrm{H}_{i-1}(H_\bullet)"] & \cdots . \end{tikzcd}$$

CHAPTER II.F

Long Exact Sequences

apter062021b

In this chapter we achieve the goal set in Section II.B.1 by proving the existence of long exact sequences for Ext and the well-definedness of Ext (see Theorems II.F.2.1, II.F.3.3, and II.F.5.2).

II.F.1. The Mother of All Long Exact Sequences

ection062921s

In this section we prove the existence of long exact sequences in general and we prove the Snake Lemma as a corollary, which we will need for future results, such as Lemmas II.F.3.1 and II.F.3.2.

def092818b DEFINITION II.F.1.1. Let $M_\bullet$, $M'_\bullet$, and $M''_\bullet$ be R-complexes. A diagram of chain maps

$$0 \longrightarrow M'_\bullet \xrightarrow{f_\bullet} M_\bullet \xrightarrow{g_\bullet} M''_\bullet \longrightarrow 0$$

is a short exact sequence of R-complexes if each row in the ladder is exact.

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial^{M'}_{i+1}"'] & \vdots \arrow[d, "\partial^{M}_{i+1}"'] & \vdots \arrow[d, "\partial^{M''}_{i+1}"'] & \\
0 \arrow[r] & M'_i \arrow[r, "f_i"] \arrow[d, "\partial^{M'}_i"'] \arrow[dr, phantom, "\circlearrowright"] & M_i \arrow[r, "g_i"] \arrow[d, "\partial^{M}_i"'] \arrow[dr, phantom, "\circlearrowright"] & M''_i \arrow[r] \arrow[d, "\partial^{M''}_i"'] & 0 \\
0 \arrow[r] & M'_{i-1} \arrow[r, "f_{i-1}"'] \arrow[d, "\partial^{M'}_{i-1}"'] & M_{i-1} \arrow[r, "g_{i-1}"'] \arrow[d, "\partial^{M}_{i-1}"'] & M''_{i-1} \arrow[r] \arrow[d, "\partial^{M''}_{i-1}"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

mother THEOREM II.F.1.2. *Consider the following short exact sequence of R-complexes.*

$$0 \longrightarrow M'_\bullet \xrightarrow{f_\bullet} M_\bullet \xrightarrow{g_\bullet} M''_\bullet \longrightarrow 0$$

Then for every $i \in \mathbb{Z}$ there exists an R-module homomorphism

$$\eth_i : H_i(M''_\bullet) \longrightarrow H_{i-1}(M'_\bullet)$$

$$\overline{m''_i} \longmapsto \overline{m'_{i-1}}$$

such that the following sequence is exact.

$$\cdots \xrightarrow{\eth_{i+1}} H_i(M'_\bullet) \xrightarrow{H_i(f_\bullet)} H_i(M_\bullet) \xrightarrow{H_i(g_\bullet)} H_i(M''_\bullet) \xrightarrow{\eth_i} H_{i-1}(M'_\bullet) \xrightarrow{H_{i-1}(f_\bullet)} \cdots$$

We call $\eth_i$ a connecting homomorphism.

PROOF. We will prove this in nine steps.

Step 1: Let us construct $\eth_i$. Let $\xi \in H_i(M''_\bullet) = \operatorname{Ker} \partial_i^{M''} / \operatorname{Im} \partial_{i+1}^{M''}$ and let $\alpha \in \operatorname{Ker} \partial_i^{M''}$ such that $\xi = \overline{\alpha} \in H_i(M''_\bullet)$. Since g_i is surjective, let $\beta \in M_i$ such that $g_i(\beta) = \alpha$. Since $g_\bullet$ is a chain map (i.e., since the partials and g_i's commute) and by definition of β we have

$$g_{i-1}\left(\partial_i^M(\beta)\right) = \partial_i^{M''}\left(g_i(\beta)\right) = \partial_i^{M''}(\alpha) = 0.$$

Therefore $\partial_i^M(\beta) \in \ker(g_{i-1}) = \operatorname{Im} f_{i-1}$, so we let $\gamma \in M'_{i-1}$ such that $f_{i-1}(\gamma) = \partial_i^M(\beta)$. We define $\eth_i$ in terms of this element γ.

$$\eth_i(\xi) = \overline{\gamma} \in \frac{\operatorname{Ker} \partial_{i-1}^{M'}}{\operatorname{Im} \partial_i^{M'}} = H_{i-1}(M'_\bullet)$$

We need to show $\gamma \in \operatorname{Ker} \partial_{i-1}^{M'}$, which we will do first in the next step.

Step 2: We show $\eth_i$ is well-defined. First we have

$$f_{i-2}\left(\partial_{i-1}^{M'}(\gamma)\right) = \partial_{i-1}^{M}\left(f_{i-1}(\gamma)\right)$$

since $F_\bullet$ is a chain map. Then

$$\partial_{i-1}^{M}\left(f_{i-1}(\gamma)\right) = \partial_{i-1}^{M}\left(\partial_i^M(\beta)\right) = 0$$

using the definition of γ and that $M_\bullet$ is an R-complex. Since f_{i-2} is injective, this implies $\partial_{i-1}^{M'}(\gamma) = 0$, i.e., $\gamma \in \operatorname{Ker} \partial_{i-1}^{M'}$, as desired.

Second we will show $\overline{\gamma} \in H_{i-1}(M'_\bullet)$ is independent of any choices made in Step 1. Let $\alpha, \alpha' \in \operatorname{Ker} \partial_i^{M''}$ such that $\overline{\alpha} = \xi = \overline{\alpha'}$, let $\beta, \beta' \in M_i$ such that $g_i(\beta) = \alpha$ and $g_i(\beta') = \alpha'$, and let $\gamma, \gamma' \in M'_{i-1}$ such that $f_{i-1}(\gamma) = \partial_i^M(\beta)$ and $f_{i-1}(\gamma') = \partial_i^M(\beta')$. We need to show $\overline{\gamma} = \overline{\gamma'}$ in $H_{i-1}(M'_\bullet) = \operatorname{Ker} \partial_{i-1}^{M'} / \operatorname{Im} \partial_i^{M'}$, or in other words, we need to show $\gamma - \gamma' \in \operatorname{Im} \partial_i^{M'}$.

By assumption $\overline{\alpha} = \overline{\alpha'} \in H_i(M'_\bullet) = \operatorname{Ker} \partial_i^{M'} / \operatorname{Im} \partial_{i+1}^{M'}$, so $\alpha - \alpha' \in \operatorname{Im} \partial_{i+1}^{M'}$ and we let $\eta \in M''_{i+1}$ such that $\partial_{i+1}^{M''}(\eta) = \alpha - \alpha'$. Since g_{i+1} is surjective, we may let $\nu \in M_{i+1}$ such that $g_{i+1}(\nu) = \eta$ and we compute the following.

$$g_i(\beta - \beta' - \partial_{i+1}^M(\nu)) = g_i(\beta) - g_i(\beta') - (g_i \circ \partial_{i+1}^M)(\nu) = \alpha - \alpha' - (\alpha - \alpha') = 0$$

In the above calculation we rely only on the definitions of our elements and the linearity of g_i. By this calculation we know $\beta - \beta' - \partial_{i+1}^M(\nu) \in \ker(g_i) = \operatorname{Im} f_i$ so let $\omega \in M'_i$ such that $f_i(\omega) = \beta - \beta' - \partial_{i+1}^M(\nu)$. Since $\gamma, \gamma', \partial_i^{M'}(\omega) \in M'_{i-1}$, we use the linearity of f_{i-1} to get

$$f_{i-1}\left(\partial_i^{M'}(\omega) - (\gamma - \gamma')\right) = \left(f_{i-1} \circ \partial_i^{M'}\right)(\omega) - f_{i-1}(\gamma) + f_{i-1}(\gamma').$$

Since $f_\bullet$ is a chain map, then

$$\left(f_{i-1} \circ \partial_i^{M'}\right)(\omega) - f_{i-1}(\gamma) + f_{i-1}(\gamma') = \left(\partial_i^M \circ f_i\right)(\omega) - \partial_i^M(\beta) + \partial_i^M(\beta').$$

The definition of ω gives a similar argument as for f_{i-2} above.

$$\begin{aligned}
\left(\partial_i^M \circ f_i\right)(\omega) - \partial_i^M(\beta) + \partial_i^M(\beta') &= \partial_i^M\left(\beta - \beta' - \partial_{i+1}^M(\nu)\right) - \partial_i^M(\beta) + \partial_i^M(\beta') \\
&= \partial_i^M\left(\beta - \beta' - \partial_{i+1}^M(\nu) - \beta + \beta'\right) \\
&= -\left(\partial_i^M \circ \partial_{i+1}^M\right)(\nu) \\
&= 0.
\end{aligned}$$

Since f_{i-1} is injective, this implies $\partial_i^{M'}(\omega) - (\gamma - \gamma') = 0$ or equivalently

$$\gamma - \gamma' = \partial_i^{M'}(\omega) \in \operatorname{Im} \partial_i^{M'}$$

completing this step.

Step 3: Here we prove $\eth_i$ is an R-module homomorphism. Let $\xi, \xi' \in H_i(M''_\bullet)$ and $r \in R$. Also let $\alpha, \alpha' \in \operatorname{Ker} \partial_i^{M''}$ such that $\overline{\alpha} = \xi$ and $\overline{\alpha'} = \xi'$, let $\beta, \beta' \in M_i$ such that $g_i(\beta) = \alpha$ and $g_i(\beta') = \alpha'$, and let $\gamma, \gamma' \in M'_{i-1}$ such that $f_{i-1}(\gamma) = \partial_i^M(\beta)$ and $f_{i-1}(\gamma') = \partial_i^M(\beta')$.

Notice that $r\alpha + \alpha' \in \operatorname{Ker} \partial_i^{M''}$ and hence it makes sense to write $\overline{r\alpha + \alpha'} = r\xi + \xi'$. Notice also that $r\beta + \beta' \in M_i$ so we have

$$g_i(r\beta + \beta') = g_i(r\beta) + g_i(\beta') = r \cdot g_i(\beta) + g_i(\beta') = r\alpha + \alpha'.$$

Finally note that $r\gamma + \gamma' \in M'_{i-1}$ for which we have

$$\begin{aligned} f_{i-1}(r\gamma + \gamma') &= f_{i-1}(r\gamma) + f_{i-1}(\gamma') = r \cdot f_{i-1}(\gamma) + f_{i-1}(\gamma') \\ &= r \cdot \partial_i^M(\beta) + \partial_i^M(\beta') = \partial_i^M(r\beta) + \partial_i^M(\beta') = \partial_i^M(r\beta + \beta'). \end{aligned}$$

Therefore we have an element satisfying the definition of $\eth_i$ described in Step 1 so we conclude this step in the following display.

$$\eth_i(r\xi + \xi') = \overline{r\gamma + \gamma'} = r \cdot \overline{\gamma} + \overline{\gamma} = r \cdot \eth_i(\xi) + \eth_i(\xi)$$

Step 4: We tackle the first of several questions of exactness. Here we show $\operatorname{Im} H_i(f_\bullet) \subseteq \operatorname{Ker} H_i(g_\bullet)$. Let $\delta \in H_i(M'_\bullet)$ and let $\rho \in \operatorname{Ker} \partial_i^{M'}$ such that $\overline{\rho} = \delta$. Therefore we have

$$H_i(g_\bullet)\left(H_i(f_\bullet)(\delta)\right) = H_i(g_\bullet)\left(\overline{f_i(\rho)}\right) = \overline{(g_i \circ f_i)(\rho)} = \overline{0} = 0$$

where the third equality comes from the exactness of the original sequence of chain maps.

Step 5: We now show $\operatorname{Im} H_i(f_\bullet) \supseteq \operatorname{Ker} H_i(g_\bullet)$. Let $\delta \in \operatorname{Ker} H_i(g_\bullet)$ and let $\rho \in \operatorname{Ker} \partial_i^M$ such that $\overline{\rho} = \delta$. This gives

$$0 = H_i(g_\bullet)(\overline{\rho}) = \overline{g_i(\rho)} \in H_i(M''_\bullet) = \frac{\operatorname{Ker} \partial_i^{M''}}{\operatorname{Im} \partial_{i+1}^{M''}}.$$

Therefore $g_i(\rho) \in \operatorname{Im} \partial_{i+1}^{M''}$ so we lift to some $\mu \in M''_{i+1}$ such that $\partial_{i+1}^{M''}(\mu) = g_i(\rho)$ and lift again to some $\sigma \in M_{i+1}$ such that $g_{i+1}(\sigma) = \mu$ (since g_{i+1} is surjective). Since $\rho, \partial_{i+1}^M(\sigma) \in M_i$, we consider the element $\rho - \partial_{i+1}^M(\sigma) \in M_i$. Using linearity and the fact that $g_\bullet$ is a chain map we compute

$$g_i(\rho - \partial_{i+1}^M(\sigma)) = g_i(\rho) - (g_i \circ \partial_{i+1}^M)(\sigma) = g_i(\rho) - (\partial_{i+1}^{M''} \circ g_{i+1})(\sigma) = g_i(\rho) - \partial_{i+1}^{M''}(\mu) = 0.$$

Hence $\rho - \partial_{i+1}^M(\sigma) \in \ker(g_i) = \operatorname{Im} f_i$ and we let $\tau \in M'_i$ such that $f_i(\tau) = \rho - \partial_{i+1}^M(\sigma)$. We claim $\tau \in \operatorname{Ker} \partial_i^{M'}$ and point out it suffices to show $\left(f_{i-1} \circ \partial_i^{M'}\right)(\tau) = 0$ since f_{i-1} is injective. We compute

$$\left(f_{i-1} \circ \partial_i^{M'}\right)(\tau) = \partial_i^M(f_i(\tau)) = \partial_i^M(\rho - \partial_{i+1}^M(\sigma)) = \partial_i^M(\rho) - \left(\partial_i^M \circ \partial_{i+1}^M\right)(\sigma) = 0$$

where the last equality holds by definition of ρ and because $M_\bullet$ is a chain complex.

We consider $\rho, \partial_{i+1}^M(\sigma) \in \operatorname{Ker} \partial_i^M$ and $\tau \in \operatorname{Ker} \partial_i^{M'}$, which represent the cosets $\overline{\rho}, \overline{\partial_{i+1}^M(\sigma)} \in H_i(M_\bullet)$ and $\overline{\tau} \in H_i(M'_\bullet)$. Therefore it makes sense to compute

$$H_i(f_\bullet)(\overline{\tau}) = \overline{f_i(\tau)} = \overline{\rho - \partial_{i+1}^M(\sigma)} = \overline{\rho} - \overline{\partial_{i+1}^M(\sigma)} = \overline{\rho} - \overline{0} = \overline{\rho} = \delta.$$

Hence $\delta \in \operatorname{Im} H_i(f_\bullet)$, completing this step.

Step 6: Continuing our proof of exactness, we show here that $\operatorname{Im} H_i(g_\bullet) \subseteq \operatorname{Ker} \eth_i$. Let $\zeta \in H_i(M_\bullet)$ and let $\beta \in \operatorname{Ker} \partial_i^M$ such that $\overline{\beta} = \zeta$. We want to show that $(\eth_i \circ H_i(g_\bullet))(\overline{\beta}) = 0$. Define $\alpha = g_i(\beta)$ and we have

$$H_i(g_\bullet)(\overline{\beta}) = \overline{g_i(\beta)} = \overline{\alpha}.$$

Computing $\eth_i(H_i(g_\bullet)(\overline{\beta})) = \eth_i(\overline{\alpha})$ requires some $\gamma \in \operatorname{Ker} \partial_{i-1}^{M'}$ such that $f_{i-1}(\gamma) = \partial_i^M(\beta)$. Since $\beta \in \operatorname{Ker} \partial_i^M$ by assumption, $\partial_i^M(\beta) = 0 = f_{i-1}(0)$, so setting $\gamma = 0$ we get

$$\eth_i(\overline{\alpha}) = \overline{\gamma} = \overline{0} = 0.$$

Step 7: We now show $\operatorname{Im} H_i(g_\bullet) \supseteq \operatorname{Ker} \eth_i$. Let $\xi \in \operatorname{Ker} \eth_i \subseteq H_i(M''_\bullet)$ and let $\alpha \in \operatorname{Ker} \partial_i^{M''}$ such that $\xi = \overline{\alpha}$. Fix some $\beta \in M_i$ such that $g_i(\beta) = \alpha$ and some $\gamma \in M'_{i-1}$ such that $f_{i-1}(\gamma) = \partial_i^M(\beta) \in \ker(g_{i-1}) = \operatorname{Im} f_{i-1}$. Our construction in Step 1 implies $\eth_i(\xi) = \overline{\gamma}$ so we have

$$0 = \eth_i(\xi) = \overline{\gamma} \in H_{i-1}(M'_\bullet) = \frac{\operatorname{Ker} \partial_{i-1}^{M'}}{\operatorname{Im} \partial_i^{M'}}.$$

Hence $\gamma \in \operatorname{Im} \partial_i^{M'}$ and we let $\omega \in M'_i$ such that $\partial_i^{M'}(\omega) = \gamma$. Moreover, $f_i(\omega)$, $\beta \in M_i$ so we compute the following.

$$\begin{aligned}
\partial_i^M(\beta - f_i(\omega)) &= \partial_i^M(\beta) - \left(\partial_i^M \circ f_i\right)(\omega) \\
&= \partial_i^M(\beta) - \left(f_{i-1} \circ \partial_i^{M'}\right)(\omega) \\
&= \partial_i^M(\beta) - f_{i-1}(\gamma) \\
&= \partial_i^M(\beta) - \partial_i^M(\beta) \\
&= 0
\end{aligned}$$

Therefore $\beta - f_i(\omega) \in \operatorname{Ker} \partial_i^M$ and hence $\overline{\beta - f_i(\omega)} \in H_i(M_\bullet)$. We may also compute

$$H_i(g_\bullet)(\overline{\beta - f_i(\omega)}) = \overline{g_i(\beta - f_i(\omega))} = \overline{g_i(\beta) - (g_i \circ f_i)(\omega)} = \overline{g_i(\beta)} = \overline{\alpha} = \xi$$

where the third equality holds by the exactness of the i^{th} row of the given diagram. Hence $\xi \in \operatorname{Im} H_i(g_\bullet)$, which completes this step.

Step 8: Here we show $\operatorname{Im} \eth_i \subseteq \operatorname{Ker} H_{i-1}(f_\bullet)$. Let $\xi \in H_i(M''_\bullet)$ and let $\alpha \in \operatorname{Ker} \partial_i^{M''}$ such that $\xi = \overline{\alpha}$. We want to show that $H_{i-1}(f_\bullet)\left(\eth_i(\overline{\alpha})\right) = 0$. Since g_i is surjective, let $\beta \in M_i$ such that $g_i(\beta) = \alpha$ and since $\partial_i^M(\beta) \in \operatorname{Ker} g_{i-1} = \operatorname{Im} f_{i-1}$, let $\gamma \in M'_{i-1}$ such that $f_{i-1}(\gamma) = \partial_i^M(\beta)$. We therefore have

$$H_{i-1}(f_\bullet)(\eth_i(\overline{\alpha})) = H_{i-1}(f_\bullet)(\overline{\gamma}) = \overline{f_{i-1}(\gamma)} = \overline{\partial_i^M(\beta)} = 0$$

which completes this step.

Step 9: We finally show that $\operatorname{Im} \eth_i \supseteq \operatorname{Ker} H_{i-1}(f_\bullet)$. Let $\lambda \in \operatorname{Ker} H_{i-1}(f_\bullet)$ and fix some element $\gamma \in \operatorname{Ker} \partial_{i-1}^{M'}$ such that $\lambda = \overline{\gamma} \in H_{i-1}(M'_\bullet)$. By assumption we have

$$0 = H_{i-1}(f_\bullet)(\lambda) = H_{i-1}(f_\bullet)(\overline{\gamma}) = \overline{f_{i-1}(\gamma)} \in H_{i-1}(M_\bullet) = \frac{\operatorname{Ker} \partial_{i-1}^M}{\operatorname{Im} \partial_i^M}.$$

It follows that $f_{i-1}(\gamma) \in \operatorname{Im} \partial_i^M$, so we may let $\beta \in M_i$ such that $\partial_i^M(\beta) = f_{i-1}(\gamma)$. Denote $g_i(\beta) = \alpha$ and notice by our construction in Step 1, this element is a good candidate on which to apply $\eth_i$. Observe that

$$\partial_i^{M''}(\alpha) = \partial_i^{M''}(g_i(\beta)) = g_{i-1}(\partial_i^M(\beta)) = (g_{i-1} \circ f_{i-1})(\gamma) = 0$$

so $\alpha \in \operatorname{Ker} \partial_i^{M''}$. Therefore $\overline{\alpha} \in H_i(M''_\bullet)$ and

$$\eth_i(\overline{\alpha}) = \overline{\gamma} = \lambda.$$

This completes the proof of the theorem. □

`SnakeLemma`

Corollary II.F.1.3 (Snake Lemma). *Consider a commutative diagram of R-modules and R-module homomorphisms with exact rows.*

$$\begin{tikzcd}
0 \arrow[r] & M'_1 \arrow[r, "f_1"] \arrow[d, "\partial'_1"'] & M_1 \arrow[r, "g_1"] \arrow[d, "\partial_1"'] & M''_1 \arrow[r] \arrow[d, "\partial''_1"'] & 0 \\
0 \arrow[r] & M'_0 \arrow[r, "f_1"] & M_0 \arrow[r, "g_1"] & M''_0 \arrow[r] & 0
\end{tikzcd}$$

There exists an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \ker(\partial'_1) \arrow[r] & \ker(\partial_1) \arrow[r] & \ker(\partial''_1) \arrow[dll, rounded corners] & \\
& \operatorname{Coker}(\partial'_1) \arrow[r] & \operatorname{Coker}(\partial_1) \arrow[r] & \operatorname{Coker}(\partial''_1) \arrow[r] & 0.
\end{tikzcd}$$

PROOF. From the given diagram, we extend to form the following short exact sequence of R-complexes.

$$0 \longrightarrow M'_\bullet \longrightarrow M_\bullet \longrightarrow M''_\bullet \longrightarrow 0$$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & 0 \arrow[d, "\epsilon'_1"'] & 0 \arrow[d, "\epsilon_1"'] & 0 \arrow[d, "\epsilon''_1"'] & \\
0 \arrow[r] & M'_1 \arrow[r, "f_1"] \arrow[d, "\partial'_1"'] & M_1 \arrow[r, "g_1"] \arrow[d, "\partial_1"'] & M''_1 \arrow[r] \arrow[d, "\partial''_1"'] & 0 \\
0 \arrow[r] & M'_0 \arrow[r, "f_0"] \arrow[d, "\tau'_1"'] & M_0 \arrow[r, "g_0"] \arrow[d, "\tau_1"'] & M''_0 \arrow[r] \arrow[d, "\tau''_1"'] & 0 \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Note that the columns in this diagram are R-complexes, because

$$\operatorname{Im} \epsilon'_1 = \{0\} \subseteq \operatorname{Ker} \partial'_1$$

$$\operatorname{Im} \partial'_1 \subseteq M'_0 = \operatorname{Ker} \tau'_1$$

and similarly for the other two columns. By Theorem II.F.1.2, we have the following long exact sequence.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_2(M''_\bullet) \arrow[dll, rounded corners, to path={[pos=0.25] -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
H_1(M'_\bullet) \arrow[r] & H_1(M_\bullet) \arrow[r] & H_1(M''_\bullet) \arrow[dll, rounded corners, to path={[pos=0.25] -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
H_0(M'_\bullet) \arrow[r] & H_0(M_\bullet) \arrow[r] & H_0(M''_\bullet) \arrow[dll, rounded corners, to path={[pos=0.25] -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
H_{-1}(M'_\bullet) \arrow[r] & \cdots &
\end{tikzcd}$$

By construction

$$H_i(M'_\bullet) = H_i(M_\bullet) = H_i(M''_\bullet) = 0$$

for all $i > 1$ and all $i < 0$. Checking definitions of the remaining six homology modules verifies the claim. $\square$

SnakeCor

REMARK II.F.1.4. In the context of Corollary II.F.1.3, we know for each $i = 0, 1, 2$, $\partial_1^{(i)}$ is injective if and only if $\ker\left(\partial_1^{(i)}\right) = 0$. For a consequence of this, suppose ∂''_1 is injective. Then in our long exact sequence we have

$$0 \longrightarrow \operatorname{Ker} \partial'_1 \longrightarrow \operatorname{Ker} \partial_1 \longrightarrow \operatorname{Ker} \partial''_1 = 0$$

and from Fact II.C.2.24, it follows that $\operatorname{Ker} \partial'_1 = 0$ if and only if $\operatorname{Ker} \partial_1 = 0$, i.e., ∂'_1 is injective if and only if ∂_1 is injective. In a similar fashion, if we suppose that ∂'_1 is surjective (i.e., $\operatorname{Coker}(\partial'_1) = 0$), then ∂_1 is surjective if and only if ∂''_1 is surjective. The proof of this is analogous using the latter half of the long exact sequence in Corollary II.F.1.3.

II.F.2. The First Long Exact Sequence in Ext

ection062021a

In this section we use Theorem II.F.1.2 to establish the first of two long exact sequences of Ext modules associated to a given short exact sequence of R-modules. We also motivate another long exact sequence in Discussion II.F.2.2.

thm112617c

THEOREM II.F.2.1. *Let L be an R-module and let*

$$0 \longrightarrow N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \longrightarrow 0$$

be a short exact sequence of R-modules. There exists the following long exact sequence associated to $\operatorname{Ext}^i_R(L,-)$.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(L,N') \arrow[r] & \operatorname{Hom}_R(L,N) \arrow[r] & \operatorname{Hom}_R(L,N'') \arrow[dll, rounded corners] \\
 & \operatorname{Ext}^1_R(L,N') \arrow[r] & \operatorname{Ext}^1_R(L,N) \arrow[r] & \operatorname{Ext}^1_R(L,N'') \arrow[dll, rounded corners] \\
 & \cdots & \cdots \arrow[r] & \operatorname{Ext}^{i-1}_R(L,N'') \arrow[dll, rounded corners] \\
 & \operatorname{Ext}^i_R(L,N') \arrow[r] & \operatorname{Ext}^i_R(L,N) \arrow[r] & \operatorname{Ext}^i_R(L,N'') \arrow[dll, rounded corners] \\
 & \operatorname{Ext}^{i+1}_R(L,N') \arrow[r] & \cdots &
\end{tikzcd}$$

PROOF. Let $P_\bullet$ be a projective resolution for L. We claim that the R-complexes $\operatorname{Hom}_R(P_\bullet,N')$, $\operatorname{Hom}_R(P_\bullet,N)$, and $\operatorname{Hom}_R(P_\bullet,N'')$ form a short exact sequence of complexes, to which we may apply Theorem II.F.1.2 to achieve the desired result. See the diagram on the following page.

Since P_i is projective for all i, $\operatorname{Hom}_R(P_i,-)$ is exact for all i and therefore the rows are all exact. Furthermore the diagrams commute by the associativity of function composition. Hence we have a short exact sequence of R-complexes and the associated long exact sequence has the desired shape, since

$$H_{-i}(\operatorname{Hom}_R(P_\bullet,N^{(j)})) = \operatorname{Ext}^i_R(L,N^{(j)})$$

for all i and $j=0,1,2$.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(P_\bullet, N') \arrow[r, "\mathrm{Hom}_R(P_\bullet,\alpha)"] & \mathrm{Hom}_R(P_\bullet, N) \arrow[r, "\mathrm{Hom}_R(P_\bullet,\beta)"] & \mathrm{Hom}_R(P_\bullet, N'') \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_R(P_0, N') \arrow[r, "\mathrm{Hom}_R(P_0,\alpha)", "\alpha_*"'] \arrow[d, "\mathrm{Hom}_R(\partial_1^P, N')"', "(\partial_1^P)^*"] & \mathrm{Hom}_R(P_0, N) \arrow[r, "\mathrm{Hom}_R(P_0,\beta)", "\beta_*"'] \arrow[d, "\mathrm{Hom}_R(\partial_1^P, N)"', "(\partial_1^P)^*"] & \mathrm{Hom}_R(P_0, N'') \arrow[r] \arrow[d, "\mathrm{Hom}_R(\partial_1^P, N'')"', "(\partial_1^P)^*"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(P_1, N') \arrow[r, "\mathrm{Hom}_R(P_1,\alpha)"] \arrow[d] & \mathrm{Hom}_R(P_1, N) \arrow[r, "\mathrm{Hom}_R(P_1,\beta)"] \arrow[d] & \mathrm{Hom}_R(P_1, N'') \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_R(P_i, N') \arrow[r] \arrow[d] & \mathrm{Hom}_R(P_i, N) \arrow[r] \arrow[d] & \mathrm{Hom}_R(P_i, N'') \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

□

disc112817a

DISCUSSION II.F.2.2. Here we describe how one might obtain the other long exact sequence from Theorem II.B.1.1, namely

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(N'', L) \arrow[r] & \mathrm{Hom}_R(N, L) \arrow[r] & \mathrm{Hom}_R(N', L) \arrow[dll, rounded corners] & \\
 & \mathrm{Ext}^i_R(N'', L) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^i_R(N, L) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^i_R(N', L) \arrow[r] \arrow[d, equal] & \cdots \\
 & H_{-i}(\mathrm{Hom}_R(Q''_\bullet, L)) & H_{-i}(\mathrm{Hom}_R(Q_\bullet, L)) & H_{-i}(\mathrm{Hom}_R(Q'_\bullet, L)) &
\end{tikzcd}$$

where $Q''_\bullet$, $Q_\bullet$, and $Q'_\bullet$ are projective resolutions of N'', N, and N', respectively. For this we would need a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(Q''_\bullet, L) \longrightarrow \mathrm{Hom}_R(Q_\bullet, L) \longrightarrow \mathrm{Hom}_R(Q'_\bullet, L) \longrightarrow 0$$

which requires a short exact sequence

$$0 \longrightarrow Q'_\bullet \longrightarrow Q_\bullet \longrightarrow Q''_\bullet \longrightarrow 0 \tag{$\dagger$}$$ eqn050818a

such that $\mathrm{Hom}_R(\dagger, L)$ is exact. Note that if there exists a short exact sequence ($\dagger$), then it actually follows that $\mathrm{Hom}_R(\dagger, L)$ is exact by the following. Consider an arbitrary row of ($\dagger$).

$$0 \longrightarrow Q'_i \longrightarrow Q_i \longrightarrow Q''_i \longrightarrow 0 \tag{$\ddagger$}$$ eqn050818b

Since Q''_i is projective, the sequence ($\ddagger$) splits, so $\mathrm{Hom}_R(\ddagger, L)$ is split exact (and therefore exact). So given a short exact sequence $0 \longrightarrow N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \longrightarrow 0$, we want to construct a short exact sequence of projective resolutions as in ($\dagger$). The good news is we already have a means of lifting α and β to chain maps $Q'_\bullet \xrightarrow{A} Q_\bullet$ and $Q_\bullet \xrightarrow{B} Q''_\bullet$, respectively. However, the resulting short sequence is not exact in general. We let this serve as motivation for the horseshoe lemma in the next section.

II.F.3. The Horseshoe Lemma and Second Long Exact Sequence in Ext

ection062921t

In this section we prove the Horseshoe Lemma (Lemma II.F.3.2) and use it to prove the existence of the long exact sequence described in Discussion II.F.2.2.

lem112817b LEMMA II.F.3.1. *Consider a short exact sequence of R-modules and R-module homomorphisms.*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

Let $\tau' : P' \to M'$ and $\tau'' : P'' \to M''$ be surjections where P' and P'' are projective. There is a commutative diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r, "\epsilon"] \arrow[d, "\tau'"] & P' \oplus P'' \arrow[r, "\pi"] \arrow[d, "\tau"] & P'' \arrow[r] \arrow[d, "\tau''"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd} \tag{II.F.3.1.1}$$

eqn112618a

where ϵ and π are the natural injection and surjection, respectively.

PROOF. Use the fact that P'' is projective (see Definition II.A.1.14) to find an R-module homomorphism $h : P'' \to M$ making the following diagram commute.

$$\begin{tikzcd}
& P'' \arrow[dl, "h"'] \arrow[d, "\tau''"] & \\
M \arrow[r, "g"] & M'' \arrow[r] & 0
\end{tikzcd}$$

Define $\tau : P' \oplus P'' \to M$ by the formula

$$\tau(p', p'') = f(\tau'(p')) + h(p'').$$

The map τ is well defined by construction. Let $\alpha, \beta \in P'$, $\xi, \zeta \in P''$, and $r \in R$. We check that τ is an R-module homomorphism below.

$$\begin{aligned}
\tau(r(\alpha, \xi) + (\beta, \zeta)) &= \tau(r\alpha + \beta\ ,\ r\xi + \zeta) \\
&= f(\tau'(r\alpha + \beta)) \ + \ h(r\xi + \zeta) \\
&= f(r \cdot \tau'(\alpha) + \tau'(\beta)) \ + \ r \cdot h(\xi) + h(\zeta) \\
&= r \cdot f(\tau'(\alpha)) + f(\tau'(\beta)) \ + \ r \cdot h(\xi) + h(\zeta) \\
&= r \cdot [f(\tau'(\alpha)) + h(\xi)] \ + \ [f(\tau'(\beta)) + h(\zeta)] \\
&= r \cdot \tau(\alpha, \xi) \ + \ \tau(\beta, \zeta)
\end{aligned}$$

We also show τ makes (II.F.3.1.1) commute. For any $p' \in P'$ we have

$$\tau(\epsilon(p')) = \tau(p', 0) = f(\tau'(p'))$$

so the square on the left side commutes. For any $(p', p'') \in P' \oplus P''$ we have

$$\begin{aligned}
&\tau''(\pi(p', p'')) = \tau''(p'') \\
&g[\tau(p', p'')] = g[f(\tau'(p')) + h(p'')] = (g \circ f)(\tau'(p')) + g[h(p'')] = 0 + g(h(p''))
\end{aligned}$$

where the zero in the last step comes from the exactness of the given short exact sequence. The two results are equal by definition of the map h. Therefore the square on the right in (II.F.3.1.1) commutes.

Since τ' and τ'' are each surjective the left and right columns of (II.F.3.1.1) are exact. Moreover, the Snake Lemma (see Remark II.F.1.4) shows that τ must be surjective as well and the center column is exact, completing the proof. $\square$

lem112817c LEMMA II.F.3.2 (Horseshoe Lemma). *Consider the short exact sequence of R-modules and R-module homomorphisms.*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

Let $P'_\bullet$ and $P''_\bullet$ be projective resolutions of M' and M'', respectively. There is a commutative diagram with exact rows

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial_2^{P'}"] & \vdots \arrow[d, "\partial_2^{P}"] & \vdots \arrow[d, "\partial_2^{P''}"] & \\
0 \arrow[r] & P'_1 \arrow[r, "F_1"] \arrow[d, "\partial_1^{P'}"] & P_1 \arrow[r, "G_1"] \arrow[d, "\partial_1^{P}"] & P''_1 \arrow[r] \arrow[d, "\partial_1^{P''}"] & 0 \\
0 \arrow[r] & P'_0 \arrow[r, "F_0"] \arrow[d, "\tau'"] & P_0 \arrow[r, "G_0"] \arrow[d, "\tau"] & P''_0 \arrow[r] \arrow[d, "\tau''"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

such that the middle column is an augmented projective resolution of M.

Proof. Note that each row of the diagram, aside from the bottom row, will be split since each P''_i is projective for all $i \in \mathbb{N}$. Using Lemma II.F.3.1 we construct a commutative diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & P'_0 \arrow[r, "F_0"] \arrow[d, "\tau'"] & P'_0 \oplus P''_0 \arrow[r, "G_0"] \arrow[d, "\tau"] & P''_0 \arrow[r] \arrow[d, "\tau''"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where F_0 and G_0 are the natural injection and surjection (ϵ and π from the lemma), respectively. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(\tau') \arrow[r, "f_1"] \arrow[d, "\subseteq"'] & \ker(\tau) \arrow[r, "g_1"] \arrow[d, "\subseteq"'] & \ker(\tau'') \arrow[r] \arrow[d, "\subseteq"'] & 0 \\
0 \arrow[r] & P'_0 \arrow[r, "F_0"] \arrow[d, "\tau'"'] & P_0 \arrow[r, "G_0"] \arrow[d, "\tau"'] & P''_0 \arrow[r] \arrow[d, "\tau''"'] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{II.F.3.2.1}$$

eqn112618b

where f_1 and g_1 are induced by f and g, respectively, and $P_0 = P'_0 \oplus P''_0$. The columns are exact by construction and the top row is exact by the Snake Lemma (II.F.1.3), because the cokernel of a surjection is zero. Hence we have exactness everywhere.

For ease of notation, let $M'_1 = \ker(\tau')$, $M_1 = \ker(\tau)$, and $M''_1 = \ker(\tau'')$. We may apply Lemma II.F.3.1 again to build another commutative diagram with exact rows and columns, defining M'_2, M_2, and M''_2

similarly.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_2' \arrow[r, "f_2"] \arrow[d, "\subseteq"'] & M_2 \arrow[r, "g_2"] \arrow[d, "\subseteq"'] & M_2'' \arrow[r] \arrow[d, "\subseteq"'] & 0 \\
0 \arrow[r] & P_1' \arrow[r, "F_1"] \arrow[d, "\tau_1'"'] & P_1 \arrow[r, "G_1"] \arrow[d, "\tau_1"'] & P_1'' \arrow[r] \arrow[d, "\tau_1''"'] & 0 \\
0 \arrow[r] & M_1' \arrow[r, "f_1"] \arrow[d] & M_1 \arrow[r, "g_1"] \arrow[d] & M_1'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd} \tag{II.F.3.2.2}$$

eqn112618c

Splicing (II.F.3.2.1) and (II.F.3.2.2) together, we obtain a slightly larger diagram with rows and columns still exact.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_2' \arrow[r, "f_2"] \arrow[d, "\subseteq"'] & M_2 \arrow[r, "g_2"] \arrow[d, "\subseteq"'] & M_2'' \arrow[r] \arrow[d, "\subseteq"'] & 0 \\
0 \arrow[r] & P_1' \arrow[r, "F_1"] \arrow[d, "\tau_1'"'] & P_1 \arrow[r, "G_1"] \arrow[d, "\tau_1"'] & P_1'' \arrow[r] \arrow[d, "\tau_1''"'] & 0 \\
0 \arrow[r] & P_0' \arrow[r, "F_0"] \arrow[d, "\tau'"'] & P_0 \arrow[r, "G_0"] \arrow[d, "\tau"'] & P_0'' \arrow[r] \arrow[d, "\tau''"'] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

We have cheated a bit by using the names τ_1', τ_1, and τ_1'', but note that there are copies of M_1', M_1, and M_1'' sitting inside of P_0', P_0, and P_0'', respectively. We may repeat this construction inductively to achieve the desired diagram. $\square$

With the Horseshoe Lemma established, we are able to give the long exact sequence we described in Discussion II.F.2.2.

thm101318a

THEOREM II.F.3.3. *Let L be an R-module and let*

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

be a sequence of R-modules. There exists the following long exact sequence associated to $\operatorname{Ext}^i_R(-,L)$.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(N'',L) \arrow[r] & \operatorname{Hom}_R(N,L) \arrow[r] & \operatorname{Hom}_R(N',L) \arrow[dll, rounded corners] \\
& \operatorname{Ext}^1_R(N'',L) \arrow[r] & \operatorname{Ext}^1_R(N,L) \arrow[r] & \operatorname{Ext}^1_R(N',L) \arrow[dll, rounded corners] \\
& \cdots & \cdots \arrow[r] & \operatorname{Ext}^{i-1}_R(N',L) \arrow[dll, rounded corners] \\
& \operatorname{Ext}^i_R(N'',L) \arrow[r] & \operatorname{Ext}^i_R(N,L) \arrow[r] & \operatorname{Ext}^i_R(N',L) \arrow[dll, rounded corners] \\
& \operatorname{Ext}^{i+1}_R(N'',L) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. By Discussion II.F.2.2 we need only justify the existence of a short exact sequence

$$0 \longrightarrow Q'_\bullet \longrightarrow Q_\bullet \longrightarrow Q''_\bullet \longrightarrow 0$$

of R-complexes where $Q'_\bullet$, $Q_\bullet$, and $Q''_\bullet$ are projective resolutions of N', N, and N'', respectively. This has just been shown in the Horseshoe Lemma above, so the proof is done. $\square$

ection062921u

II.F.4. Mapping Cones

In this section we explore mapping cones and quasiisomorphisms. Both are needed for Lemmas II.F.5.1 and II.F.5.3, which are each used directly to prove Ext is well-defined (Theorem II.F.5.2). Proposition II.F.4.9 and Lemma II.F.4.13 from Schanuel are also used directly in the proof of the well-definedness of Ext.

def120417i

Definition II.F.4.1. Let $X_\bullet$ be an R-complex. The shift of $X_\bullet$, or the suspension of $X_\bullet$, is denoted $\Sigma X_\bullet$ where

$$(\Sigma X)_i = X_{i-1} \qquad \text{and} \qquad \partial_i^{\Sigma X} = -\partial_{i-1}^X.$$

rmk120417j

Remark II.F.4.2. We line up the R-complex $X_\bullet$ with its shift.

$$X_\bullet = \qquad \cdots \xrightarrow{\partial_{i+1}^X} X_i \xrightarrow{\partial_i^X} X_{i-1} \xrightarrow{\partial_{i-1}^X} \cdots$$

$$\Sigma X_\bullet = \qquad \cdots \xrightarrow{-\partial_i^X} X_{i-1} \xrightarrow{-\partial_{i-1}^X} X_{i-2} \xrightarrow{-\partial_{i-2}^X} \cdots$$

We now verify that the shift of $X_\bullet$ is itself an R-complex and that

$$H_i(\Sigma X_\bullet) = H_{i-1}(X_\bullet).$$

Colloquially, we want to verify that the homology of a shift is just a shift in the homology. Certainly $\Sigma X_\bullet$ is a sequence of R-module homomorphisms and since $X_\bullet$ is an R-complex we also have

$$-\partial_{i-1}^X \circ -\partial_i^X = \partial_{i-1}^X \circ \partial_i^X = 0.$$

Hence $\Sigma X_\bullet$ is an R-complex. By definition of homology we have

$$H_{i-1}(X_\bullet) = \frac{\operatorname{Ker} \partial_{i-1}^X}{\operatorname{Im} \partial_i^X} \qquad\qquad H_i(\Sigma X_\bullet) = \frac{\operatorname{Ker} -\partial_{i-1}^X}{\operatorname{Im} -\partial_i^X}$$

and these two are equal since $\operatorname{Ker} -\partial_{i-1}^X = \operatorname{Ker} \partial_{i-1}^X$ and $\operatorname{Im} -\partial_i^X = \operatorname{Im} \partial_i^X$.

def120417k

Definition II.F.4.3. Let $f_\bullet : X_\bullet \longrightarrow Y_\bullet$ be a chain map. We define the mapping cone of $f_\bullet$ as

$$\operatorname{Cone}(f_\bullet) = \qquad \cdots \longrightarrow \begin{matrix} Y_i \\ \oplus \\ X_{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix}} \begin{matrix} Y_{i-1} \\ \oplus \\ X_{i-2} \end{matrix} \longrightarrow \cdots$$

where for every $i \in \mathbb{Z}$

$$\operatorname{Cone}(f_\bullet)_i = \begin{matrix} Y_i \\ \oplus \\ X_{i-1} \end{matrix}$$

and

$$\partial_i^{\operatorname{Cone}(f_\bullet)} \begin{pmatrix} y_i \\ x_{i-1} \end{pmatrix} = \begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix} \begin{pmatrix} y_i \\ x_{i-1} \end{pmatrix} = \begin{pmatrix} \partial_i^Y(y_i) + f_{i-1}(x_{i-1}) \\ -\partial_{i-1}^X(x_{i-1}) \end{pmatrix}.$$

prop120517a

Proposition II.F.4.4. *If $f_\bullet : X_\bullet \longrightarrow Y_\bullet$ is a chain map, then* $\operatorname{Cone}(f_\bullet)$ *is an R-complex.*

Proof. First we verify that the cone is a sequence of R-module homomorphisms. Each element $\operatorname{Cone}(f_\bullet)_i$ is a direct sum of two R-modules so is itself an R-module. Taking an arbitrary $r \in R$ and two elements from $\operatorname{Cone}(f_\bullet)_i$ we observe

$$\begin{aligned}
\partial_i^{\operatorname{Cone}(f_\bullet)} \left(r \begin{pmatrix} y_i \\ x_{i-1} \end{pmatrix} + \begin{pmatrix} y_i' \\ x_{i-1}' \end{pmatrix} \right) &= \begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix} \begin{pmatrix} r y_i + y_i' \\ r x_{i-1} + x_{i-1}' \end{pmatrix} \\
&= \begin{pmatrix} \partial_i^Y(r y_i + y_i') + f_{i-1}(r x_{i-1} + x_{i-1}') \\ -\partial_{i-1}^X(r x_{i-1} + x_{i-1}') \end{pmatrix} \\
&= \begin{pmatrix} r\partial_i^Y(y_i) + \partial_i^Y(y_i') + r f_{i-1}(x_{i-1}) + f_{i-1}(x_{i-1}') \\ r \cdot -\partial_{i-1}^X(x_{i-1}) - \partial_{i-1}^X(x_{i-1}') \end{pmatrix} \\
&= \begin{pmatrix} r\partial_i^Y(y_i) + r f_{i-1}(x_{i-1}) \\ r \cdot -\partial_{i-1}^X(x_{i-1}) \end{pmatrix} + \begin{pmatrix} \partial_i^Y(y_i') + f_{i-1}(x_{i-1}') \\ -\partial_{i-1}^X(x_{i-1}') \end{pmatrix} \\
&= r \cdot \begin{pmatrix} \partial_i^Y(y_i) + f_{i-1}(x_{i-1}) \\ -\partial_{i-1}^X(x_{i-1}) \end{pmatrix} + \begin{pmatrix} \partial_i^Y(y_i') + f_{i-1}(x_{i-1}') \\ -\partial_{i-1}^X(x_{i-1}') \end{pmatrix} \\
&= r \cdot \partial_i^{\operatorname{Cone}(f_\bullet)} \begin{pmatrix} y_i \\ x_{i-1} \end{pmatrix} + \partial_i^{\operatorname{Cone}(f_\bullet)} \begin{pmatrix} y_i' \\ x_{i-1}' \end{pmatrix}.
\end{aligned}$$

Since the well-definedness of $\partial_i^{\operatorname{Cone}(f_\bullet)}$ is a direct consequence of the well-definedness of the maps ∂_i^Y, f_{i-1}, and ∂_{i-1}^X for each $i \in \mathbb{Z}$, we conclude each $\partial_i^{\operatorname{Cone}(f_\bullet)}$ is an R-module homomorphism. Moreover

$$\begin{aligned}
\partial_i^{\operatorname{Cone}(f_\bullet)} \circ \partial_{i+1}^{\operatorname{Cone}(f_\bullet)} &= \begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix} \begin{pmatrix} \partial_{i+1}^Y & f_i \\ 0 & -\partial_i^X \end{pmatrix} \\
&= \begin{pmatrix} \partial_i^Y \circ \partial_{i+1}^Y & \partial_i^Y \circ f_i - f_{i-1} \circ \partial_i^X \\ 0 & \partial_{i-1}^X \circ \partial_i^X \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

The $(1,1)$-entry of the composition is zero, because $Y_\bullet$ is an R-complex and similarly for the $(2,2)$-entry. The $(1,2)$-entry is zero, because $f_\bullet$ is a chain map. This concludes the proof. $\square$

ex120517b

Example II.F.4.5. Here we introduce some special cases of the Koszul complex. (See Section II.G.3 for more on this topic.) Fix an element $x \in R$ and define the R-complex

$$X_\bullet \qquad 0 \longrightarrow R \xrightarrow{x\cdot} R \longrightarrow 0\,.$$

Fix another element $y \in R$ and define the following chain map.

$$\begin{tikzcd}
X_\bullet \arrow[d, "y_\bullet"'] & & 0 \arrow[r] & R \arrow[r, "x\cdot"] \arrow[d, "y\cdot"'] \arrow[dr, phantom, "\circlearrowleft"] & R \arrow[r] \arrow[d, "y\cdot"] & 0 \arrow[r, dashed] & 0 \\
X_\bullet & 0 \arrow[r, dashed] & 0 \arrow[r] & R \arrow[r, "x\cdot"] & R \arrow[r] & 0 &
\end{tikzcd}$$

We can compute the mapping cone of $f_\bullet$.

$$
\begin{aligned}
\operatorname{Cone}(y_\bullet) &= \quad \begin{array}{c}0\\ \oplus\\ 0\end{array} \longrightarrow \begin{array}{c}0\\ \oplus\\ R\end{array} \xrightarrow{\begin{pmatrix}0 & y\\ 0 & -x\end{pmatrix}} \begin{array}{c}R\\ \oplus\\ R\end{array} \xrightarrow{\begin{pmatrix}x & y\\ 0 & 0\end{pmatrix}} \begin{array}{c}R\\ \oplus\\ 0\end{array} \longrightarrow \begin{array}{c}0\\ \oplus\\ 0\end{array} \\
&\cong \quad 0 \longrightarrow R \xrightarrow{\begin{pmatrix}y\\ -x\end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix}x & y\end{pmatrix}} R \longrightarrow 0
\end{aligned}
$$

It is sensible that we should call this a complex, since

$$
\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}y\\ -x\end{pmatrix} = xy - xy = 0.
$$

prop120517c PROPOSITION II.F.4.6. *Let $f_\bullet : X_\bullet \longrightarrow Y_\bullet$ be a chain map.*

prop120517c.a (a) *There is a chain map $\epsilon_\bullet : Y_\bullet \longrightarrow \operatorname{Cone}(f_\bullet)$ defined as the sequence of natural injections*

$$
\epsilon_i : Y_i \hookrightarrow \begin{array}{c}Y_i\\ \oplus\\ X_{i-1}\end{array} = \operatorname{Cone}(f_\bullet)_i\ .
$$

prop120517c.b (b) *There is a chain map $\tau_\bullet : \operatorname{Cone}(f_\bullet) \longrightarrow \Sigma X_\bullet$ defined as the sequence of natural surjections*

$$
\tau_i : \operatorname{Cone}(f_\bullet) = \begin{array}{c}Y_i\\ \oplus\\ X_{i-1}\end{array} \longrightarrow X_{i-1} = (\Sigma X_\bullet)_i\ .
$$

prop120517c.c (c) *The following sequence is exact.*

$$
0 \longrightarrow Y_\bullet \xrightarrow{\epsilon_\bullet} \operatorname{Cone}(f_\bullet) \xrightarrow{\tau_\bullet} \Sigma X_\bullet \longrightarrow 0
$$

prop120517c.d (d) *In the associated long exact sequence, the connecting map*

$$
\eth_i : H_i\left(\Sigma X_\bullet\right) \longrightarrow H_{i-1}(Y_\bullet)
$$

is equal to $H_{i-1}(f_\bullet)$.

PROOF. To prove the first three parts, it suffices to fix an arbitrary $i \in \mathbb{Z}$ and show the following diagram is commutative with exact rows.

$$
\begin{tikzcd}
0 \arrow[r] & Y_i \arrow[r, "\begin{pmatrix}1\\ 0\end{pmatrix}", "\epsilon_i"'] \arrow[d, "\partial_i^Y"'] & \begin{array}{c}Y_i\\ \oplus\\ X_{i-1}\end{array} \arrow[r, "\begin{pmatrix}0 & 1\end{pmatrix}", "\tau_i"'] \arrow[d, "\begin{pmatrix}\partial_i^Y & f_{i-1}\\ 0 & -\partial_{i-1}^X\end{pmatrix}"'] & X_{i-1} \arrow[r] \arrow[d, "-\partial_{i-1}^X"'] & 0 \\
0 \arrow[r] & Y_{i-1} \arrow[r, "\epsilon_{i-1}"'] & \begin{array}{c}Y_{i-1}\\ \oplus\\ X_{i-2}\end{array} \arrow[r, "\tau_{i-2}"'] & X_{i-2} \arrow[r] & 0
\end{tikzcd}
$$

The rows are exact by Example II.A.1.4. We check commutivity of the two squares by tracking arbitrary elements around each and thereby complete the proof of parts (a), (b), and (c).

$$\begin{tikzcd}
y \arrow[r, mapsto] \arrow[d, mapsto] & \begin{pmatrix} y \\ 0 \end{pmatrix} \arrow[d, mapsto] & & \begin{pmatrix} y \\ x \end{pmatrix} \arrow[r, mapsto] \arrow[d, mapsto] & x \arrow[d, mapsto] \\
\partial_i^Y(y) \arrow[r] & \begin{pmatrix} \partial_i^Y(y) \\ 0 \end{pmatrix} & & \begin{pmatrix} \partial_i^Y(y) + f_{i-1}(x) \\ -\partial_{i-1}^X(x) \end{pmatrix} \arrow[r, mapsto] & -\partial_{i-1}^X(x)
\end{tikzcd}$$

(d) Note that $H_i(\Sigma X_\bullet) = H_{i-1}(X_\bullet)$ and let $\overline{x} \in H_{i-1}(X_\bullet)$ be arbitrary. Thus $x \in \operatorname{Ker} -\partial_{i-1}^X \subseteq X_{i-1}$ and we begin what has become standard for calculating $\eth$. We lift back to the element

$$\begin{pmatrix} 0 \\ x \end{pmatrix} \in \begin{array}{c} Y_i \\ \oplus \\ X_{i-1} \end{array}$$

which is a preimage of x under τ_i. It also holds

$$\begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} f_{i-1}(x) \\ -\partial_i^X(x) \end{pmatrix} = \begin{pmatrix} f_{i-1}(x) \\ 0 \end{pmatrix}$$

so we may lift to the element $f_{i-1}(x) \in Y_{i-1}$ for which we have

$$\epsilon_{i-1}(f_{i-1}(x)) = \begin{pmatrix} f_{i-1}(x) \\ 0 \end{pmatrix}.$$

Hence $\eth_i(\overline{x}) = \overline{f_{i-1}(x)} = H_{i-1}(f_\bullet)(\overline{x})$ as desired. $\square$

def120517d DEFINITION II.F.4.7. A chain map $f_\bullet \colon X_\bullet \longrightarrow Y_\bullet$ is a <u>quasiisomorphism</u> if the induced map on homology

$$H_i(f_\bullet) \colon H_i(X_\bullet) \longrightarrow H_i(Y_\bullet)$$

is an isomorphism, for all $i \in \mathbb{Z}$.

ex120517e EXAMPLE II.F.4.8. If $f_\bullet : X_\bullet \longrightarrow Y_\bullet$ is an isomorphism, then it is also a quasiisomorphism. To see the reason for this, consider that if $g_\bullet : Y_\bullet \longrightarrow X_\bullet$ is a two-sided inverse for $f_\bullet$, then the induced map on homology $H_i(g_\bullet) : H_i(Y_\bullet) \longrightarrow H_i(X_\bullet)$ is a two-sided inverse for $H_i(f_\bullet)$.

The converse of this, however, fails in general. By way of demonstration, let M be an R-module and let $P_\bullet$ be a projective resolution of M.

$$P_\bullet^+ = \qquad \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \longrightarrow 0$$

We may define also the following chain map, call it $\tau_\bullet$.

$$\begin{tikzcd}
P_\bullet = \arrow[d, "\tau_\bullet"'] & \cdots \arrow[r, "\partial_2^P"] & P_1 \arrow[r, "\partial_1^P"] \arrow[d] & P_0 \arrow[r] \arrow[d, "\tau"] & 0 \\
M_\bullet = & \cdots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

While $\tau_\bullet$ is not an isomorphism (since $P_1 \neq 0$ and M not projective, in general), we claim $\tau_\bullet$ is a quasiisomorphism. Since $P_\bullet$ is exact at P_i for all $i \neq 0$, for these i we have the silly isomorphism below.

$$H_i(\tau_\bullet) : 0 \longrightarrow 0$$

It suffices then to study the $i = 0$ position.

$$\begin{tikzcd}
H_0(\tau_\bullet) : H_0(P_\bullet) \arrow[r] \arrow[d, equal] & H_0(M_\bullet) \arrow[d, equal] \\
\frac{P_0}{\operatorname{Im} \partial_1^P} \arrow[r, "\hat{\tau}"] & \frac{M}{0}
\end{tikzcd}$$

Here $\hat{\tau}$ denotes the map induced by τ (see Proposition II.E.1.3). Since τ is surjective, $\hat{\tau}$ must also be surjective. Since $P_\bullet^+$ is exact, $\operatorname{Ker}\tau = \operatorname{Im}\partial_1^P$ and therefore $\hat{\tau}$ is also injective. Hence $H_0(\tau_\bullet)$ is an isomorphism and $\tau_\bullet$ is a quasiisomorphism as claimed.

prop120517f PROPOSITION II.F.4.9. *A chain map $f_\bullet : X_\bullet \longrightarrow Y_\bullet$ is a quasiisomorphism if and only if* $\operatorname{Cone}(f_\bullet)$ *is exact.*

PROOF. Consider the long exact sequence

$$\cdots \longrightarrow H_i(\operatorname{Cone}(f_\bullet)) \longrightarrow H_{i-1}(X_\bullet) \longrightarrow H_{i-1}(Y_\bullet) \longrightarrow H_{i-1}(\operatorname{Cone}(f_\bullet)) \longrightarrow H_{i-2}(X_\bullet) \longrightarrow H_{i-2}(Y_\bullet) \longrightarrow \cdots$$

from the mapping cone (see Proposition II.F.4.6 and Theorem II.F.1.2), where the connecting homomorphisms are $\eth_i = H_{i-1}(f_\bullet)$. If we suppose $f_\bullet$ is a quasiisomorphism, then by definition $\eth_i$ is an isomorphism for all i and it follows from Lemma II.F.4.10 that

$$H_{i-1}(\operatorname{Cone}(f_\bullet)) = 0.$$

On the other hand, if we suppose $\operatorname{Cone}(f_\bullet)$ is exact, then each section of our long exact sequence is of the form

$$0 \longrightarrow H_{i-1}(X_\bullet) \xrightarrow{H_i(f_\bullet)} H_{i-1}(Y_\bullet) \longrightarrow 0$$

where exactness at $H_{i-1}(X_\bullet)$ and $H_{i-1}(Y_\bullet)$ forces $\operatorname{Ker} H_i(f_\bullet) = 0$ and $\operatorname{Im} H_i(f_\bullet) = G$, respectively. Hence $H_i(f_\bullet)$ is an isomorphism and $f_\bullet$ is a quasiisomorphism by definition. $\square$

lem120617a LEMMA II.F.4.10. *Given A, B, C, D, and E are R-modules and given the exact sequence*

$$A \xrightarrow{\cong} B \xrightarrow{f} C \xrightarrow{g} D \xrightarrow{\cong} E$$

it follows that $C = 0$.

PROOF. The isomorphism on the left forces $\ker(f) = B$, implying $\ker(g) = \operatorname{Im} f = 0$. The other isomorphism forces $\operatorname{Im} g = 0$ and it follows that $C = 0$. $\square$

lem120617b LEMMA II.F.4.11. *Consider the following exact sequence with $n \geq 1$.*

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

If $P_0, \ldots, P_{n-1}$ are all projective, then K_n is projective as well.

PROOF. We tackle a few base cases first. If $n = 1$, then the exactness of the sequence implies $K_1 \cong P_0$ and K_1 is therefore projective. If $n = 2$, then since P_0 is projective, the sequence below splits.

$$0 \longrightarrow K_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

That is, $P_1 \cong K_2 \oplus P_0$. Since P_1 is projective, it follows that K_2 must also be projective (see Lemma II.F.4.12).

Assume now that $n \geq 3$ and the result holds for all sequences of length $n-1$. Our exact sequence is therefore of the form

$$0 \longrightarrow K_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow 0$$

which we may ‘slice’ using the kernel of the 1^{st} differential.

$$\begin{tikzcd}
 & & & & & & & 0 & \\
 & & & & & & P_0 \arrow[ur] & & \\
0 \arrow[r] & K_n \arrow[r] & \cdots \arrow[r] & P_2 \arrow[rr, "\partial_2"] \arrow[dr] & & P_1 \arrow[ur, "\partial_1"] \arrow[rr, "\partial_1"'] & & P_0 \arrow[r] & 0 \\
 & & & & K_2 \arrow[ur, "\subseteq"'] \arrow[dr] & = \operatorname{Im}\partial_2 = \operatorname{Ker}\partial_1 & & & \\
 & & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

The diagonal we have constructed guarantees K_2 is projective by the $n = 2$ base case. Since the exact sequence

$$0 \longrightarrow K_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow K_2 \longrightarrow 0$$

is therefore covered under the induction hypothesis, we conclude K_n is projective as well. □

lem120617c

LEMMA II.F.4.12. *Two R-modules A and B are projective if and only if $A \oplus B$ is projective.*

PROOF. Let $\mathcal{S}$ be an arbitrary exact sequence of R-modules.

$$\mathcal{S} = \qquad \cdots \xrightarrow{\partial^{\mathcal{S}}_{i+2}} S_{i+1} \xrightarrow{\partial^{\mathcal{S}}_{i+1}} S_i \xrightarrow{\partial^{\mathcal{S}}_{i}} S_{i-1} \xrightarrow{\partial^{\mathcal{S}}_{i-1}} \cdots$$

We will show

$$\operatorname{Hom}_R(A \oplus B, \mathcal{S}) \cong \operatorname{Hom}_R(A, \mathcal{S}) \oplus \operatorname{Hom}_R(B, \mathcal{S})$$

as R-complexes. For each $i \in \mathbb{Z}$ define the map

$$\begin{aligned} F_i : \operatorname{Hom}_R(A \oplus B, S_i) &\longrightarrow \operatorname{Hom}_R(A, S_i) \oplus \operatorname{Hom}_R(B, S_i) \\ \rho &\longmapsto (\rho_A, \rho_B) \end{aligned}$$

where

$$\begin{aligned} \rho_A : A &\longrightarrow S_i \\ a &\longmapsto \rho(a, 0) \end{aligned} \qquad\qquad \begin{aligned} \rho_B : B &\longrightarrow S_i \\ b &\longmapsto \rho(0, b). \end{aligned}$$

Since ρ_A and ρ_B are compositions of ρ with natural inclusions, each is a well-defined R-module homomorphism and therefore F_i is also a well-defined function. It is straightforward to show that F_i is also R-linear. Each F_i is also surjective since for any $(\alpha, \beta) \in \operatorname{Hom}_R(A, S_i) \oplus \operatorname{Hom}_R(B, S_i)$ we may define

$$\begin{aligned} \gamma : A \oplus B &\longrightarrow S_i \\ (a, b) &\longmapsto \alpha(a) + \beta(b) \end{aligned}$$

for which

$$F_i(\gamma) = (\gamma_A, \gamma_B) = (\alpha, \beta).$$

Consider also that if $F_i(\rho) = 0$, then $\rho_A = 0^A_{S_i}$ and $\rho_B = 0^B_{S_i}$ and hence $\rho = 0^{A\oplus B}_{S_i}$, so F_i is injective (refer to Fact II.C.5.10 for 0^-_- notation).

Therefore the isomorphism of R-complexes will follow once we have verified the commutivity of the following diagram.

$$\begin{tikzcd}
& \vdots \arrow[dd] & \vdots \arrow[dd] & \\
\gamma \arrow[rrr, mapsto, bend left] \arrow[ddd, mapsto, bend right] & & & (\gamma_A, \gamma_B) \arrow[dd, mapsto, bend left] \\
& \operatorname{Hom}_R(A \oplus B, S_i) \arrow[r, "F_i"] \arrow[d, "(\partial_i^S)_*"'] & \operatorname{Hom}_R(A, S_i) \oplus \operatorname{Hom}_R(B, S_i) \arrow[d, "(\partial_i^S)_* \oplus (\partial_i^S)_*"] & \\
& \operatorname{Hom}_R(A \oplus B, S_{i-1}) \arrow[r, "F_{i-1}"'] \arrow[dd] & \operatorname{Hom}_R(A, S_{i-1}) \oplus \operatorname{Hom}_R(B, S_{i-1}) \arrow[dd] & \big((\partial_i^S \circ \gamma_A), (\partial_i^S \circ \gamma_B)\big) \arrow[d, phantom, "\|?"] \\
\partial_i^S \circ \gamma \arrow[rrr, mapsto, bend right] & & & \big((\partial_i^S \circ \gamma)_A, (\partial_i^S \circ \gamma)_B\big) \\
& \vdots & \vdots &
\end{tikzcd}$$

To this end, consider that for any $a \in A$ we have

$$(\partial_i^S \circ \gamma_A)(a) = \partial_i^S(\gamma_A(a)) = \partial_i^S(\gamma(a, 0)) = (\partial_i^S \circ \gamma)(a, 0) = (\partial_i^S \circ \gamma)_A(a).$$

Similarly, for any $b \in B$ we have

$$(\partial_i^S \circ \gamma_B)(b) = \partial_i^S(\gamma_B(b)) = \partial_i^S(\gamma(b, 0)) = (\partial_i^S \circ \gamma)(b, 0) = (\partial_i^S \circ \gamma)_B(b).$$

Hence $F_\bullet$ is an isomorphism of R-complexes. Therefore we have

$$\begin{aligned}
\operatorname{Hom}_R(A \oplus B, \mathcal{S}) \text{ exact } &\iff \operatorname{Hom}_R(A, \mathcal{S}) \oplus \operatorname{Hom}_R(B, \mathcal{S}) \text{ exact} \\
&\iff \operatorname{Hom}_R(A, \mathcal{S}),\ \operatorname{Hom}_R(B, \mathcal{S}) \text{ both exact}
\end{aligned}$$

and therefore $A \oplus B$ is projective if and only if A and B are both projective. □

lem120617d

Lemma II.F.4.13 (Schanuel). *Consider exact sequences*

$$0 \longrightarrow K_n \xrightarrow{\partial_n^P} P_{n-1} \xrightarrow{\partial_{n-1}^P} \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \longrightarrow 0$$

$$0 \longrightarrow L_n \xrightarrow{\partial_n^Q} Q_{n-1} \xrightarrow{\partial_{n-1}^Q} \cdots \xrightarrow{\partial_2^Q} Q_1 \xrightarrow{\partial_1^Q} Q_0 \xrightarrow{\pi} M \longrightarrow 0$$

such that $P_0, \ldots, P_{n-1}, Q_0, \ldots, Q_{n-1}$ are all projective. Then

$$K_n \textit{ projective} \iff L_n \textit{ projective.}$$

Proof. By the proof of Proposition II.E.2.2, we can lift the identity map id_M to build a chain map between the two sequences. That is, there exist R-module homomorphisms $f_0, \ldots, f_n$ that make the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r] \arrow[d, "f_n"] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
0 \arrow[r] & L_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Let f_i be the zero map for all $i \notin \{0, \ldots, n\}$ and truncate the two resolutions. We have a chain map $f_\bullet \colon P_\bullet \longrightarrow Q_\bullet$ in the display below.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & K_n \arrow[r] \arrow[d, "f_n"] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & L_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

As in Example II.F.4.8, one can check that $f_\bullet$ is a quasiisomorphism and thus by Proposition II.F.4.9 we know $\mathrm{Cone}(f_\bullet)$ is exact, which we write below.

$$0 \longrightarrow K_n \longrightarrow \begin{array}{c} L_n \\ \oplus \\ P_{n-1} \end{array} \longrightarrow \begin{array}{c} Q_{n-1} \\ \oplus \\ P_{n-2} \end{array} \longrightarrow \cdots \longrightarrow \begin{array}{c} Q_1 \\ \oplus \\ P_0 \end{array} \longrightarrow Q_0 \longrightarrow 0$$

If we assume L_n is projective, then $L_n \oplus P_{n-1}$ is projective and K_n must also be projective under Lemma II.F.4.11.

Running this entire argument again having placed $Q_\bullet^+$ in the top of our ladder diagram would yield an identical result, so the forward implication is proven by symmetry. □

ection062921v

II.F.5. Well-Definedness of Ext

With all the necessary tools now in place, we finally prove that Ext is well-defined.

lem120617e

LEMMA II.F.5.1. *If $P_\bullet$ is an exact R-complex such that each P_i is projective and $P_i = 0$ for all $i < i_0$ for some fixed $i_0 \in \mathbb{Z}$, then for any R-module N, $\mathrm{Hom}_R(P_\bullet, N)$ is exact.*

PROOF. The given complex has the following form around the i_0 position.

$$P_\bullet = \qquad \cdots \xrightarrow{\partial^P_{i_0+2}} P_{i_0+1} \xrightarrow{\partial^P_{i_0+1}} P_{i_0} \longrightarrow 0 \longrightarrow \cdots$$

Let K_t denote $\mathrm{Ker}\,\partial^P_{t-1}$ and 'slice' the above exact sequence.

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & & & 0 & \\
& K_{t+1} \arrow[dr] & & & & K_{t-1} \arrow[ur] \arrow[dr] & & & & P_{i_0} \arrow[ur] \arrow[dr, "="] & & \\
\cdots \arrow[rr] & & P_t \arrow[rr, "\partial^P_t"] \arrow[dr] & & P_{t-1} \arrow[rr, "\partial^P_{t-1}"'] \arrow[ur] & & \cdots \arrow[rr, "\partial^P_{i_0+2}"] \arrow[dr] & & P_{i_0+1} \arrow[rr, "\partial^P_{i_0+1}"'] \arrow[ur] & & P_{i_0} \arrow[r] & 0 \\
& & & K_t \arrow[ur] \arrow[dr] & & & & K_{i_0+2} \arrow[ur] \arrow[dr] & & & & \\
& & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd} \tag{II.F.5.1.1}$$

eqn112618d

The diagonals are all exact and both P_{i_0}, P_{i_0+1} are projective, so by Lemma II.F.4.11, the module K_{i_0+2} is projective. If we let $t > i_0 + 2$ and assume K_{t-1} is projective, then since P_{t-1} is projective, the same lemma guarantees K_t is projective as well. Hence by induction K_t is projective for all $t \geq i_0 + 2$, implying $\mathrm{Hom}_R(D, N)$ is split exact for any R-module N, where D is any diagonal sequence in (II.F.5.1.1). Let $(-)^* = \mathrm{Hom}_R(-, N)$ and we have the following commutative diagram.

$$\begin{tikzcd}
0 & & & & 0 & & 0 \arrow[dl] & & & & 0 \arrow[dl] & \\
& K^*_{t+1} \arrow[ul] & & & & K^*_{t-1} \arrow[ul] \arrow[dl] & & & & P^*_{i_0} \arrow[dl] & & \\
\cdots & & P^*_t \arrow[ll] \arrow[ul] & & P^*_{t-1} \arrow[ll, "\partial^{P*}_t"'] \arrow[dl] & & \cdots \arrow[ll, "\partial^{P*}_{t-1}"] \arrow[ul] & & P^*_{i_0+1} \arrow[ll, "\partial^{P*}_{i_0+2}"'] \arrow[dl] & & P^*_{i_0} \arrow[ll, "\partial^{P*}_{i_0+1}"] \arrow[ul, "="'] & 0 \arrow[l] \\
& & & K^*_t \arrow[ul] \arrow[dl] & & & & K^*_{i_0+2} \arrow[ul] \arrow[dl] & & & & \\
& & 0 & & 0 \arrow[ul] & & 0 & & 0 \arrow[ul] & & &
\end{tikzcd}$$

Since the diagonals are exact and the diagrams all commute, a diagram chase shows that the row must also be exact. That is, $P^*_\bullet = \mathrm{Hom}_R(P_\bullet, N)$ is exact, as desired. □

Ladies and gentlemen, we have arrived:

tWellDefined

THEOREM II.F.5.2. Ext *is independent of choice of projective resolution.*

PROOF. Let $P_\bullet$ and $Q_\bullet$ be two projective resolutions of an R-module M and let $f_\bullet\colon P_\bullet \longrightarrow Q_\bullet$ be a lift of the identity on M (see Proposition II.E.2.2). From the work done in the proof of Lemma II.F.4.13, this implies $\mathrm{Cone}(f_\bullet)$ is exact.

$$\mathrm{Cone}(f_\bullet) = \qquad \cdots \longrightarrow \begin{matrix} Q_2 \\ \oplus \\ P_1 \end{matrix} \longrightarrow \begin{matrix} Q_1 \\ \oplus \\ P_0 \end{matrix} \longrightarrow Q_0 \longrightarrow 0$$

Since every module $Q_i \oplus P_{i-1}$ is projective, by Lemma II.F.5.1 we have $\mathrm{Hom}_R(\mathrm{Cone}(f_\bullet), N)$ exact for any R-module N. Moreover the shift $\Sigma\,\mathrm{Hom}_R(\mathrm{Cone}(f_\bullet), N)$ is also exact since

$$H_i\left(\Sigma\star\right) = H_{i-1}(\star).$$

By Lemma II.F.5.3

$$\Sigma\,\mathrm{Hom}_R(\mathrm{Cone}(f_\bullet), N) \cong \mathrm{Cone}\left(\mathrm{Hom}_R(f_\bullet, N)\right).$$

Hence by Proposition II.F.4.9 it follows that $\mathrm{Hom}_R(f_\bullet, N)$ is a quasiisomorphism and therefore the following is an isomorphism for any $i \in \mathbb{Z}$.

$$H_{-i}(\mathrm{Hom}_R(f_\bullet, N)) \;:\; H_{-i}(\mathrm{Hom}_R(Q_\bullet, N)) \xrightarrow{\cong} H_{-i}(\mathrm{Hom}_R(P_\bullet, N))\,.$$

This completes the proof. $\square$

lem051818a

LEMMA II.F.5.3. *Let R be a commutative ring with identity, let M be an R-module, and consider a chain map $F_\bullet : X_\bullet \to Y_\bullet$. Then*

$$\mathrm{Cone}(\mathrm{Hom}_R(F_\bullet, M)) \cong \Sigma\,\mathrm{Hom}_R(\mathrm{Cone}(F_\bullet), M).$$

PROOF. From our chain map

$$\begin{tikzcd} X_\bullet \arrow[d, "F_\bullet"'] & \cdots \arrow[r, "\partial^X_{i+1}"] & X_i \arrow[r, "\partial^X_i"] \arrow[d, "F_i"'] & X_{i-1} \arrow[r, "\partial^X_{i-1}"] \arrow[d, "F_{i-1}"'] & \cdots \\ Y_\bullet & \cdots \arrow[r, "\partial^Y_{i+1}"'] & Y_i \arrow[r, "\partial^Y_i"'] & Y_{i_1} \arrow[r, "\partial^Y_{i-1}"'] & \cdots \end{tikzcd}$$

we are able to write

$$\mathrm{Cone}(F_\bullet) = \qquad \cdots \longrightarrow \begin{matrix} Y_i \\ \oplus \\ X_{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} \partial^Y_i & F_{i-1} \\ 0 & -\partial^X_{i-1} \end{pmatrix}} \begin{matrix} Y_{i-1} \\ \oplus \\ X_{i-2} \end{matrix} \longrightarrow \cdots.$$

Applying the contravariant functor $\mathrm{Hom}_R(-, M)$ we write $\mathrm{Hom}_R(\mathrm{Cone}(F_\bullet), M)$ below.

$$\cdots \longrightarrow \mathrm{Hom}_R\left(\begin{matrix} Y_{-i} \\ \oplus \\ X_{-i-1} \end{matrix}, M\right) \xrightarrow{\begin{pmatrix} \partial^Y_{-i+1} & F_{-i} \\ 0 & -\partial^X_{-i} \end{pmatrix}^*} \mathrm{Hom}_R\left(\begin{matrix} Y_{-i+1} \\ \oplus \\ X_{-i} \end{matrix}, M\right) \longrightarrow \cdots$$

The shift $\Sigma\,\mathrm{Hom}_R(\mathrm{Cone}(F_\bullet), M)$ follows readily, which we write below.

$$\cdots \longrightarrow \mathrm{Hom}_R\left(\begin{matrix} Y_{-i+1} \\ \oplus \\ X_{-i} \end{matrix}, M\right) \xrightarrow{-\begin{pmatrix} \partial^Y_{-i+2} & F_{-i+1} \\ 0 & -\partial^X_{-i+1} \end{pmatrix}^*} \mathrm{Hom}_R\left(\begin{matrix} Y_{-i+2} \\ \oplus \\ X_{-i+1} \end{matrix}, M\right) \longrightarrow \cdots$$

Now we write down $\mathrm{Cone}(\mathrm{Hom}_R(F_\bullet, M))$. We begin with the induced chain map

$$\begin{tikzcd} \mathrm{Hom}_R(Y_\bullet, M) \arrow[d, "\mathrm{Hom}_R(F_\bullet, M)"'] & \cdots \arrow[r, "(\partial^Y_{-i})^*"] & \mathrm{Hom}_R(Y_{-i}, M) \arrow[r, "(\partial^Y_{-i+1})^*"] \arrow[d, "F^*_{-i}"'] & \mathrm{Hom}_R(Y_{-i+1}, M) \arrow[r, "(\partial^Y_{-i+2})^*"] \arrow[d, "F^*_{-i+1}"'] & \cdots \\ \mathrm{Hom}_R(X_\bullet, M) & \cdots \arrow[r, "(\partial^X_{-i})^*"'] & \mathrm{Hom}_R(X_{-i}, M) \arrow[r, "(\partial^X_{-i+1})^*"'] & \mathrm{Hom}_R(X_{-i+1}, M) \arrow[r, "(\partial^X_{-i+2})^*"'] & \cdots \end{tikzcd}$$

and take the cone

$$\mathrm{Cone}(\mathrm{Hom}_R(F_\bullet, M)) = \cdots \to \begin{array}{c}\mathrm{Hom}_R(X_{-i}, M)\\ \oplus \\ \mathrm{Hom}_R(Y_{-i+1}, M)\end{array} \xrightarrow{\begin{pmatrix} \left(\partial^X_{-i+1}\right)^* & F^*_{-i+1} \\ 0 & -\left(\partial^Y_{-i+2}\right)^* \end{pmatrix}} \begin{array}{c}\mathrm{Hom}_R(X_{-i+1}, M)\\ \oplus \\ \mathrm{Hom}_R(Y_{-i+2}, M)\end{array} \to \cdots$$

On the next page we write down explicitly the isomorphism between these two complexes, because what this document needs is another large commutative diagram. The vertical maps send $\begin{pmatrix} a & b \end{pmatrix}$ to $\begin{pmatrix} b \\ -a \end{pmatrix}$ and each is an isomorphism. Since the commutivity of the diagram is depicted as well, the diagram completes the proof. $\square$

$$
\begin{tikzcd}[column sep=small]
 & & \begin{pmatrix}\phi & \psi\end{pmatrix} \arrow[rr, mapsto, bend left] \arrow[d, phantom, sloped, "\in"] \arrow[ddd, mapsto, bend right=40] & & \begin{pmatrix}-\phi\circ\partial^Y_{-i+2} & -\phi\circ F_{-i+1}+\psi\circ\partial^X_{-i+1}\end{pmatrix} \arrow[ddd, mapsto, bend left=40] & \\
\Sigma\operatorname{Hom}_R(\operatorname{Cone}(F_\bullet),M) = \arrow[d] & \cdots \arrow[r] & \operatorname{Hom}_R\left(\begin{matrix}Y_{-i+1}\\ \oplus\\ X_{-i}\end{matrix}, M\right) \arrow[rr, "\begin{pmatrix}-\partial^Y_{-i+2} & -F_{-i+1}\\ 0 & \partial^X_{-i+1}\end{pmatrix}^*"] \arrow[d] & & \operatorname{Hom}_R\left(\begin{matrix}Y_{-i+2}\\ \oplus\\ X_{-i+1}\end{matrix}, M\right) \arrow[r] \arrow[d] & \cdots \\
\operatorname{Cone}(\operatorname{Hom}_R(F_\bullet,M)) = & \cdots \arrow[r] & \begin{matrix}\operatorname{Hom}_R(X_{-i},M)\\ \oplus\\ \operatorname{Hom}_R(Y_{-i+1},M)\end{matrix} \arrow[rr, "\begin{pmatrix}\left(\partial^X_{-i+1}\right)^* & F^*_{-i+1}\\ 0 & -\left(\partial^Y_{-i+2}\right)^*\end{pmatrix}"'] & & \begin{matrix}\operatorname{Hom}_R(X_{-i+1},M)\\ \oplus\\ \operatorname{Hom}_R(Y_{-i+2},M)\end{matrix} \arrow[r] & \cdots \\
 & & \begin{pmatrix}\psi\\ -\phi\end{pmatrix} \arrow[r, mapsto] & \begin{pmatrix}\psi\circ\partial^X_{-i+1}-\phi\circ F_{-i+1}\\ \phi\circ\partial^Y_{-i+2}\end{pmatrix} = & \begin{pmatrix}-\phi\circ F_{-i+1}+\psi\circ\partial^X_{-i+1}\\ \phi\circ\partial^Y_{-i+2}\end{pmatrix} &
\end{tikzcd}
$$

Exercises

exer171121a EXERCISE II.F.5.4. Let M, M', N be R-modules. Let $P_\bullet$ be a projective resolution of M. Let $P'_\bullet$ be a projective resolution of M'. Let $Q_\bullet$ be a projective resolution of N.

exer171121a1 (a) Prove that $P_\bullet \oplus P'_\bullet$ is a projective resolution of $M \oplus M'$.

exer171121a2 (b) Prove that $\operatorname{Ext}^i_R(M \oplus M', N) \cong \operatorname{Ext}^i_R(M, N) \oplus \operatorname{Ext}^i_R(M', N)$ for all i.

exer171121a3 (c) Prove that $\operatorname{Ext}^i_R(N, M \oplus M') \cong \operatorname{Ext}^i_R(N, M) \oplus \operatorname{Ext}^i_R(N, M')$ for all i.

exer171121a4 (d) State and prove versions of parts (b) and (c) for modules $M_1, \ldots, M_n$.

exer060202q EXERCISE II.F.5.5. Consider a short exact sequence of chain maps.

$$0 \to M'_\bullet \xrightarrow{F_\bullet} M_\bullet \xrightarrow{G_\bullet} M''_\bullet \to 0$$

exer060202qa (a) Prove that $M'_\bullet$ is exact if and only if $G_\bullet$ is a quasiisomorphism.

exer060202qb (b) Prove that $M''_\bullet$ is exact if and only if $F_\bullet$ is a quasiisomorphism.

exer040301 EXERCISE II.F.5.6. (Functoriality of long exact sequences) Let $\phi \colon L \to N$ be an R-module homomorphism. Given an exact sequence of R-module homomorphisms

$$0 \to K \xrightarrow{g} M \xrightarrow{f} C \to 0$$

show that there are commutative diagrams of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Ext}^i_R(N,K) \arrow[r] \arrow[d] & \operatorname{Ext}^i_R(N,M) \arrow[r] \arrow[d] & \operatorname{Ext}^i_R(N,C) \arrow[r] \arrow[d] & \operatorname{Ext}^{i+1}_R(N,K) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \operatorname{Ext}^i_R(L,K) \arrow[r] & \operatorname{Ext}^i_R(L,M) \arrow[r] & \operatorname{Ext}^i_R(L,C) \arrow[r] & \operatorname{Ext}^{i+1}_R(L,K) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Ext}^i_R(C,L) \arrow[r] \arrow[d] & \operatorname{Ext}^i_R(M,L) \arrow[r] \arrow[d] & \operatorname{Ext}^i_R(K,L) \arrow[r] \arrow[d] & \operatorname{Ext}^{i+1}_R(C,L) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \operatorname{Ext}^i_R(C,N) \arrow[r] & \operatorname{Ext}^i_R(M,N) \arrow[r] & \operatorname{Ext}^i_R(K,N) \arrow[r] & \operatorname{Ext}^{i+1}_R(C,N) \arrow[r] & \cdots
\end{tikzcd}$$

where the vertical maps are induced by ϕ.

CHAPTER II.G

Additional Topics

hapter062921e

In this chapter, we give a colloquial treatment of some further properties of Ext. We also briefly discuss the Koszul complex and some further homological constructions.

ection062921w

II.G.1. Other Derived Functors

To obtain $\mathrm{Ext}^i_R(M, N)$, we know to take a projective resolution of M, apply $\mathrm{Hom}_R(-, N)$ to the resolution, and take homology. More generally, given a functor $\mathfrak{F}$, one can take an appropriate resolution, apply $\mathfrak{F}$ to the resolution, and take homology. Here the type of resolution depends entirely on the type of exactness and the variance of the functor to be applied. In this section we explore some such functors.

ex122017b

Example II.G.1.1. The functor we already know is Ext.

$$\mathrm{Ext}^i_R(M, N) = H_{-i}(\mathrm{Hom}_R(P_\bullet, N))$$

We say Ext is the <u>right-derived functor</u> of $\mathrm{Hom}_R(-, N)$ and we use i as a superscript, because $\mathrm{Hom}_R(-, N)$ is contravariant (i.e., arrow-reversing).

ex122017c

Example II.G.1.2. Closely related to Ext is Tor, the <u>left-derived functor</u> of the tensor product $- \otimes_R N$.

$$\mathrm{Tor}^R_i(M, N) = H_{-i}(P_\bullet \otimes_R N)$$

Here we use i as a subscript, because $- \otimes_R N$ is covariant (i.e., arrow-preserving).

Other constructions require different resolutions, which we define next.

def122017d

Definition II.G.1.3. An <u>augmented injective resolution</u> of N is an exact sequence

$$^+I_\bullet = \qquad 0 \longrightarrow N \xrightarrow{\varepsilon} I_0 \xrightarrow{\partial^I_0} I_{-1} \xrightarrow{\partial^I_{-1}} I_{-2} \xrightarrow{\partial^I_{-2}} \cdots$$

where I_i is injective for all $i \in \mathbb{Z}$. The corresponding <u>truncated injective resolution</u> is

$$I_\bullet = \qquad 0 \longrightarrow I_0 \xrightarrow{\partial^I_0} I_{-1} \xrightarrow{\partial^I_{-1}} I_{-2} \xrightarrow{\partial^I_{-2}} \cdots$$

which is not exact in general.

fact122017e

Fact II.G.1.4. *For all R-modules N, the exists an injective module I_0 and an injective R-module homomorphism $\varepsilon : N \longrightarrow I_0$. Colloquially, we say every R-module N is a 'submodule' of an injective R-module. A consequence of this is the existence of an injective resolution for any R-module N, built inductively as the following diagram suggests.*

$$\begin{tikzcd}
 & & & 0 \arrow[dr] & & 0 & & & & 0 \\
 & & & & N' \arrow[ur] \arrow[dr, "\varepsilon'"] & & & & N''' \arrow[ur] & \\
0 \arrow[r] & N \arrow[rr] \arrow[dr, "="'] & & I_0 \arrow[rr] \arrow[ur] & & I_{-1} \arrow[rr] \arrow[dr] & & I_{-2} \arrow[rr] \arrow[ur] & & \cdots \\
 & & N \arrow[ur, "\varepsilon"'] & & & & N'' \arrow[ur, "\varepsilon''"'] \arrow[dr] & & & \\
 & 0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where $N^{(i)} = \operatorname{Coker}\left(\varepsilon^{(i-1)}\right)$.

ex122017f EXAMPLE II.G.1.5. The i^{th} right-derived functor of $\operatorname{Hom}_R(M,-)$ is $H_{-i}(\operatorname{Hom}_R(M, I_\bullet))$.

The following result says we can compute Ext modules from injective resolutions as well as projective resolutions.

thm122017g THEOREM II.G.1.6 (Balance for Ext). *Let M and N by two R-modules, let $P_\bullet$ be a projective resolution for M, and let $I_\bullet$ be an injective resolution for N. Then $H_{-i}(\operatorname{Hom}_R(M, I_\bullet)) \cong \operatorname{Ext}^i_R(M,N)$.*

PROOF. We give only a sketch of this proof. There exists a notion of $\operatorname{Hom}_R(P_\bullet, I_\bullet)$ and one uses mapping cones as in Theorem II.F.5.2 to show that the induced chain maps

$$\operatorname{Hom}_R(P_\bullet, N) \xrightarrow{\simeq} \operatorname{Hom}_R(P_\bullet, I_\bullet) \xleftarrow{\simeq} \operatorname{Hom}_R(M, I_\bullet)$$

are quasiisomorphisms. From this one concludes directly that

$$\operatorname{Ext}^i_R(M,N) \cong \operatorname{Hom}_R(P_\bullet, N) \cong \operatorname{Hom}_R(P_\bullet, I_\bullet) \cong \operatorname{Hom}_R(M, I_\bullet).$$

□

Similarly, we have the following.

thm122017h THEOREM II.G.1.7 (Balance for Tor). *For any R-modules M and N with respective projective resolutions $P_\bullet$ and $Q_\bullet$, we have the following isomorphisms.*

$$H_i(P_\bullet \otimes_R N) \cong H_i(P_\bullet \otimes_R Q_\bullet) \cong H_i(M \otimes_R Q_\bullet)$$

Next, we consider Grothendieck's local cohomology.

def122017i DEFINITION II.G.1.8. Let $\mathfrak{a} \leq R$ be an ideal and let M be an R-module. The $\mathfrak{a}$-torsion functor, denoted $\Gamma_{\mathfrak{a}}$, is defined on modules as

$$\Gamma_{\mathfrak{a}}(M) = \{m \in M \mid \mathfrak{a}^n m = 0,\ \forall n \gg 0\}.$$

See Facts II.G.1.10 and II.G.1.11 for functorial properties.

ex122017j EXAMPLE II.G.1.9. Given the ring $\mathbb{Z}$ and an ideal $p\mathbb{Z}$ the p-torsion functor can be written

$$\Gamma_{p\mathbb{Z}}(M) = \{m \in M \mid p^n m = 0,\ \forall n \gg 0\}.$$

In particular, let $p = 2$ and let $M = \mathbb{Z}/144\mathbb{Z}$. We compute the 2-torsion functor as follows.

$$\begin{aligned}
\Gamma_{p\mathbb{Z}}\left(\frac{\mathbb{Z}}{144\mathbb{Z}}\right) &\cong \Gamma_{p\mathbb{Z}}\left(\frac{\mathbb{Z}}{2^4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3^2\mathbb{Z}}\right) \\
&\cong \Gamma_{2\mathbb{Z}}\left(\frac{\mathbb{Z}}{2^4\mathbb{Z}}\right) \oplus \Gamma_{2\mathbb{Z}}\left(\frac{\mathbb{Z}}{3^2\mathbb{Z}}\right) \\
&\cong \frac{\mathbb{Z}}{2^4\mathbb{Z}} \oplus 0 \\
&\cong \frac{\mathbb{Z}}{2^4\mathbb{Z}}
\end{aligned}$$

Note that $\Gamma_{2\mathbb{Z}}(\mathbb{Z}/3^2\mathbb{Z}) \cong 0$ since 2^n acts as a unit on $\mathbb{Z}/3^2\mathbb{Z}$ for all $n \in \mathbb{N}$.

fact122017k FACT II.G.1.10. *For all R-module homomorphisms $\phi : M \longrightarrow M'$, we have*

$$\phi(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(M').$$

A result of this fact is the following commutative diagram, where $\Gamma_{\mathfrak{a}}(\phi)$ is induced from ϕ by restricting the domain and codomain.

$$\begin{tikzcd}
M \arrow[r, "\phi"] & M' \\
\Gamma_{\mathfrak{a}}(M) \arrow[u, hook, "\subseteq"] \arrow[r, dashed, "\Gamma_{\mathfrak{a}}(\phi)"'] \arrow[ur, phantom, "\circlearrowright"] & \Gamma_{\mathfrak{a}}(M') \arrow[u, hook, "\subseteq"']
\end{tikzcd}$$

PROOF. Let $n \in \mathbb{N}$. If $\mathfrak{a}^n m = 0$, then we also have $0 = \phi(\mathfrak{a}^n m) = \mathfrak{a}^n \cdot \phi(m)$. □

fact122017l FACT II.G.1.11. *$\Gamma_{\mathfrak{a}}$ is a covariant functor and is left-exact.*

ex122017m

EXAMPLE II.G.1.12. The functor $\Gamma_{\mathfrak{a}}$ is not right-exact in general. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

to which we apply $\Gamma_{2\mathbb{Z}}$ to obtain

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{2\mathbb{Z}}(\mathbb{Z}) \arrow[r, "2\cdot"] \arrow[d, equal] & \Gamma_{2\mathbb{Z}}(\mathbb{Z}) \arrow[r] \arrow[d, equal] & \Gamma_{2\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\
& 0 & 0 & \mathbb{Z}/2\mathbb{Z} \neq 0 &
\end{tikzcd}$$

which is not exact.

def122017n

DEFINITION II.G.1.13. Let $I_\bullet$ be an injective resolution of an R-module N. The i^{th} local cohomology module associated to N with support in $\mathfrak{a} \leq R$ is the i^{th} right-derived functor of $\Gamma_{\mathfrak{a}}$.

$$H^i_{\mathfrak{a}}(N) = H_{-i}(\Gamma_{\mathfrak{a}}(I_\bullet))$$

ex122017o

EXAMPLE II.G.1.14. Let $\mathbb{Z}$ be both the ring and module in this example and let $\mathfrak{a} = 2\mathbb{Z}$. The following is an augmented injective resolution for $\mathbb{Z}$.

$$^{+}I_\bullet = \qquad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Applying $\Gamma_{\mathfrak{a}}$ to the truncated resolution we get the following.

$$\Gamma_{\mathfrak{a}}(I_\bullet) = \qquad 0 \longrightarrow \Gamma_{2\mathbb{Z}}(\mathbb{Q}) \longrightarrow \Gamma_{2\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

Since $2^n \in \mathbb{Q}$ is a unit for $n = 1, 2, 3, \dots$ we write equivalently

$$\Gamma_{\mathfrak{a}}(I_\bullet) = \qquad 0 \longrightarrow 0 \longrightarrow \Gamma_{2\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

where $\Gamma_{2\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) = \left\{\overline{(a/2^n)} \mid a \in \mathbb{Z},\ n \in \mathbb{N}\right\} \neq 0$. We now compute the cohomology as follows.

$$H^i_{2\mathbb{Z}} = \begin{cases} 0 & i \neq 1 \\ \Gamma_{2\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) & i = 1 \end{cases}$$

ection062921x

II.G.2. Ext and Extensions

The point of this section is that one can define an equivalence relation on sets of short exact sequences in such a way that the set of equivalence classes is naturally in bijection with an Ext^1_R-module.

def122117g

DEFINITION II.G.2.1. An extension of M by N is a short exact sequence

$$\zeta = \qquad 0 \longrightarrow N \xrightarrow{f} A \xrightarrow{g} M \longrightarrow 0.$$

We also define an equivalence relation on the set of extensions of M by N. If ζ' is another extension of M, then $\zeta \sim \zeta'$ if there exists a commutative diagram of the following form.

$$\begin{tikzcd}
\zeta = & 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "1"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[r, "g"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r] \arrow[d, "1"'] & 0 \\
\zeta' = & 0 \arrow[r] & N \arrow[r, "f'"'] & A' \arrow[r, "g'"'] & M \arrow[r] & 0
\end{tikzcd}$$

The collection of all equivalence classes of such extensions is a set which we denote $\mathrm{E}^1_R(M, N)$.

thm122317a

THEOREM II.G.2.2 (Yoneda). *For any R-modules M and N, there exists a bijective function*

$$\Phi : \mathrm{E}^1_R(M, N) \longrightarrow \mathrm{Ext}^1_R(M, N)$$

which we construct next.

con122317b CONSTRUCTION II.G.2.3. Let an extension ξ be given:

$$\xi = \qquad 0 \longrightarrow N \xrightarrow{f} A \xrightarrow{g} M \longrightarrow 0\ .$$

If $P_\bullet^+$ is a projective resolution of M, by Proposition II.E.2.2 we can lift the identity map on M to build the following ladder diagram.

$$\begin{tikzcd}
P_\bullet^+ = & \cdots \arrow[r, "\partial_3^P"] & P_2 \arrow[r, "\partial_2^P"] \arrow[d, dashed] \arrow[dr, phantom, "\circlearrowleft"] & P_1 \arrow[r, "\partial_1^P"] \arrow[d, dashed, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & P_0 \arrow[r, "\tau"] \arrow[d, dashed, "\alpha"] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
\xi = & & 0 \arrow[r] & N \arrow[r, "f"'] & A \arrow[r, "g"'] & M \arrow[r] & 0
\end{tikzcd}$$

The commutivity of the diagram implies $0 = \beta \circ \partial_2^P = \left(\partial_2^P\right)^* (\beta)$ and therefore

$$\overline{\beta} \in \frac{\operatorname{Ker}\left(\partial_2^P\right)^*}{\operatorname{Im}\left(\partial_1^P\right)^*} = \operatorname{Ext}_R^1(M, N).$$

Hence we define the bijection proposed in Theorem II.G.2.2 as follows.

$$\Phi([\xi]) = \overline{\beta}$$

We give a sketch of the proof that this is well-defined. We suppose $\xi \sim \xi'$ and we want to show $\overline{\beta} = \overline{\beta'}$, where ξ' is

$$\xi' = \qquad 0 \longrightarrow N \xrightarrow{f'} A' \xrightarrow{g'} M \longrightarrow 0\ .$$

and β' is in the following ladder diagram.

$$\begin{tikzcd}
P_\bullet^+ = & \cdots \arrow[r, "\partial_3^P"] & P_2 \arrow[r, "\partial_2^P"] \arrow[d, dashed] \arrow[dr, phantom, "\circlearrowleft"] & P_1 \arrow[r, "\partial_1^P"] \arrow[d, dashed, "\beta'"] \arrow[dr, phantom, "\circlearrowleft"] & P_0 \arrow[r, "\tau"] \arrow[d, dashed, "\alpha'"] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
\xi' = & & 0 \arrow[r] & N \arrow[r, "f'"'] & A' \arrow[r, "g'"'] & M \arrow[r] & 0
\end{tikzcd}$$

For this it suffices to show $\beta - \beta' \in \operatorname{Im}(\partial_1^P)^*$. Consider the following diagram, where all the rectangular diagrams commute, but the triangular ones need not commute.

$$\begin{tikzcd}
& P_2 \arrow[rr, "\partial_2^P"] \arrow[dl] \arrow[dd] & & P_1 \arrow[rr, "\partial_1^P"] \arrow[dl, "\beta"'] \arrow[dd, "\beta'"] & & P_0 \arrow[rr, "\tau"] \arrow[dl, "\alpha"'] \arrow[dd, "\alpha'"] & & M \arrow[rr] \arrow[dl, "\mathrm{id}_M"'] \arrow[dd, "\mathrm{id}_M"] & & 0 \arrow[dl] \arrow[dd] \\
0 \arrow[rr] \arrow[dr] & & N \arrow[rr, "f"] \arrow[dr, "\mathrm{id}_N"'] & & A \arrow[rr, "g"] \arrow[dr, "\phi"'] & & M \arrow[rr] \arrow[dr, "\mathrm{id}_M"'] & & 0 \arrow[dr] & \\
& 0 \arrow[rr] & & N \arrow[rr, "f'"'] & & A' \arrow[rr, "g'"'] & & M \arrow[rr] & & 0
\end{tikzcd}$$

Here the map ϕ comes from the equivalence $\xi \sim \xi'$. One can apply $\operatorname{Hom}_R(P_0, -)$ to ξ', which preserves exactness, and select a map $\gamma \in \operatorname{Hom}_R(P_0, N)$ such that $f' \circ \gamma = \phi \circ \alpha - \alpha'$. One then shows that $\beta - \beta' = (\partial_1^P)^*(\gamma)$.

Proving the injectivity and surjectivity of this map is beyond the scope of this document. The crux of the latter is that given any extension ζ we can lift the identity map on M to find an appropriate β.

One can obtain the next result as a corollary of Theorem II.G.2.2. We present a partial alternate proof that uses technology we have developed completely.

thm110817b THEOREM II.G.2.4. *For all R-modules M and N, the following are equivalent.*

thm110817b.i (i) $\operatorname{Ext}_R^1(M, N) = 0$

thm110817b.ii (ii) *Every short exact sequence* $0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0$ *splits.*

PROOF. (i) $\Longrightarrow$ (ii). Consider the short exact sequence

$$0 \longrightarrow N \xrightarrow{\rho} X \xrightarrow{\phi} M \longrightarrow 0. \tag{II.G.2.4.1}$$ eqn102618a

An associated long exact sequence is

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_R(M,N) \arrow[r] & \operatorname{Hom}_R(M,X) \arrow[r, "\phi_*"] & \operatorname{Hom}_R(M,M) \arrow[dll, rounded corners] \\ & \operatorname{Ext}^1_R(M,N) \arrow[r] & \cdots . & \end{tikzcd}$$

If we assume $\operatorname{Ext}^1_R(M,N) = 0$, then ϕ_* is surjective and for $\mathrm{id}_M \in \operatorname{Hom}_R(M,M)$, there exists some $\alpha \in \operatorname{Hom}_R(M,X)$ such that $\mathrm{id}_M = \phi_*(\alpha) = \phi \circ \alpha$. Therefore by Fact II.A.1.10, the sequence (II.G.2.4.1) splits and (ii) holds.

$\square$

ection062921y

II.G.3. The Koszul Complex

Here we introduce the Koszul complex in full generality (Defintion II.G.3.5) and study its homology. In Theorem II.G.3.17 we give a means of detecting regular sequences and in Theorem II.G.3.21 we give three significant characteristics of R modulo a regular sequence.

rec122317c

RECALL II.G.3.1. In Proposition II.F.4.6 we saw for any chain map $f_\bullet : M_\bullet \longrightarrow N_\bullet$ we have the following short exact sequence and associated long exact sequence.

$$0 \longrightarrow N_\bullet \xrightarrow{\varepsilon_\bullet} \operatorname{Cone}(f_\bullet) \xrightarrow{\pi_\bullet} \Sigma M_\bullet \longrightarrow 0$$

$$\begin{tikzcd} \cdots \arrow[r, "H_i(\varepsilon_\bullet)"] & H_i(\operatorname{Cone}(f_\bullet)) \arrow[r, "H_i(\pi_\bullet)"] & H_{i-1}(M_\bullet) \arrow[r, "\eth_i ="', "H_{i-1}(f_\bullet)"'] \arrow[d, equal] & H_{i-1}(N) \arrow[r] & \cdots \\ & & H_i(\Sigma M_\bullet) & & \end{tikzcd}$$

Here $\varepsilon_\bullet$ and $\pi_\bullet$ are the natural injection and surjection, respectively. (See also Definition II.F.4.3.)

ex122317d

EXAMPLE II.G.3.2. If M is an R-module, then we say

$$M_\bullet = \qquad 0 \longrightarrow M \longrightarrow 0$$

is a chain complex concentrated in degree zero. For any $r \in R$ we may also define a chain map

$$\begin{tikzcd} M_\bullet = \arrow[d, "\mu^r_\bullet"'] & 0 \arrow[r] & M \arrow[r] \arrow[d, "r\cdot"] & 0 \\ M_\bullet = & 0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

which yields the cone

$$\operatorname{Cone}(\mu^r_\bullet) = \qquad 0 \longrightarrow M \xrightarrow{r} M \longrightarrow 0.$$

def122317e

DEFINITION II.G.3.3. For any R-module M and any $r \in R$, define the following submodule.

$$(0 \underset{M}{:} r) = \{m \in M \mid rm = 0\}$$

This is the largest submodule of M annihilated by r, called the <u>annihilator</u> of r in M. More generally, for any $S \subseteq R$ we have

$$(0 \underset{M}{:} S) = \{m \in M \mid sm = 0,\ \forall s \in S\}.$$

prop122317g

PROPOSITION II.G.3.4. *Let $X_\bullet$ be an R-complex and let $r \in R$ be fixed. Consider the <u>homothety</u> map $\mu^r_\bullet : X_\bullet \longrightarrow X_\bullet$ defined as in Discussion II.E.2.4 and the short exact sequence*

$$0 \longrightarrow X_\bullet \longrightarrow \operatorname{Cone}(\mu^r_\bullet) \longrightarrow \Sigma X_\bullet \longrightarrow 0\,.$$

rop122317g.a

(a) *In the associated long exact sequence, the connecting map is also a multiplication map, i.e.,*

$$\eth_i(\overline{x_{i-1}}) = r \cdot \overline{x_{i-1}}.$$

prop122317g.b (b) *For any $i \in \mathbb{Z}$, there exists a short exact sequence*

$$0 \longrightarrow \frac{H_i(X_\bullet)}{r \cdot H_i(X_\bullet)} \longrightarrow H_i\left(\mathrm{Cone}(\mu^r_\bullet)\right) \longrightarrow (\underset{H_{i-1}(X_\bullet)}{0} : r) \longrightarrow 0\ .$$

Proof. (a) This follows directly from the definition of the connecting map, properties of cosets, and Recall II.G.3.1.

$$\eth_i(\overline{x_{i-1}}) = H_{i-1}(\mu^r_\bullet)(\overline{x_{i-1}}) = \overline{\mu^r_{i-1}(x_{i-1})} = \overline{rx_{i-1}} = r \cdot \overline{x_{i-1}}$$

(b) By part (a) and from our comments in II.G.3.1, the associated long exact sequence is as follows.

$$\cdots \longrightarrow H_i(X_\bullet) \xrightarrow{r\cdot} H_i(X_\bullet) \xrightarrow{H_i(\varepsilon_\bullet)} H_i(\mathrm{Cone}(\mu^r_\bullet)) \xrightarrow{H_i(\pi_\bullet)} H_{i-1}(X_\bullet) \xrightarrow{r\cdot} H_{i-1}(X_\bullet) \longrightarrow \cdots$$

From the First Isomorphism Theorem for modules we have

$$\mathrm{Im}\, H_i(\varepsilon_\bullet) \cong \frac{H_i(X_\bullet)}{\mathrm{Ker}\, H_i(\varepsilon_\bullet)} = \frac{H_i(X_\bullet)}{r \cdot H_i(X_\bullet)}$$

since the kernel of $H_i(\varepsilon_\bullet)$ is the image of $r\cdot$ by the exactness of the sequence. Therefore when we 'slice' the long exact sequence around $H_i(\mathrm{Cone}(\mu^r_\bullet))$ we get the following.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}\, H_i(\varepsilon_\bullet) \arrow[r] \arrow[d, phantom, "\wr\|"] & H_i(\mathrm{Cone}(\mu^r_\bullet)) \arrow[r] & \mathrm{Im}\, H_i(\pi_\bullet) \arrow[r] \arrow[d, phantom, "\|"] & 0 \\
& \dfrac{H_i(X_\bullet)}{r \cdot H_i(X_\bullet)} & & \mathrm{Ker}\, r\cdot \arrow[d, phantom, "\|"] & \\
& & & (\underset{H_{i-1}(X_\bullet)}{0} : r) &
\end{tikzcd}$$

$\square$

def122517a Definition II.G.3.5. Here we define a particular R-complex, called the Koszul complex. Given an R-module M and $x_1, \ldots, x_n \in R$, we define $\mathrm{K}_\bullet(\underline{x}; M)$ inductively on the length of the sequence. Let $\underline{x} = x_1, \ldots, x_n$ and $\underline{x}' = x_1, \ldots, x_{n-1}$.

$$\begin{array}{lll}
n = 0 & \mathrm{K}_\bullet(\emptyset; M) = 0 \longrightarrow M \longrightarrow 0 & = M_\bullet \\
n = 1 & \mathrm{K}_\bullet(x_1; M) = 0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow 0 & = \mathrm{Cone}\left(M_\bullet \xrightarrow{x_1\cdot} M_\bullet \right) \\
n \geq 2 & \mathrm{K}_\bullet(\underline{x}; M) = \mathrm{Cone}\left(\mathrm{K}_\bullet(\underline{x}'; M) \xrightarrow{x_n\cdot} \mathrm{K}_\bullet(\underline{x}'; M) \right) &
\end{array}$$

We define also the following shorthand notations.

$$H_i(\underline{x}; M) = H_i(\mathrm{K}_\bullet(\underline{x}; M)) \qquad \mathrm{K}_\bullet(\underline{x}) = \mathrm{K}_\bullet(\underline{x}; R) \qquad H_i(\underline{x}) = H_i(\underline{x}; R)$$

We will use the above notation for $\underline{x}$ and $\underline{x}'$ throughout the rest of this section.

ex122917a Example II.G.3.6. By the previous definition $K_\bullet(x, y; M)$ is the cone of the following chain map.

$$\begin{tikzcd}
\mathrm{K}_\bullet(x; M) \arrow[d, "y\cdot"'] & & 0 \arrow[r] & M \arrow[r, "x\cdot"] \arrow[d, "y\cdot"'] & M \arrow[r] \arrow[d, "y\cdot"] & 0 \arrow[r] & 0 \\
\mathrm{K}_\bullet(x; M) & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r, "x\cdot"'] & M \arrow[r] & 0 &
\end{tikzcd}$$

This yields

$$\mathrm{K}_\bullet(x, y; M) = \quad \begin{matrix} 0 \\ \oplus \\ 0 \end{matrix} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} \begin{matrix} 0 \\ \oplus \\ M \end{matrix} \xrightarrow{\begin{pmatrix} 0 & y \\ 0 & -x \end{pmatrix}} \begin{matrix} M \\ \oplus \\ M \end{matrix} \xrightarrow{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}} \begin{matrix} M \\ \oplus \\ 0 \end{matrix} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} \begin{matrix} 0 \\ \oplus \\ 0 \end{matrix}$$

or more simply

$$\mathrm{K}_\bullet(x, y; M) = \quad 0 \longrightarrow M \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} M^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} M \longrightarrow 0\ .$$

To find $\mathrm{K}_\bullet(x, y, z; M)$ we take the cone of the following chain map.

$$\begin{tikzcd}
\mathrm{K}_\bullet(x,y;M) = \arrow[d, "z\cdot"'] & & 0 \arrow[r] \arrow[d] & M \arrow[r, "\begin{pmatrix} y \\ -x \end{pmatrix}"] \arrow[d, "z\cdot"'] & M^2 \arrow[r, "\begin{pmatrix} x & y \end{pmatrix}"] \arrow[d, "z\cdot I_2"'] & M \arrow[r] \arrow[d, "z\cdot"'] & 0 \arrow[r] \arrow[d] & 0 \\
\mathrm{K}_\bullet(x,y;M) = & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r, "\begin{pmatrix} y \\ -x \end{pmatrix}"'] & M^2 \arrow[r, "\begin{pmatrix} x & y \end{pmatrix}"'] & M \arrow[r] & 0 &
\end{tikzcd}$$

So we have

$$\mathrm{K}_\bullet(x,y,z;M) = \begin{matrix} 0 \\ \oplus \\ 0 \end{matrix} \longrightarrow \begin{matrix} 0 \\ \oplus \\ M \end{matrix} \xrightarrow{\left(\begin{array}{c|c} 0 & z \\ \hline 0 & -y \\ 0 & x \end{array}\right)} \begin{matrix} M \\ \oplus \\ M^2 \end{matrix} \xrightarrow{\left(\begin{array}{c|cc} y & z & 0 \\ -x & 0 & z \\ \hline 0 & -x & -y \end{array}\right)} \begin{matrix} M^2 \\ \oplus \\ M \end{matrix} \xrightarrow{\left(\begin{array}{cc|c} x & y & z \\ \hline 0 & 0 & 0 \end{array}\right)} \begin{matrix} M \\ \oplus \\ 0 \end{matrix} \longrightarrow \begin{matrix} 0 \\ \oplus \\ 0 \end{matrix}$$

which we can simplify to write

$$\mathrm{K}_\bullet(x,y,z;M) = \qquad 0 \longrightarrow M \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} M^3 \xrightarrow[\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}]{} M^3 \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} M \longrightarrow 0.$$

ex122917b EXAMPLE II.G.3.7. Consider the polynomial ring $R = A[x]$ where A is a commutative ring with identity and x is an indeterminate. The Koszul complex for this singleton sequence is

$$\mathrm{K}_\bullet(x) = \qquad 0 \longrightarrow \underset{1}{R} \xrightarrow{x\cdot} \underset{0}{R} \longrightarrow 0$$

and we may calculate the homology modules of this complex. Since x is a non-zero-divisor

$$H_1(x) \cong \ker(x\cdot) = 0.$$

At the only other position of any potential interest we have

$$H_0(x) = \frac{R}{\operatorname{Im} x\cdot} = \frac{R}{xR} \cong A.$$

ex122917c EXAMPLE II.G.3.8. Now consider the polynomial ring in two variables $R = A[x, y]$, and we again calculate the homology modules of this complex.

$$\mathrm{K}_\bullet(x,y) = \qquad 0 \longrightarrow \underset{2}{R} \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \underset{1}{R^2} \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \underset{0}{R} \longrightarrow 0$$

The zero position and second position are each straightforward.

$$H_0(x,y) = \frac{R}{\operatorname{Im}\begin{pmatrix} x & y \end{pmatrix}} = \frac{R}{(x,y)} \cong A$$

$$H_2(x,y) = \frac{\operatorname{Ker}\begin{pmatrix} y & -x \end{pmatrix}^T}{0} \cong (0 \underset{R}{:} y) \cap (0 \underset{R}{:} x) = 0$$

We claim the homology is zero at the first position as well, for which it suffices to show $\operatorname{Ker}\begin{pmatrix} x & y \end{pmatrix} = \operatorname{Im}\begin{pmatrix} y & -x \end{pmatrix}^T$. The reverse containment holds because $K_\bullet(x, y)$ is an R-complex.

For any $\begin{pmatrix} f & g \end{pmatrix}^T \in \operatorname{Ker}\begin{pmatrix} x & y \end{pmatrix}$ we have $gy = -fx$, so $x|g$ and $y|f$. Therefore let $g_1, f_1 \in R$ such that $g = xg_1$ and $f = yf_1$. It follows that

$$xy(f_1 + g_1) = xyf_1 + xyg_1 = xf + yg = 0$$

and hence $f_1 + g_1 = 0$, so $g_1 = -f_1$ and $g = -xf_1$. Finally this gives

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} yf_1 \\ -xf_1 \end{pmatrix} = f_1 \begin{pmatrix} y \\ -x \end{pmatrix} \in \left\langle \begin{pmatrix} y \\ -x \end{pmatrix} \right\rangle = \operatorname{Im} \begin{pmatrix} y \\ -x \end{pmatrix}$$

so the forward containment holds.

ex122917d

EXAMPLE II.G.3.9. Let A be a field and define the ring

$$R = \frac{A[X,Y]}{(XY)}$$

where X and Y are indeterminates. Let $x, y \in R$ denote $\overline{X}, \overline{Y}$, respectively. The Koszul complex is then written the same as in the previous example.

$$\mathrm{K}_\bullet(x,y) = \qquad 0 \longrightarrow \underset{2}{R} \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \underset{1}{R^2} \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \underset{0}{R} \longrightarrow 0$$

Also as in the previous example, the homology modules in the zeroth and second positions are straightforward to calculate.

$$H_0(x,y) \cong \frac{R}{(x,y)R} \cong A$$

$$H_2(x,y) = \{r \in R \mid xr = 0 = yr\} = yR \cap xR = xyR = 0$$

We claim $H_1(x,y) \cong A$. As in Example II.G.3.8, let $\begin{pmatrix} f & g \end{pmatrix}^T \in \operatorname{Ker} \begin{pmatrix} x & y \end{pmatrix}$. Using the canonical basis $\{1, x, y, x^2, y^2, \dots\}$ for R over A we may write f and g as the following finite sums.

$$f = a + x \sum_i b_i x^i + y \sum_j c_j y^j$$

$$g = d + x \sum_i e_i x^i + y \sum_j v_j y^j$$

By virtue of being in the kernel we have

$$\begin{aligned} 0 &= fx + gy \\ &= ax + x^2 \sum_i b_i x^i + xy \sum_j c_j y^j + dy + yx \sum_i e_i x^i + y^2 \sum_j v_j y^j \\ &= ax + x^2 \sum_i b_i x^i + dy + y^2 \sum_j v_j y^j \end{aligned}$$

since $xy = 0 \in R$. Therefore by the linear independence of our basis we have $a, d, b_i, v_j = 0 \in A$ for all i and j, so we write $f = y \sum_j c_j y^j$ and $g = x \sum_i e_i x^i$. From this we have

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix} = \sum_j c_j y^j \begin{pmatrix} y \\ 0 \end{pmatrix} + \sum_i e_i x^i \begin{pmatrix} 0 \\ x \end{pmatrix} \in \left\langle \begin{pmatrix} y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix} \right\rangle$$

so $\operatorname{Ker} \begin{pmatrix} x & y \end{pmatrix} \subseteq \left\langle \begin{pmatrix} y & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & x \end{pmatrix}^T \right\rangle$. Since the generators of the right-hand side are in the kernel (because $xy = 0$), we actually have equality. Thus we compute

$$H_1(x,y) = \frac{\left\langle \begin{pmatrix} y & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & x \end{pmatrix}^T \right\rangle}{\left\langle \begin{pmatrix} y & -x \end{pmatrix}^T \right\rangle} = \frac{\left\langle \begin{pmatrix} y & -x \end{pmatrix}^T, \begin{pmatrix} 0 & x \end{pmatrix}^T \right\rangle}{\left\langle \begin{pmatrix} y & -x \end{pmatrix}^T \right\rangle}.$$

Hence $H_1(x,y)$ is cyclic generated by $\overline{\begin{pmatrix} 0 & x \end{pmatrix}^T}$, so we can surject onto $H_1(x,y)$ by the following R-module homomorphism.

$$\begin{aligned} R &\xrightarrow{\phi} H_1(x,y) \\ r &\longmapsto r\overline{\begin{pmatrix} 0 \\ x \end{pmatrix}} \end{aligned}$$

Since $\begin{pmatrix}0 & x\end{pmatrix}^T \notin \left\langle \begin{pmatrix}y & -x\end{pmatrix}^T \right\rangle$, we have $H_1(x,y) \neq 0$ and therefore $\ker(\phi) \neq R$. On the other hand, $x, y \in \ker(\phi)$ by the following.

$$\phi(x) = x\overline{\begin{pmatrix}0\\x\end{pmatrix}} = \overline{\begin{pmatrix}0\\x^2\end{pmatrix}} = \overline{\begin{pmatrix}-xy\\x^2\end{pmatrix}} = -x\overline{\begin{pmatrix}y\\-x\end{pmatrix}} = 0$$
$$\phi(y) = y\overline{\begin{pmatrix}0\\x\end{pmatrix}} = \overline{\begin{pmatrix}0\\xy\end{pmatrix}} = \overline{\begin{pmatrix}0\\0\end{pmatrix}} = 0$$

Therefore the ideal (x,y) is contained in the kernel of ϕ, which is strictly contained in the ring R. Since A is a field, (x,y) is maximal and is therefore equal to $\ker(\phi)$. Hence

$$H_1(x,y) \cong \frac{R}{\ker(\phi)} = \frac{R}{(x,y)} \cong A.$$

prop122917e

PROPOSITION II.G.3.10. *For any R-module M, $\mathrm{K}_i(\underline{x}; M) \cong M^{\binom{n}{i}}$.*

PROOF. This is proven by induction on n. The base cases $n = 0, 1, 2, 3$ have already been seen in Definition II.G.3.5 and Example II.G.3.6. Assume $n \geq 4$ and the claim holds for $1, \ldots, n-1$. Then we use Definition II.F.4.3 and the induction hypothesis to get

$$\begin{aligned}\mathrm{K}_i(\underline{x}; M) &\cong \mathrm{K}_i(\underline{x}'; M) \oplus \mathrm{K}_{i-1}(\underline{x}'; M)\\ &\cong M^{\binom{n-1}{i}} \oplus M^{\binom{n-1}{i-1}}\\ &\cong M^{\binom{n-1}{i}+\binom{n-1}{i-1}}\\ &= M^{\binom{n}{i}}.\end{aligned}$$

□

prop122917f

PROPOSITION II.G.3.11. *Let M be an R-module.*

prop122917f.a

(a) *The differential $\partial_1^{\mathrm{K}_\bullet(\underline{x};M)}$ is the following map.*

$$M^n \xrightarrow{\begin{pmatrix}x_1 & x_2 & \cdots & x_n\end{pmatrix}} M$$

prop122917f.b

(b) *The differential $\partial_n^{\mathrm{K}_\bullet(\underline{x};M)}$ is the following map.*

$$M \xrightarrow{\begin{pmatrix}x_n & -x_{n-1} & \cdots & (-1)^{n-1}x_1\end{pmatrix}^T} M^n$$

prop122917f.c

(c) *The homologies of the two 'ends' of the complex will be as follows.*

$$H_0(\underline{x}; M) \cong \frac{M}{(\underline{x})M}$$
$$H_n(\underline{x}; M) \cong \bigcap_{i=1}^{n} (0 \underset{M}{:} x_i) = \{m \in M \mid x_i m = 0,\ \forall i = 1, \ldots, n\} = (0 \underset{M}{:} \langle \underline{x} \rangle)$$

PROOF. (a) We prove this by induction on n. The base cases $n = 1, 2$ have already been seen in Definition II.G.3.5 and Example II.G.3.6, so assume $n \geq 3$ and that the claim holds for $\underline{x}' = x_1, \ldots, x_{n-1}$. By definition, the Koszul complex, $\mathrm{K}_\bullet(\underline{x}; M)$, is the cone of

$$\begin{tikzcd} \mathrm{K}_\bullet(\underline{x}'; M) \arrow[d, "x_n\cdot"'] & \cdots \arrow[r] & M^{n-1} \arrow[r, "(x_1 \cdots x_{n-1})"] \arrow[d, "x_n \cdot I_{n-1}"'] & M \arrow[r] \arrow[d, "x_n\cdot"'] & 0 \\ \mathrm{K}_\bullet(\underline{x}'; M) & \cdots \arrow[r] & M^{n-1} \arrow[r, "(x_1 \cdots x_{n-1})"] & M \arrow[r] & 0 \end{tikzcd}$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix. Taking the cone yields the top row of the following diagram.

$$\begin{tikzcd}
\mathrm{K}_\bullet(\underline{x};M) = & \cdots \arrow[r] & \begin{matrix} M^{n-1} \\ \oplus \\ M \end{matrix} \arrow[rr, "\begin{pmatrix} x_1 & \cdots & x_{n-1} & x_n \\ 0 & \cdots & 0 & 0 \end{pmatrix}"] \arrow[d, "\cong"] \arrow[drr, phantom, "\circlearrowleft"] & & \begin{matrix} M \\ \oplus \\ 0 \end{matrix} \arrow[r] \arrow[d, "\cong"] & 0 \\
& & M^n \arrow[rr, "\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}"'] & & M \arrow[r] & 0
\end{tikzcd}$$

This proves part (a). Moreover, taking homology at the zeroth position of the bottom row (the M beneath $M\oplus 0$), the commutivity of the diagram and the isomorphisms depicted allow us to conclude

$$H_0(\underline{x};M) \cong \frac{M}{(x_1,\ldots,x_n)M}$$

which is among the claims of part (c).
(b) This is likewise proven by induction on n and the base cases $n = 1, 2$ have likewise already been shown. Therefore we assume $n \geq 3$ and that the claim holds for $\underline{x}'$. Again $\mathrm{K}_\bullet(\underline{x};M)$ is the cone of the chain map

$$\begin{tikzcd}
\mathrm{K}_\bullet(\underline{x}';M) = \arrow[d, "x_n\cdot"'] & 0 \arrow[r] & M \arrow[r, "\xi"] \arrow[d, "x_n\cdot"'] & M^{n-1} \arrow[r] \arrow[d, "x_n I_{n-1}"'] & \cdots \\
\mathrm{K}_\bullet(\underline{x}';M) = & 0 \arrow[r] & M \arrow[r, "\xi"'] & M^{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

where $\xi = \begin{pmatrix} x_{n-1} & -x_{n-2} & \cdots & (-1)^{n-2}x_1 \end{pmatrix}^T$. Taking the cone yields

$$\begin{tikzcd}
0 \arrow[r] & \begin{matrix} 0 \\ \oplus \\ M \end{matrix} \arrow[r, "\begin{pmatrix} 0 & x_n \\ 0 & -\xi \end{pmatrix}"] \arrow[d, "\cong"'] & \begin{matrix} M \\ \oplus \\ M^{n-1} \end{matrix} \arrow[r] \arrow[d, "\cong"] & \cdots \\
0 \arrow[r] & M \arrow[r, "\overline{\xi}=\begin{pmatrix} x_n \\ -\xi \end{pmatrix}"'] & M^n \arrow[r] & \cdots
\end{tikzcd}$$

which proves the desired result. Taking homology at the n^{th} position allows us to complete the proof of part (c) as well:

$$H_n(\underline{x};M) \cong \operatorname{Ker}\overline{\xi} = \bigcap_{i=1}^{n} \{m \in M \mid x_i m = 0\} = (0 \underset{M}{:} \langle \underline{x} \rangle).$$

$\square$

rmk010418a REMARK II.G.3.12. In the context of Proposition II.G.3.11, a similar analysis shows that each differential $\partial_j^{\mathrm{K}_\bullet(\underline{x};M)}$ can be expressed as a matrix consisting entirely of zeros and $\pm x_1, \ldots, \pm x_n$.

prop010418b PROPOSITION II.G.3.13. *For every $i \in \mathbb{Z}$, there exists a short exact sequence*

$$0 \longrightarrow \frac{H_i(\underline{x}';M)}{x_n \cdot H_i(\underline{x}';M)} \longrightarrow H_i(\underline{x};M) \longrightarrow (0 \underset{H_{i-1}(\underline{x}';M)}{:} x_n) \longrightarrow 0.$$

PROOF. By part (b) of Proposition II.G.3.4 and by the definitions of the mapping cone and the Koszul complex (II.F.4.3 and II.G.3.5, respectively) it suffices to show there exists a short exact sequence of R-complexes

$$0 \longrightarrow K_\bullet(\underline{x}';M) \longrightarrow K_\bullet(\underline{x};M) \longrightarrow \Sigma K_\bullet(\underline{x}';M) \longrightarrow 0\ .$$

This is given by Proposition II.F.4.6, so the proof is complete. $\square$

The following fact is used with the preceding proposition to explain some annihilation properties of Koszul homology modules in the subsequent proposition.

fact010418c

FACT II.G.3.14. *Consider the following exact sequence of R-modules.*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

If $r, s \in R$ annihilate A and C, respectively, then $rs \in R$ annihilates B.

PROOF. Let $b \in B$ be given. Since s annihilates C we have

$$\beta(sb) = s\beta(b) = 0$$

so $sb \in \ker(\beta) = \operatorname{Im}\alpha$. Let $a \in A$ such that $\alpha(a) = sb$ and we have

$$rsb = r\alpha(a) = \alpha(ra) = \alpha(0) = 0.$$

□

prop010418d

PROPOSITION II.G.3.15. *In the context of Proposition II.G.3.11, for any $i \in [n]$ and any $j \in \mathbb{Z}$ one has*

$$x_i^{2^{n-1}} \cdot H_j(\underline{x}; M) = 0.$$

PROOF. This is yet another proof by induction on n. When $n = 1$ we have $H_1(x; M) = (0 \underset{M}{:} x)$ and $H_0(x; M) = M/xM$ by Proposition II.G.3.11(c) and $H_j(x; M) = 0$ for all $j \neq 0, 1$. Note these are indeed annihilated by x, so the result holds for the base case.

Assume $n \geq 2$ and that

$$x_i^{2^{n-2}} \cdot H_j(\underline{x}'; M) = 0$$

for any $i \leq n-1$ and any j. Let $i \in [n]$ and $j \in \mathbb{Z}$ be given and consider the short exact sequence given by Proposition II.G.3.13.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{H_j(\underline{x}'; M)}{x_n \cdot H_j(\underline{x}'; M)} & \longrightarrow & H_j(\underline{x}; M) & \longrightarrow & (0 \underset{H_{j-1}(\underline{x}'; M)}{:} x_n) & \longrightarrow & 0 \\ & & A & & B & & C & & \end{array}$$

By the induction hypothesis, the two modules at the A and C positions are each annihilated by both x_n and $x_i^{2^{n-2}}$ for all $i \leq n-1$. Thus $H_j(\underline{x}; M)$ is annihilated by both x_n^2 and $x_i^{2^{n-1}}$ for all $i \leq n-1$, by Fact II.G.3.14. □

rmk052118a

REMARK II.G.3.16. The conclusion of Proposition II.G.3.15 can be strengthened to say $x_i H_j(\underline{x}; M) = 0$. However, the proof of this stronger result requires technology beyond the scope of this document.

The next result leads to one of the most important properties of the Koszul complex. See Theorem II.G.3.21.

thm010418e

THEOREM II.G.3.17. *If $\underline{x}$ is M-regular, then $H_i(\underline{x}; M) = 0$ for all non-zero i.*

PROOF. Another proof by induction. The base case $n = 1$ follows from Proposition II.G.3.11(c). Assume $n \geq 2$ and the claim holds for regular sequences of length $n - 1$. If $\underline{x}$ is M-regular, then by definition of the shorter sequence $\underline{x}'$ is M-regular as well. Therefore by the induction hypothesis $H_i(\underline{x}'; M) = 0$ for all $i \neq 0$. Let $i \geq 1$ be given and consider the short exact sequence given in Proposition II.G.3.13.

$$0 \longrightarrow \frac{H_i(\underline{x}'; M)}{x_n \cdot H_i(\underline{x}'; M)} \longrightarrow H_i(\underline{x}; M) \longrightarrow (0 \underset{H_{i-1}(\underline{x}'; M)}{:} x_n) \longrightarrow 0$$

By the induction hypothesis this can be rewritten

$$0 \longrightarrow 0 \longrightarrow H_i(\underline{x}; M) \longrightarrow (0 \underset{H_{i-1}(\underline{x}'; M)}{:} x_n) \longrightarrow 0.$$

Note also that as long as $i \geq 2$, by our induction hypothesis we have $(0 \underset{H_{i-1}(\underline{x}'; M)}{:} x_n) \subseteq H_{i-1}(\underline{x}'; M) = 0$, so by Fact II.A.1.5 it suffices to show $(0 \underset{H_{i-1}(\underline{x}'; M)}{:} x_n) = 0$ when $i = 1$. In the case when $i = 1$ we have

$$H_{i-1}(\underline{x}'; M) = H_0(\underline{x}'; M) \cong M/(\underline{x}')M$$

by Proposition II.G.3.11. Since $\underline{x}$ is M-regular, x_n is regular on $M/(\underline{x}')M$ and is therefore not a zero-divisor on $H_0(\underline{x}';M)$. Hence

$$(0 \underset{H_0(\underline{x}';M)}{:} x_n) = 0.$$

□

def120217c

Definition II.G.3.18. An R-module M has <u>finite projective dimension</u> (written $\mathrm{pd}_R(M) < \infty$) if there exists an exact sequence

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that $P_0, \dots, P_n$ are each projective. Given such a sequence we also write $\mathrm{pd}_R(M) \leq n$; we have equality in the case when the above is the shortest such sequence.

ex122117b

Example II.G.3.19. By the above definition, an R-module M is projective if and only if its projective dimension is zero.

ex122117c

Example II.G.3.20. We claim if M is a finitely generated abelian group (i.e., a finitely generated $\mathbb{Z}$-module), then $\mathrm{pd}_{\mathbb{Z}}(M) \leq 1$. If M has generators $m_1, \dots, m_r$, then by the Fundamental Theorem of Finitely Generated Abelian Groups we write

$$M \cong \mathbb{Z}^{r-n} \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z} \qquad \text{(II.G.3.20.1)}$$ eqn112618e

for some integers $d_1, \dots, d_n$. Hence one can surject onto M from the free module $\mathbb{Z}^r$:

$$\mathbb{Z}^r \xrightarrow{\tau} M \longrightarrow 0$$

where $\tau(e_i) = m_i$ for each standard basis vector e_i. Using the isomorphism (II.G.3.20.1) we complete the projective resolution as a short exact sequence.

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{D} \mathbb{Z}^r \xrightarrow{\tau} M \longrightarrow 0$$

Here D can be represented as a matrix mapping generators of $\mathbb{Z}^n$ to generators of $\ker(\tau)$.

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}$$

A takeaway from this example is that the Fundamental Theorem gives us a way to build free resolutions.

thm010418f

Theorem II.G.3.21. *Assume $\underline{x}$ is R-regular.*

thm010418f.a (a) $\mathrm{K}_\bullet(\underline{x})$ *is a free resolution of* $R/(\underline{x})R$ *over* R.

thm010418f.b (b) $\mathrm{Ext}^i_R(R/(\underline{x})R, R/(\underline{x})R) \cong (R/(\underline{x})R)^{\binom{n}{i}}$

thm010418f.c (c) $\mathrm{pd}_R(R/(\underline{x})R) = n$

Proof. (a) Theorem II.G.3.17 tells us we have vanishing homologies for all $i \neq 0$ and Proposition II.G.3.11(c) tells us $H_0(\underline{x}) \cong R/(\underline{x})R$. It follows readily that the following augmented Koszul complex is exact.

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & R & \longrightarrow & R^n & \longrightarrow & \cdots & \longrightarrow & R^n & \longrightarrow & R & \xrightarrow{\tau} & \dfrac{R}{(\underline{x})R} & \longrightarrow & 0 \\ & & n & & n-1 & & & & 1 & & 0 & & -1 & & \end{array}$$

Note we have incidentally shown $\mathrm{pd}_R(R/(\underline{x})) \leq n$.

(b) The free resolution of $R/(\underline{x})R$ from part (a) is a projective resolution so we consider

$$\begin{array}{rccccccccccc} \mathrm{Hom}_R(\mathrm{K}_\bullet(\underline{x}), R/(\underline{x})) = & 0 & \longrightarrow & R^* & \longrightarrow & (R^n)^* & \longrightarrow \cdots \longrightarrow & (R^n)^* & \longrightarrow & R^* & \longrightarrow & 0 \\ & & & 0 & & -1 & & -(n-1) & & -n & & \end{array}$$

which is isomorphic to

$$\begin{array}{ccccccccccccccccc}
0 & \to & R/(\underline{x}) & \to & (R/(\underline{x}))^n & \to & \cdots & \to & (R/(\underline{x}))^{\binom{n}{i}} & \to & \cdots & \to & (R/(\underline{x}))^n & \to & R/(\underline{x}) & \to & 0 \\
 & & 0 & & -1 & & & & i & & & & -(n-1) & & -n & &
\end{array}$$

by Hom-cancellation. Let the above complex be denoted $\diamond$. The differentials of $\diamond$ are the transposes of the matrices representing the differentials in the original free resolution, which are composed entirely of zeroes and $\pm x_1, \ldots, \pm x_n$. Hence every differential in $\diamond$ is a zero map and therefore

$$\operatorname{Ext}^i_R(R/(\underline{x}), R/(\underline{x})) \cong H_{-i}(\diamond) \cong (R/(\underline{x}))^{\binom{n}{i}}$$

for all i.

(c) Suppose the projective dimension of $R/(\underline{x})$ is less than n. Then there exists a projective resolution

$$0 \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0\ .$$

Since Ext is independent of choice of resolution by Theorem II.F.5.2, this implies

$$0 \cong \operatorname{Ext}^n_R(R/(\underline{x}), R/(\underline{x})) \cong (R/(\underline{x}))^{\binom{n}{n}} \cong R/(\underline{x}) \neq 0$$

where the non-vanishing holds since $\underline{x}$ is R-regular. Hence part (c) is proven by contradiction. □

ex122117d

EXAMPLE II.G.3.22. Let $\mathbb{K}$ be a field and let R be one of the following rings.

$$\mathbb{K}[X_1, \ldots, X_n] \qquad \mathbb{K}[X_1, \ldots, X_n]_{(X_1, \ldots, X_n)} \qquad \mathbb{K}[\![X_1, \ldots, X_n]\!]$$

In any case, the sequence $\underline{X} = X_1, \ldots, X_n$ is R-regular and as we saw in Theorem II.G.3.21, the augmented Koszul complex is therefore exact

$$\begin{array}{ccccccccccccc}
0 & \to & R & \to & R^n & \to & \cdots & \to & R^n & \to & R & \to & R/(\underline{X}) \to 0 \\
 & & n & & n-1 & & & & 1 & & 0 & &
\end{array}$$

and $\operatorname{pd}_R(R/(\underline{X})) = n$. This is a noteworthy example, because in general writing out projective resolutions is very hard. In fact, even detecting finite projective dimension is difficult.

II.G.4. Additional Discussions on Ext

ection062921z

In the first theorem of the section, we strengthen part of Proposition II.D.2.3, which we will subsequently generalize in Theorem II.G.4.3. This is related to the very important Hilbert Syzygy Theorem (II.G.4.4) and results of Auslander, Buchsbaum, and Serre (II.G.4.11), and Auslander and Bridger (II.G.4.18).

thm110817a

THEOREM II.G.4.1. *Let R be a commutative ring with identity and let M be an R-module. The following are equivalent.*

thm110817a.i (i) *M is a projective module over R.*

hm110817a.ii (ii) $\operatorname{Ext}^i_R(M, N) = 0$ *for all $i \geq 1$ and for all R-modules N.*

m110817a.iii (iii) $\operatorname{Ext}^1_R(M, N) = 0$ *for all R-modules N.*

PROOF. It is obvious that (ii) implies (iii). The implication (i) implies (ii) is Proposition II.D.2.3(a). The implication (iii) implies (i) follows from Theorem II.G.2.4 and Definition II.A.1.14(d). □

lem052718a

LEMMA II.G.4.2 (Dimension Shifting). *Assume*

$$0 \longrightarrow A \xrightarrow{\varepsilon} L_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} L_0 \xrightarrow{\tau} B \longrightarrow 0 \qquad \text{(II.G.4.2.1)}$$ eqn052718a

is an exact sequence of R-modules and that L_i is projective for each $i = 0, \ldots, n-1$. Then for all $i \geq 1$ and for any R-module X we have

$$\operatorname{Ext}^{n+i}_R(B, X) \cong \operatorname{Ext}^i_R(A, X).$$

Proof. Let the following be a projective resolution of A, indexed rather suggestively.

$$\cdots \xrightarrow{d_{n+2}} L_{n+1} \xrightarrow{d_{n+1}} L_n \xrightarrow{\pi} A \longrightarrow 0$$

We can splice this with (II.G.4.2.1) to get

$$\begin{tikzcd}
\cdots \arrow[r] & L_{n+1} \arrow[r, "d_{n+1}"] & L_n \arrow[rr, "d_n"] \arrow[dr, "\pi"'] & & L_{n-1} \arrow[r, "d_{n-1}"] & \cdots \arrow[r, "d_1"] & L_0 \arrow[r, "\tau"] & B \arrow[r] & 0 \\
& & & A \arrow[ur, "\varepsilon"'] \arrow[dr] & & & & & \\
& & 0 \arrow[ur] & & 0 & & & &
\end{tikzcd}$$

where $d_n = \varepsilon \circ \pi$. A diagram chase shows that the top row of this diagram is an augmented projective resolution of B. Calculating Ext using this we have

$$\operatorname{Ext}_R^{n+i}(B, X) = \frac{\operatorname{Ker}\ L^*_{n+i} \xrightarrow{d^*_{n+i}} L^*_{n+i+1}}{\operatorname{Im}\ L^*_{n+i-1} \xrightarrow{d^*_{n+i-1}} L^*_{n+i}} = \operatorname{Ext}_R^i(A, X)$$

for any $i \geq 1$. (Note that there is an alternative proof using long exact sequences associated with (II.G.4.2.1).) $\square$

The following theorem generalizes Theorem II.G.4.1.

thm110817c Theorem II.G.4.3. *Let $n \in \mathbb{N}$ and let M be an R-module. The following are equivalent.*

thm110817c.i (i) *There exists an exact sequence*

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that each P_i is projective, i.e., $\operatorname{pd}_R(M) \leq n$.

hm110817c.ii (ii) $\operatorname{Ext}_R^i(M, -) = 0$ *for all $i \geq n+1$.*

m110817c.iii (iii) $\operatorname{Ext}_R^{n+1}(M, -) = 0$.

hm110817c.iv (iv) *For every augmented projective resolution of M*

$$Q_\bullet^+ = \qquad \cdots \longrightarrow Q_{n+1} \xrightarrow{\partial^Q_{n+1}} Q_n \xrightarrow{\partial^Q_n} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

the module $\operatorname{Im} \partial^Q_n$ is projective. That is, the augmented resolution above can be "softly truncated" to form a new projective resolution, written below.

$$0 \longrightarrow \operatorname{Im} \partial^Q_n \xrightarrow{\subseteq} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

Proof. Showing (iv) implies (i) and showing (ii) implies (iii) are each trivial, and (i) implies (ii) follows from Note II.B.1.7. So we will endeavor only to show that (iii) implies (iv). Assume (iii) holds and let $Q_\bullet^+$ be an augmented projective resolution of M. By Lemma II.G.4.2 and our assumption we have

$$0 = \operatorname{Ext}_R^{n+1}(M, N) \cong \operatorname{Ext}_R^1\left(\operatorname{Im} \partial^Q_n, N\right)$$

for all R-modules N. Therefore $\operatorname{Im} \partial^Q_n$ is projective by Theorem II.G.4.1. $\square$

The next result gives some rings over which all modules have finite projective dimension. Its proof is outside the scope of this document. See the subsequent example for rings that have modules of infinite projective dimension.

thm112617d Theorem II.G.4.4 (Hilbert Syzygy Theorem). *Let $R = \mathbb{K}[x_1, \ldots, x_d]$ where $\mathbb{K}$ is a field, let M be an R-module, and let $P_\bullet^+$ be an augmented projective resolution of M. Under these assumptions $\operatorname{Im} \partial^P_d$ is projective. This is called a $\underline{d^{th}\text{ syzygy of }M}$. If $R = \mathbb{Z}[x_1, \ldots, x_d]$, then $\operatorname{Im} \partial^P_{d+1}$ is projective. If we localize either of these two rings, then the respective conclusions still hold.*

ex112617e

Example II.G.4.5. Define the following two rings.

$$R_1 = \frac{\mathbb{K}[x]}{(x^2)} \qquad\qquad R_2 = \frac{\mathbb{K}[x,y]}{(xy)}$$

The rings R_1 and R_2 are not integral domains, so they are not (localizations of) polynomial rings over fields (and the hypotheses of Theorem II.G.4.4 are therefore not satisfied). It is a fact (beyond the scope of this document) that if M_1 is an R_1-module and not free, then given a projective resolution $P_\bullet$ of M_1, the module $\operatorname{Im}\partial_n^P$ is never projective.

For example, consider the module

$$K = \frac{R_1}{xR_1}$$

for which we construct an augmented projective resolution.

$$P_\bullet^+ = \quad \begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & \\
 & & & xR_1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[r] & R_1 \arrow[r, "x\cdot"] & R_1 \arrow[ur] \arrow[rr, "x\cdot"] & & R_1 \arrow[rr, "x\cdot"] \arrow[dr] & & R_1 \arrow[r, "\tau"] & K \arrow[r] & 0 \\
 & & & & & xR_1 \arrow[ur] \arrow[dr] & & & \\
 & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

The map τ is the natural surjection and we can observe immediately that at no point does this resolution terminate, which is a result of the fact that $\operatorname{Im} x\cdot = xR_1$ is not projective. Indeed if xR_1 were free, then $\operatorname{Ann}_R(xR_1) = \{0\}$, but $0 \neq x \in \operatorname{Ann}_R(xR_1)$ since $x^2 = 0$ implies $x \cdot xR_1 = 0$. Moreover, xR_1 is not projective by Corollary II.C.4.18, because R_1 is local. Hence $xR_1 = \operatorname{Im}\partial_1^P = \operatorname{Im}\partial_n^P$ is not projective, for all $n \geq 1$.

Let us justify our claim that R_1 is not local. Recall the prime correspondence under quotients.

$$\begin{aligned}
\{\mathfrak{p} \in \operatorname{Spec}(R_1)\} \rightleftarrows \; & \left\{\mathfrak{p} \in \operatorname{Spec}(\mathbb{K}[x]) \;\middle|\; x^2 \in \mathfrak{p}\right\} \\
=& \{\mathfrak{p} \in \operatorname{Spec}(\mathbb{K}[x]) \mid x \in \mathfrak{p}\} \\
=& \{\mathfrak{p} \in \operatorname{Spec}(\mathbb{K}[x]) \mid (x) \subseteq \mathfrak{p}\}
\end{aligned}$$

Since (x) is maximal in $\mathbb{K}[x]$, there is only one ideal on the right and therefore only one ideal on the left, so R_1 is local.

It follows that for all $n \in \mathbb{Z}$, there exists some R-module N_n such that $\operatorname{Ext}^n_{R_1}(K, N_n)$ is non-zero. In fact, for any n we may set $N_n = K$. Consider $\operatorname{Hom}_{R_1}(P_\bullet, K)$ below.

$$0 \longrightarrow \operatorname{Hom}_{R_1}(R_1, K) \xrightarrow{x\cdot^*} \operatorname{Hom}_{R_1}(R_1, K) \xrightarrow{x\cdot^*} \operatorname{Hom}_{R_1}(R_1, K) \xrightarrow{x\cdot^*} \cdots$$

By Hom-cancellation this is isomorphic to the following.

$$0 \longrightarrow K \xrightarrow[=0]{x\cdot} K \xrightarrow[=0]{x\cdot} K \xrightarrow[=0]{x\cdot} \cdots$$

Therefore we may compute Ext for any $n \geq 1$.

$$\operatorname{Ext}^n_{R_1}(K,K) = \frac{\operatorname{Ker}\; K \xrightarrow{0} K}{\operatorname{Im}\; K \xrightarrow{0} K} = \frac{K}{0} \cong K \neq 0$$

Now let us play with R_2, defining the modules $K = R_2/(x, y)R_2$ and $M = R_2/xR_2 \cong K[y]$. Constructing an augmented projective resolution $P_\bullet^+$ of M, we observe a periodic behavior.

$$
\begin{tikzcd}
 & & & 0 \arrow[dr] & & 0 & & & & \\
 & & & & yR_2 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[r, "y"] & R_2 \arrow[rr, "x"] \arrow[dr, "x"'] & & R_2 \arrow[ur, "y"] \arrow[rr, "y"'] & & R_2 \arrow[rr, "x"] \arrow[dr, "x"'] & & R_2 \arrow[r] & M \arrow[r] & 0 \\
 & & xR_2 \arrow[ur] \arrow[dr] & & & & xR_2 \arrow[ur] \arrow[dr] & & & \\
 & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

We may also construct an augmented projective resolution $Q_\bullet^+$ of K that exhibits a similar periodic behavior, but not immediately.

$$
\begin{tikzcd}
\cdots \arrow[r, "\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}"] & R_2^2 \arrow[r, "\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}"] & R_2^2 \arrow[r, "\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}"] & R_2^2 \arrow[r, "\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}"] & R_2^2 \arrow[rr, "\begin{pmatrix} x & y \end{pmatrix}"] \arrow[dr] & & R_2 \arrow[r] & K \arrow[r] & 0 \\
 & & & & & (x, y)R_2 \arrow[ur] \arrow[dr] & & & \\
 & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

As above, we know

$$\operatorname{Ext}^n_{R_2}(M, K) \neq 0 \neq \operatorname{Ext}^n_{R_2}(K, K).$$

Applying the $\operatorname{Hom}_{R_2}(-, M)$ functor to $P_\bullet$, we compute $\operatorname{Ext}^n_{R_2}(M, M)$ precisely. Skipping over the Hom-cancellation step we have

$$\operatorname{Hom}_{R_2}(P_\bullet, M) = \qquad 0 \longrightarrow M \xrightarrow[=0]{x\cdot} M \xrightarrow[\neq 0]{y\cdot} M \xrightarrow[=0]{x\cdot} M \xrightarrow[\neq 0]{y\cdot} \cdots$$

so we compute as follows.

$$\operatorname{Ext}^0_{R_2}(M, M) = \frac{\operatorname{Ker}\ M \xrightarrow{x\cdot} M}{\operatorname{Im}\ 0 \longrightarrow M} = \frac{M}{0} \cong M$$

$$\forall n \geq 1 \qquad \operatorname{Ext}^{2n-1}_{R_2}(M, M) = \operatorname{Ext}^1_{R_2}(M, M) = \frac{\operatorname{Ker}\ M \xrightarrow{y\cdot} M}{\operatorname{Im}\ M \xrightarrow{x\cdot} M} = \frac{0}{0} = 0$$

$$\forall n \geq 1 \qquad \operatorname{Ext}^{2n}_{R_2}(M, M) = \operatorname{Ext}^2_{R_2}(M, M) = \frac{\operatorname{Ker}\ M \xrightarrow{x\cdot} M}{\operatorname{Im}\ M \xrightarrow{y\cdot} M} = \frac{M}{yM} \cong \frac{K[y]}{yK[y]} \cong K$$

It is natural to ask whether one can say anything nice (as in Theorem II.G.4.3(iv)) about the image modules occurring in the resolutions from Example II.G.4.5. In fact, we can, using the following notion; see Theorem II.G.4.8.

def120117a DEFINITION II.G.4.6. An R-module G is totally reflexive if

(a) G is finitely generated and the map

$$\begin{aligned} \delta_R^G : G &\xrightarrow{\cong} \operatorname{Hom}_R(\operatorname{Hom}_R(G,R),R) \\ g &\longmapsto \Psi_g \end{aligned}$$

is an R-module isomorphism, where

$$\begin{aligned} \Psi_g : \operatorname{Hom}_R(G,R) &\longrightarrow R \\ \psi &\longmapsto \psi(g). \end{aligned}$$

(b) For all $i \geq 1$ we have
$$\operatorname{Ext}^i_R(G,R) = 0 = \operatorname{Ext}^i_R(\operatorname{Hom}_R(G,R),R).$$

Using the notation $(-)^* := \operatorname{Hom}_R(-,R)$ we may write more succinctly

(a) G is finitely generated and $\delta_R^G \colon G \xrightarrow{\cong} G^{**}$.

(b) $\operatorname{Ext}^i_R(G,R) = 0 = \operatorname{Ext}^i_R(G^*,R)$ for all $i \geq 1$.

ex111218a EXAMPLE II.G.4.7. Let $n \in \mathbb{N}$. The finitely generated free module R^n is totally reflexive as is any finitely generated projective R-module.

thm120217b THEOREM II.G.4.8 (Auslander-Bridger). *Let $\mathbb{K}$ be a field. If R is either of the two rings*

$$\mathbb{K}[x_0,\ldots,x_d]/(f) \qquad\qquad \mathbb{Z}[x_1,\ldots,x_d]/(f)$$

(where f is a non-zero, non-unit polynomial) or a localization of either of these, then there exists an exact sequence

$$0 \longrightarrow G_d \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

such that $G_0,\ldots,G_d$ are each totally reflexive. Moreover for every projective resolution $P_\bullet$ of M such that each P_i is finitely generated, the module $\operatorname{Im}\partial_d^P$ is totally reflexive. Therefore the sequence

$$0 \longrightarrow \operatorname{Im}\partial_d^P \longrightarrow P_{d-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is exact and $P_0,\ldots,P_{d-1},\operatorname{Im}\partial_d^P$ are all totally reflexive.

Now we return to projective dimension.

fact120217f FACT II.G.4.9. *For any local noetherian ring $(R,\mathfrak{m})$, the number of generators of $\mathfrak{m}$ is no smaller than the Krull dimension of R,*

$$\dim(R) = \sup\{n \in \mathbb{N} \mid \exists \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \operatorname{Spec}(R)\}.$$

Next we restate part of Definition II.B.3.4 and give a more complete version of Theorem II.B.3.8.

def120217e DEFINITION II.G.4.10. A local noetherian ring $(R,\mathfrak{m})$ is <u>regular</u> if the number of generators of $\mathfrak{m}$ is equal to the Krull dimension of R.

thm120217g THEOREM II.G.4.11 (Auslander, Buchsbaum, Serre). *Let $(R,\mathfrak{m},k)$ be a noetherian local ring. The following are equivalent.*

(i) *R is regular.*
(ii) $\operatorname{pd}_R(M) < \infty$ *for all R-modules M.*
(iii) $\operatorname{pd}_R(k) < \infty$.
(iv) $\operatorname{pd}_R(k) = \dim(R)$, *i.e., the finite projective dimension is equivalent to the Krull dimension.*
(v) $\operatorname{pd}_R(M) \leq \dim(R)$ *for all R-modules M.*
(vi) $\operatorname{Ext}^{d+1}_R(M,-) = 0$ *for all R-modules M, where $d = \dim(R)$.*
(vii) *For every R-module M and for every projective resolution $Q_\bullet$ of M, $\operatorname{Im}\partial_d^Q$ is projective, where $d = \dim(R)$.*

While R is always projective as an R-module (in fact, R^n is projective for all $n \geq 1$), R is rarely injective as an R-module (defined in II.A.1.15), as we see next.

thm120417b THEOREM II.G.4.12. *Assume R is noetherian and that R is either local or an integral domain. If R has a non-zero, finitely generated injective module, then R is artinian. That is, R satisfies the following equivalent conditions.*

(i) *R satisfies the descending chain condition on ideals.*
(ii) *R is a noetherian ring with Krull dimension zero.*
(iii) *R is a noetherian ring and every prime ideal is maximal.*

ex120417c

EXAMPLE II.G.4.13. If R is any field, then the only two ideals are R and 0, implying each of the following also hold.

(a) R satisfies both the ascending and descending chain conditions on ideals (the only non-trivial chain of ideals is $0 \subsetneq R$).
(b) The sole prime ideal of R is the zero ideal, so the Krull dimension of R is 0.
(c) Every R-module is a free module (i.e., of the form $R^{(\Lambda)}$) and therefore all R-modules are both injective and projective.

ex120417d

EXAMPLE II.G.4.14. Consider the ring of integers $R = \mathbb{Z}$, for which the ascending chain condition holds, but for which the descending chain condition fails. Ti see why the ascending chain condition holds, consider an arbitrary ascending chain of ideals.

$$n_1\mathbb{Z} \subseteq n_2\mathbb{Z} \subseteq n_3\mathbb{Z} \subseteq \dots$$

The integer n_1 has a finite list of prime factors. In order for the chain above to be one of proper containments, one must remove at least one prime factor from the list at each step. Since the list is finite, the chain has to stabilize.

We can confirm the descending chain condition fails by giving the following example.

$$\mathbb{Z} \supsetneq 10\mathbb{Z} \supsetneq 20\mathbb{Z} \supsetneq 40\mathbb{Z} \supsetneq \cdots \supsetneq 10 \cdot 2^k\mathbb{Z} \supsetneq \cdots$$

We can also confirm that $\mathbb{Z}$ is not injective as a $\mathbb{Z}$-module, which we do by showing $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ is not exact. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{13\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}/13\mathbb{Z} \longrightarrow 0 \qquad \text{(II.G.4.14.1)}$$ eqn111218b

and apply $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$.

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/13\mathbb{Z}, \mathbb{Z}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{13\cdot} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \longrightarrow 0 \qquad \text{(II.G.4.14.2)}$$ eqn052218b

By Hom-cancellation the labeled map above can be written

$$\mathbb{Z} \xhookrightarrow{13\cdot} \mathbb{Z}\,.$$

Since the multiplication map is not onto, (II.G.4.14.2) is not exact.

Alternatively, our short exact sequence (II.G.4.14.1) is also an augmented projective resolution for the $\mathbb{Z}$-module $\mathbb{Z}/13\mathbb{Z}$, yielding the sequences

$$P_\bullet = \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{13\cdot} \mathbb{Z} \longrightarrow 0$$

$$\operatorname{Hom}_{\mathbb{Z}}(P_\bullet, \mathbb{Z}) \cong \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{13\cdot} \mathbb{Z} \longrightarrow 0$$

by Hom-cancellation. We calculate $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/13\mathbb{Z}, \mathbb{Z})$ below.

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/13\mathbb{Z}, \mathbb{Z}) = \frac{\operatorname{Ker}\ \mathbb{Z} \longrightarrow 0}{\operatorname{Im}\ \mathbb{Z} \xrightarrow{13\cdot} \mathbb{Z}} = \frac{\mathbb{Z}}{13\mathbb{Z}} \neq 0$$

If $\mathbb{Z}$ were injective, then for an arbitrary $\mathbb{Z}$-module M, $\operatorname{Ext}^i_{\mathbb{Z}}(M, \mathbb{Z}) = 0$ for all $i \geq 1$ (Definition II.A.1.15 part (d)). Since this has just been shown not to be the case, $\mathbb{Z}$ is not injective as a $\mathbb{Z}$-module.

rmk120417e

REMARK II.G.4.15. A similar result as in the previous example can be obtained for any integral domain that is not a field, as the construction requires only that we have a ring R with a non-zero-divisor. In general R is not injective as an R-module. Moreover, in general there does not exist an exact sequence

$$0 \longrightarrow R \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow 0$$

such that $I_0, \dots, I_n$ are injective. This prompts the following definition.

def120417f

Definition II.G.4.16. A noetherian ring R is Gorenstein if there exists an exact sequence

$$0 \longrightarrow R \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow 0$$

such that $I_0, \ldots, I_n$ are injective, i.e., R has finite injective dimension and we write $\mathrm{id}_R(R) < \infty$.

ex120417g

Example II.G.4.17. Each of the following rings are Gorenstein.

$$\frac{\mathbb{K}[x_0, \ldots, x_n]}{(f)} \qquad\qquad \frac{\mathbb{Z}[x_1, \ldots, x_n]}{(f)}$$

In Theorem II.G.4.11 we see that given a regular local ring R, one can take any R-module M along with any projective resolution $P_\bullet$ of M, and it follows that the kernel of every differential past the d^{th} spot will be projective, where $d = \dim(M)$. What if the ring is only Gorenstein? The answer comes in the next result by Auslander and Bridger. Compare it to Theorem II.G.4.8 with Example II.G.4.17 in mind.

thm120417h

Theorem II.G.4.18 (Auslander-Bridger). *If $(R, \mathfrak{m}, k)$ is a local noetherian ring, then the following are equivalent.*

(i) *R is a Gorenstein ring.*

(ii) *For every finitely generated R-module M, there exists an exact sequence*

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

such that $G_0, \ldots, G_n$ are all totally reflexive.

(iii) *For every finitely generated R-module M and for every projective resolution $P_\bullet$ where each P_i is finitely generated, the module $\operatorname{Im} \partial_d^P$ is totally reflexive.*

(iv) *There exists an exact sequence*

$$0 \longrightarrow G_d \longrightarrow G_{d-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow k \longrightarrow 0$$

such that $G_0, \ldots, G_d$ are totally reflexive and where $d = \dim(R)$ is the Krull dimension of R.

Part III

Free Resolutions

part210604a

Introduction

ection063021a

This part will be broken into three chapters and we outline them here.

The reader will be familiar with the notion of using small amounts of summary data to gain insight into sometimes exceedingly complicated mathematical objects. For the statistician, one uses measures of center and spread, for instance, to understand immense data sets. In graduate-level abstract algebra, students are exposed to the Sylow Theorems, whereby significant structural information is gleaned from knowing only the order of a finite group. We know that one way to determine two groups G and H are not isomorphic is to show their respective orders are different, however, we also know that merely knowing $|G| = |H|$ does not imply $G \cong H$. So tools like the Sylow Theorems are powerful, yet are quite limited. In this course we are interested in similar tools that allow us to understand rings and modules.

In general, throughout the course we will use the following notation. We let k be a field and let $R = k[X_1, \ldots, X_d]$ be the polynomial ring in d variables with coefficients in k. We will write $I \leq R$ to denote an ideal I of the ring R. We want to understand the quotient ring $S = R/I$ and one of the aforementioned tools for doing so is free resolutions, the existence of which will be explored in Chapter III.A, along with Hilbert's Syzygy Theorem, presented below.

hilbert0 THEOREM 1 (Hilbert's Syzygy Theorem). *Let k be a field and $R = k[X_1, \ldots, X_d]$ the polynomial ring in d variables.*

hilbert0.a (a) *If $I \leq R$ is $I = \langle f_1, \ldots, f_{\beta_1} \rangle$ where f_i is a polynomial in R for $i = 1, \ldots, \beta_1$, then there exists an exact sequence*

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & R^{\beta_d} & \xrightarrow{\partial_d} \cdots \xrightarrow{\partial_3} & R^{\beta_2} & \xrightarrow{\partial_2} & R^{\beta_1} & \xrightarrow[\begin{pmatrix} f_1 & \cdots & f_{\beta_1} \end{pmatrix}]{\partial_1} & R & \xrightarrow{\tau} & R/I & \longrightarrow & 0. \\ d+1 & & d & & 2 & & 1 & & 0 & & & & \end{array}$$

This is an <u>*augmented free resolution*</u> *of R/I over R. The* <u>*free resolution*</u> *omits the module R/I. The maps ∂_i are the* <u>*differentials*</u> *in the resolution and the (homological) degree of each module in the resolution is given beneath it. It is common to write simply ∂ when the degree is understood.*

hilbert0.b (b) *If f_i is homogeneous for $i = 1, \ldots, d$, then this resolution can be built minimally and the β_j's are independent of the choice of minimal free resolution. The integer*

$$\beta_j = \beta_j^R(R/I)$$

is the <u>*j^{th} Betti number*</u> *of R/I over R. This notion is originally from algebraic topology where it was named after Enrico Betti by Poincaré and modernized by Emmy Noether.*

It should be noted that part (a) of the theorem guarantees the sequence will vanish beyond (homological) degree d, but it is not necessarily the case that $R^{\beta_i} \neq 0$ for $i = 1, \ldots, d$. An application of the theorem is as follows, and it resembles our finite group example above. If $J \leq R$ is another ideal generated by polynomials in R and $\beta_j^R(R/I) \neq \beta_j^R(R/J)$ for some j, then $R/I \ncong R/J$. However, if $\beta_j^R(R/I) = \beta_j^R(R/J)$ for all j, then R/I may or may not be isomorphic to R/J.

Chapter III.B of the course will be spent exploring examples of free resolutions, including the Koszul complex named after J. L. Koszul, which we partially present here.

koszul0 EXAMPLE 2 (Koszul Complex). Let $R = k[X_1, \ldots, X_d]$ and let $\{i_1, \ldots, i_{\beta_1}\}$ be a subset of $\{1, \ldots, d\}$. Then let $I = \left\langle X_{i_1}, \ldots, X_{i_{\beta_1}} \right\rangle \leq R$ be an ideal of R and we have the following free resolution.

$$0 \longrightarrow R^{\binom{\beta_1}{\beta_1}} \xrightarrow{\partial_{\beta_1}} \cdots \xrightarrow{\partial_4} R^{\binom{\beta_1}{3}} \xrightarrow{\partial_3} R^{\binom{\beta_1}{2}} \xrightarrow{\partial_2} R^{\binom{\beta_1}{1}} \xrightarrow[\begin{pmatrix} X_{i_1} & \cdots & X_{i_{\beta_1}} \end{pmatrix}]{\partial_1} R \longrightarrow 0$$

So the Betti numbers

$$\beta_j^R(R/I) = \binom{\beta_1}{j}$$

are independent of the list $\{i_1, \ldots, i_{\beta_1}\}$. This demonstrates that if I and J are ideals of R with equivalent Betti numbers $\beta_j^R(R/I) = \beta_j^R(R/J)$, then we need not have $R/J \cong R/I$.

In Chapter III.C of this course we will explore the theory of differential graded algebra (DGA) resolutions, as well as examples and applications of such resolutions. The Koszul complex is one such DGA resolution, as it admits a unital ring structure in the following way.

dga0

Example 3. Consider the ideal $I = \langle X_1, X_2, X_3 \rangle \leq R = k[X_1, \ldots, X_d]$. Then the Koszul complex as seen in the previous example is

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & R & \xrightarrow{\partial_3} & R^3 & \xrightarrow{\partial_2} & R^3 & \xrightarrow[\begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}]{\partial_1} & R & \longrightarrow & 0 \\ & & e_{123} & & e_{12} & & e_1 & & 1 & & \\ & & & & e_{13} & & e_2 & & & & \\ & & & & e_{23} & & e_3 & & & & \end{array}$$

where we have denoted the basis elements below each R-module. We see that $\partial_1(e_i) = X_i$ for $i = 1, 2, 3$ and we also have the following.

$$\begin{aligned} \partial_2(e_{12}) &= X_1 e_2 - X_2 e_1 & \qquad \partial_3(e_{123}) &= X_1 e_{23} - X_2 e_{13} + X_3 e_{12} \\ \partial_2(e_{13}) &= X_1 e_3 - X_3 e_1 & & \\ \partial_2(e_{23}) &= X_2 e_3 - X_3 e_2 & & \end{aligned}$$

In general we have

$$\partial_m(e_{i_1 \cdots i_m}) = \sum_{j=1}^{m} (-1)^{j-1} X_{i_j} e_{i_1 \cdots \hat{i}_j \cdots i_m}$$

where $i_1 \cdots \hat{i}_j \cdots i_m$ denotes the ordered list $i_1 \cdots i_m$ with i_j omitted. This determines an R-linear map by respecting linear combinations of basis vectors with coefficients in R (this is the UMP for free modules). The multiplication goes as follows.

$$\begin{aligned} e_1 e_2 &= e_{12} & \qquad e_1 e_{12} &= 0 \\ e_1 e_1 &= 0 & e_2 e_{13} &= -e_{123} \\ e_2 e_1 &= -e_1 e_2 = -e_{12} & & \end{aligned}$$

This so-called "wedge product" is unital, associative and graded commutative, i.e.,

$$e_A e_B = (-1)^{|A| \cdot |B|} e_B e_A$$

where $A, B \subseteq \{1, 2, \ldots, m\}$, and $|A|$ and $|B|$ denote the homological degrees of e_A and e_B, respectively. The differentials and multiplication also satisfy the Leibniz rule:

$$\partial_{|A|+|B|}(e_A e_B) = \partial_{|A|}(e_A) e_B + (-1)^{|A|} e_A \partial_{|B|}(e_B).$$

For instance, we have

$$\partial_2(e_1 e_2) = \partial_2(e_{12}) = X_1 e_2 - X_2 e_1 \stackrel{\dagger}{=} \partial_1(e_1) e_2 + (-1)^1 e_1 \partial_1(e_2)$$

where † holds since degree-zero elements commute.

CHAPTER III.A

Homological Algebra

apter072921a

ection063021b

III.A.1. Linear Algebra

Throughout the chapter, we will assume R is a commutative ring with identity, unless stated otherwise.

def190827a

Definition III.A.1.1. Let M be an R-module.

(a) A sequence $e_1, \ldots, e_n \in M$ is a finite basis for M if it generates M as an R-module and it is linearly independent over R, i.e., for every $m \in M$ there exist unique $r_1, \ldots, r_n \in R$ such that $m = \sum_{i=1}^{n} r_i e_i$.

(b) M is a finite rank free R-module if it has a finite basis.

ex190827b

Example III.A.1.2. (a) We define R^n to be the R-module whose elements are column vectors of size n with entries in R, i.e.,

$$R^n = \left\{ \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \middle| \; r_1, \ldots, r_n \in R \right\}.$$

This is a finite rank free R-module with standard basis $e_1, \ldots, e_n$ where $e_i = (\delta_{ij})_j$ and δ_{ij} is the Kronecker delta.

(b) If $0 \neq I \lneq R$, then R/I is not free, because it fails linear independence over R in the following way. If $0 \neq r \in I$ (which exists since $I \neq 0$), then for every $s \in R \setminus I$ we have $0 \neq \overline{s} = s + I \in R/I$ and $rs \in I$, which implies $\overline{rs} = 0 \in R/I$. Therefore $r \cdot \overline{s} = \overline{rs} = 0$ and we have thus exhibited a linear combination which sums to zero, but has a non-zero coefficient.

fact190827a

Fact III.A.1.3. (a) *(Universal Mapping Property) Let F be a free R-module with basis $e_1, \ldots, e_n \in F$. For every R-module M and any collection of elements $m_1, \ldots, m_n \in M$, there exists a unique R-module homomorphism $\phi\colon F \to M$ such that $\phi(e_i) = m_i$ for $i = 1, \ldots, n$.*

$$\begin{tikzcd} \{e_1, \ldots, e_n\} \arrow[rr, "\subseteq"] \arrow[drr] & & F \arrow[d, dashed, "\exists!"] & \sum_i r_i e_i \arrow[d, mapsto] \\ e_i \arrow[r, mapsto] & m_i & M & \sum_i r_i m_i \end{tikzcd}$$

act190827c.b

(b) *If F and G are finite rank free R-modules with bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$, respectively, then $F \cong G$ as R-modules.*

act190827c.c

(c) *If F is a finite rank free R-module with basis $e_1, \ldots, e_n$, then $F \cong R^n$ as R-modules.*

Proof. Part (a) is standard. We prove part (b) and then part (c) will follow as a corollary. Since F and G are free, by the universal mapping property we have R-module homomorphisms $\phi\colon F \to G$ and $\psi\colon G \to F$ such that $\phi(e_i) = f_i$ and $\psi(f_i) = e_i$ for $i = 1, \ldots, n$.

$$\begin{tikzcd} \{e_1, \ldots, e_n\} \arrow[r, "\subseteq"] \arrow[dr] & F \arrow[d, dashed, "\phi"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} \{f_1, \ldots, f_n\} \arrow[r, "\subseteq"] \arrow[dr] & G \arrow[d, dashed, "\psi"] \\ & F \end{tikzcd}$$

Since the composition $\psi \circ \phi$ is a homomorphism from F to F and id_F is likewise a homomorphism from F to F, by the uniqueness given in the universal mapping property we have $\psi \circ \phi = \mathrm{id}_F$. By the same reasoning we know $\phi \circ \psi = \mathrm{id}_G$ and we conclude both ϕ and ψ are isomorphisms, i.e., $F \cong G$.

$$\begin{tikzcd} \{e_1, \ldots, e_n\} \arrow[r, "\subseteq"] \arrow[dr] & F \arrow[d, shift right, "\mathrm{id}_F"'] \arrow[d, shift left, "\psi\circ\phi"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} \{f_1, \ldots, f_n\} \arrow[r, "\subseteq"] \arrow[dr] & G \arrow[d, shift right, "\mathrm{id}_G"'] \arrow[d, shift left, "\phi\circ\psi"] \\ & G \end{tikzcd}$$

□

notn190827d NOTATION III.A.1.4. Let $\phi\colon R^n \to R^m$ be R-linear. We represent ϕ by a matrix A where the j^{th} column of A consists of the coefficients needed to represent $\phi(e_j)$. That is, if $e_1, \ldots, e_n \in R^n$ and $f_1, \ldots, f_m \in R^m$ form the standard bases, then we let $A = (a_{ij})$ where

$$\phi(e_j) = \sum_{i=1}^{m} a_{ij} f_i$$

for $j = 1, \ldots, n$. This partially justifies the following:

$$\operatorname{Hom}_R(R^n, R^m) \cong \operatorname{Mat}_{m \times n}(R) \cong R^{mn}.$$

fact190827e FACT III.A.1.5. *An R-module M is finitely generated over R if and only if there exists an R-module epimorphism (surjective homomorphism) $\tau\colon R^n \to M$ for some $n \in \mathbb{N}$, in which case M is generated by $\tau(e_1), \ldots, \tau(e_n) \in M$.*

thm190827f THEOREM III.A.1.6. *The following are equivalent.*

(i) *Every ideal of R is finitely generated.*
(ii) *R satisfies the ascending chain condition for ideals, i.e., for every chain of ideals $I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$, there exists an integer $N \in \mathbb{N}$ such that $I_N = I_{N+1} = I_{N+2} = \ldots$.*
(iii) *R satisfies the maximum condition for ideals, i.e., every non-empty set of ideals of R contains a maximal element with respect to containment.*
(iv) *For every $n \in \mathbb{N}$, every submodule of R^n is finitely generated.*
(v) *For every $n \in \mathbb{N}$, R^n satisfies the ascending chain condition for submodules.*
(vi) *For every $n \in \mathbb{N}$, R^n satisfies the maximum condition for submodules.*

def190827g DEFINITION III.A.1.7. R is <u>noetherian</u> if it satisfies the equivalent conditions of Theorem III.A.1.6.

fact190827h FACT III.A.1.8. (a) (Hilbert's Basis Theorem) *If R is noetherian, then $R[X]$ is noetherian.*

(b) *If R is noetherian, then for every $n \in \mathbb{N}$ and for every ideal $I \leq R[X_1, \ldots, X_n]$, the quotient ring $R[X_1, \ldots, X_n]/I$ is noetherian.*

(c) *If k is a field, then k is noetherian and by Hilbert's Basis Theorem, $k[X_1, \ldots, X_n]$ is noetherian as well for every $n \in \mathbb{N}$.*

III.A.2. Exact Sequences

ection063021c

def190827i DEFINITION III.A.2.1. (a) A sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ of R-module homomorphisms is <u>exact</u> if $\operatorname{Im}\alpha = \operatorname{Ker}\beta$.

(b) A sequence

$$\cdots \xrightarrow{f_{i+2}} A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \xrightarrow{f_{i-1}} \cdots$$

is <u>exact</u> if $\operatorname{Im} f_{i+1} = \operatorname{Ker} f_i$ for all $i \in \mathbb{Z}$.

(c) A <u>short exact sequence</u> is an exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

fact190827j FACT III.A.2.2. (a) $0 \longrightarrow A \xrightarrow{\alpha} B$ *is exact if and only if α is injective.*

(b) $B \xrightarrow{\beta} C \longrightarrow 0$ *is exact if and only if β is surjective.*

act190827j.c (c) $0 \longrightarrow A \longrightarrow 0$ *is exact if and only if $A = 0$.*

act190827j.d (d) $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$ *is exact if and only if α is an isomorphism.*

thm190827k THEOREM III.A.2.3. *If R is noetherian and M is a finitely generated R-module, then there exists an exact sequence*

$$\cdots \xrightarrow{\partial_{i+1}} R^{\beta_i} \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \xrightarrow{\tau} M \longrightarrow 0.$$

PROOF. Since M is assumed to be finitely generated, by Fact III.A.1.5 the exists an integer $\beta_0 \in \mathbb{N}$ and an epimorphism $\tau\colon R^{\beta_0} \to M$. Since $\operatorname{Ker}\tau$ is a submodule of R^{β_0} and R is noetherian, by Theorem III.A.1.6

and Fact III.A.1.5 there exists an integer $\beta_1 \in \mathbb{N}$ and epimorphism $\tau_1\colon R^{\beta_2} \to \operatorname{Ker}\tau$. This procedure continues and yields the following commutative diagram of short exact diagonal sequences.

$$
\begin{tikzcd}
 & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & & \\
 & & \operatorname{Ker}\tau_2 \arrow[ur] \arrow[dr, "\iota_2"] \arrow[d, phantom, "\circlearrowleft\ \subseteq"] & & & & \operatorname{Ker}\tau \arrow[ur] \arrow[dr, "\iota_0"] \arrow[d, phantom, "\circlearrowleft\ \subseteq"] & & & & 0 \\
\cdots \arrow[r] & R^{\beta_3} \arrow[ur, "\tau_3"] \arrow[rr, "\partial_3=\iota_2\tau_3"'] & {} & R^{\beta_2} \arrow[rr, "\partial_2=\iota_1\tau_2"] \arrow[dr, "\tau_2"'] & {} & R^{\beta_1} \arrow[ur, "\tau_1"] \arrow[rr, "\partial_1=\iota_0\tau_1"'] & {} & R^{\beta_0} \arrow[rr, "\tau"] \arrow[dr, "\tau"'] & {} & M \arrow[ur] \arrow[r] & 0 \\
 & & & & \operatorname{Ker}\tau_1 \arrow[ur, "\iota_1"'] \arrow[dr] \arrow[u, phantom, "\circlearrowleft\ \subseteq"] & & & & M \arrow[ur, "\mathrm{id}_M"'] \arrow[dr] \arrow[u, phantom, "\circlearrowleft"] & & \\
 & & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 &
\end{tikzcd}
$$

A diagram chase shows the horizontal sequence is exact. □

def190827l

Definition III.A.2.4. The exact sequence in Theorem III.A.2.3 is an augmented free resolution of M.

rmk190827m

Remark III.A.2.5. In general, these are difficult to compute. Thus, the following examples are particularly nice. A main point of this course is to construct other examples explicitly.

ex190827n

Example III.A.2.6. We give three examples of free resolutions.

(a) From the fundamental theorem of finitely generated abelian groups, if G is a finitely generated abelian group, there exist positive integers $d_1, \dots, d_n, r \in \mathbb{N}$ such that

$$G \cong \frac{\mathbb{Z}}{(d_1)} \oplus \cdots \oplus \frac{\mathbb{Z}}{(d_n)} \oplus \mathbb{Z}^r.$$

The following is an augmented free resolution of G (as a $\mathbb{Z}$-module).

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow[\begin{pmatrix} d_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & d_n \\ & \mathbf{0} & \end{pmatrix}]{\qquad\qquad} \mathbb{Z}^{n+r} \longrightarrow G \longrightarrow 0$$

(b) If R is an integral domain and $0 \neq r \in R \setminus R^\times$, then the following is an augmented free resolution of $R/(r)$.

$$
\begin{tikzcd}
0 \arrow[r] & R \arrow[rr, "r\cdot"] \arrow[dr] & & R \arrow[r] & \frac{R}{(r)} \arrow[r] & 0 \\
 & & (r) \arrow[ur] \arrow[dr] & & & \\
 & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

(c) These resolutions need not be finite. Consider the ring $R = k[X,Y]/(XY)$ and the R-module $M = R/(\overline{X})$. Then we have the following augmented free resolution of M.

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & \\
 & & & (\overline{X}) \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[r, "\overline{X}\cdot"] & R \arrow[r, "\overline{Y}\cdot"] & R \arrow[ur] \arrow[rr, "\overline{X}\cdot"'] & & R \arrow[rr, "\overline{Y}\cdot"] \arrow[dr] & & R \arrow[r] & \frac{R}{(\overline{X})} \arrow[r] & 0 \\
 & & & & & (\overline{Y}) \arrow[ur] \arrow[dr] & & & \\
 & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

One can find a similar resolution when $R = k[X]/(X^2)$ and $M = R/\left(\overline{X}\right)$.

hilbert

Theorem III.A.2.7 (Hilbert's Syzygy Theorem). *Let $R = k[X_1, \ldots, X_d]$ where k is a field and let M be a finitely generated R-module. Then there exists a free resolution*

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow M \longrightarrow 0$$

with $\beta_i \geq 0$ for $i = 0, \ldots, d$. (One can replace k with a principal ideal domain and the conclusion holds if one also replaces d with $d+1$ in the above sequence.)

rmk190827o

Remark III.A.2.8. (a) If $d = 0$, then R is a field and $M \cong R^{\beta_0}$, so we have the augmented free resolution

$$0 \longrightarrow R^{\beta_0} \longrightarrow M \longrightarrow 0.$$

(b) If $d = 1$, then R is a principal ideal domain and we can construct the following augmented free resolution.

$$\begin{tikzcd}
0 \arrow[dr] & & & & & \\
0 \arrow[r] & R^{\beta_1} \arrow[rr] \arrow[dr, "\cong"'] & & R^{\beta_0} \arrow[r, "\tau"] & M \arrow[r] & 0 \\
 & & \operatorname{Ker}\tau \arrow[ur] \arrow[dr] & & & \\
 & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Here we use the assumptions that $\operatorname{Ker}\tau$ is a submodule of R^{β_0} and R is a principal ideal domain to conclude that $\operatorname{Ker}\tau \cong R^{\beta_1}$ for some $\beta_1 \leq \beta_0$.

The following results were given as exercises. For each exercise we consider the following sequence of R-modules and R-module homomorphisms.

$$A = \qquad \cdots \xrightarrow{\partial^A_{i+1}} A_i \xrightarrow{\partial^A_i} \cdots$$

Assume that each R-module A_i is free with finite basis B_i.

exr190829a

Exercise III.A.2.9. Fix an integer i. If $\partial^A_{i-1}\left(\partial^A_i(b)\right) = 0$ for all $b \in B_i$, then we have $\partial^A_{i-1}\partial^A_i = 0$.

PROOF. Set $\{b_1, \dots, b_r\} = B_i$. By assumption, the following diagram commutes for each of the vertical maps.

$$\begin{tikzcd} \{b_1,\dots,b_r\} \arrow[r, "\subseteq"] \arrow[dr, "0"'] & A_i \arrow[d, shift right, "0"'] \arrow[d, shift left, "\partial^A_{i-1}\circ\partial^A_i"] \\ & A_{i-2} \end{tikzcd}$$

Therefore by the uniqueness given in the Universal Mapping Property (Fact III.A.1.3) we have $\partial^A_{i-1} \circ \partial^A_i = 0$. $\square$

exr190829b EXERCISE III.A.2.10. Fix integers i and j, and let $f\colon B_i \times B_j \to A_{i+j}$ be a function. Then there is a unique well-defined R-bilinear map $\mu_{i,j}\colon A_i \times A_j \to A_{i+j}$ such that $\mu_{i,j}(b, b') = f(b, b')$ for all $b \in B_i$ and $b' \in B_j$.

PROOF. We define $\mu_{i,j}$ the only way we can, because of the requirement of bilinearity.

$$\mu_{i,j}\left(\sum_{b\in B_i} r_b b\,,\ \sum_{b'\in B_j} s_{b'} b'\right) := \sum_{b\in B_i}\sum_{b'\in B_j} r_b s_{b'}\cdot f(b,b')$$

Well-definedness follows readily from the linear independence of B_i and of B_j. If we suppose that ρ is another R-bilinear map satisfying $\rho(b, b') = f(b, b')$ for all $b \in B_i$ and all $b' \in B_j$, then we have

$$\rho\left(\sum_{b\in B_i} r_b b\,,\ \sum_{b'\in B_j} s_{b'} b'\right) = \sum_{b\in B_i}\sum_{b'\in B_j} r_b s_{b'}\rho(b,b') = \sum_{b\in B_i}\sum_{b'\in B_j} r_b s_{b'} f(b,b') = \mu_{i,j}\left(\sum_{b\in B_i} r_b b\,,\ \sum_{b'\in B_j} s_{b'} b'\right),$$

so $\mu_{i,j}$ is unique. $\square$

exr190829c EXERCISE III.A.2.11. Fix integers i and j, and let $\mu_{i,j}\colon A_i \times A_j \to A_{i+j}$ be an R-bilinear map. For all $a \in A_i$ and $a' \in A_j$, set $aa' = \mu_{i,j}(a, a')$.

(a) If $i = 0$ and there exists and element $1 \in A_0$ such that $1b' = b'$ for all $b' \in B_j$, then $1a' = a'$ for all $a' \in A_j$.
(b) If $bb' = (-1)^{ij}b'b$ for all $b \in B_i$ and $b' \in B_j$, then $aa' = (-1)^{ij}a'a$ for all $a \in A_i$ and $a' \in A_j$.
(c) If $b(b' + b'') = bb' + bb''$ for all $b \in B_i$ and $b', b'' \in B_j$ (with the standard order of operations), then $a(a' + a'') = aa' + aa''$ for all $a \in A_i$ and $a', a'' \in A_j$.

exr190829d EXERCISE III.A.2.12. For all integers i and j, let $\mu_{i,j}\colon A_i \times A_j \to A_{i+j}$ be an R-bilinear map. For all $a \in A_i$ and $a' \in A_j$, set $aa' = \mu_{i,j}(a, a')$. Fix integers i, j, and k. If we have $b(b'b'') = (bb')b''$ for all $b \in B_i$, $b' \in B_j$, and $b'' \in B_k$, then $a(a'a'') = (aa')a''$ for all $a \in A_i$, $a' \in A_j$, and $a'' \in A_k$.

ection063021d

III.A.3. Graded Resolutions

In this chapter we are interested in being able to keep track of finer information about certain resolutions.

ass190903a ASSUMPTION III.A.3.1. In this chapter, assume k is a field and that $R = k[X_1, \dots, X_d]$ is the polynomial ring in d variables with the standard grading, i.e., $\deg X_i = 1$ for all i.

def190903b DEFINITION III.A.3.2. A homogeneous (or graded) ideal in R is an ideal generated by homogeneous polynomials (not necessarily of the same degree).

ex190903c EXAMPLE III.A.3.3. Let $R = k[X, Y]$ and consider the ideal $I = \langle X^2, XY^2\rangle$. Since X^2 and XY^2 are each homogeneous, I is a homogeneous ideal. Note that $X^2 - XY^2$ is not homogeneous, yet we have $I = \langle X^2 - XY^2, XY^2\rangle$, so the existence of a non-homogeneous generator in the representation of an ideal does not imply the ideal is not graded.

ilbertgraded THEOREM III.A.3.4 (Hilbert's Syzygy Theorem (graded version)). *If $I = \langle f_1, \dots, f_{\beta_1}\rangle$ such that each f_i is homogeneous, then there exists an (augmented) free resolution*

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_1} \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

such that each differential in the resolution is represented by a matrix of homogeneous polynomials.

ex190903d EXAMPLE III.A.3.5. Let $R = k[X,Y]$ and consider the ideals $I_1 = \langle X,Y\rangle$, $I_2 = \langle X^a, Y^b\rangle$, and $J = \langle X^a, XY, Y^b\rangle$. In the case of I_2 we assume $a,b \geq 1$ and in the case of J we assume $a,b \geq 2$. Then we claim the following are respective free resolutions of R/I_1, R/I_2, and R/J.

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -Y \\ X \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} X & Y \end{pmatrix}} R \longrightarrow \frac{R}{I_1} \longrightarrow 0$$

$$0 \longrightarrow R \xrightarrow[\partial_2^I]{\begin{pmatrix} -Y^b \\ X^a \end{pmatrix}} R^2 \xrightarrow[\partial_1^I]{\begin{pmatrix} X^a & Y^b \end{pmatrix}} R \longrightarrow \frac{R}{I_2} \longrightarrow 0$$

$$0 \longrightarrow R^2 \xrightarrow[\partial_2^J]{\begin{pmatrix} -Y & 0 \\ X^{a-1} & -Y^{b-1} \\ 0 & X \end{pmatrix}} R^3 \xrightarrow[\partial_1^J]{\begin{pmatrix} X^a & XY & Y^b \end{pmatrix}} R \longrightarrow \frac{R}{J} \longrightarrow 0$$

Since the first diagram is just a special case of the second, we need only justify the exactness of the resolutions of R/I_2 and R/J. The exactness at the (homological) degree -1 and 0 positions are by construction. The exactness at the degree 2 position in the second resolution follows from the fact that R is an integral domain and ∂_2^I amounts to the standard scalar multiplication of $(-Y^b\ X^a)^T$ by elements $r \in R$. Also for the second resolution, argue as we do for the third resolution to show that one also has exactness at the degree 1 position.

To show the third resolution is exact at the degree 1 position, we will show $\operatorname{Ker} \partial_1^J = \operatorname{Im} \partial_2^J$, i.e.,

$$\operatorname{Ker} \partial_1^J = \left\langle \begin{pmatrix} -Y \\ X^{a-1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -Y^{b-1} \\ X \end{pmatrix} \right\rangle.$$

The proof of the reverse containment is short.

$$\begin{pmatrix} X^a & XY & Y^b \end{pmatrix} \cdot \begin{pmatrix} -Y \\ X^{a-1} \\ 0 \end{pmatrix} = 0 \qquad \begin{pmatrix} X^a & XY & Y^b \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -Y^{b-1} \\ X \end{pmatrix} = 0$$

For the forward containment, let $(f\ g\ h)^T \in \operatorname{Ker} \partial_1^J$ and note this implies

$$X^a f + XYg + Y^b h = 0. \qquad \text{(III.A.3.5.1)}$$ eqn190903a

Since $X | X^a f$ and $X | XYg$, it follows that $X | Y^b h$ and therefore $X | h$. Let $h' \in R$ such that $h = Xh'$. By similar reasoning, we let $f' \in R$ such that $f = Yf'$. Hence (III.A.3.5.1) becomes

$$0 = X^a Y f' + XYg + Y^b X h' = XY(X^{a-1} f' + g + Y^{b-1} h').$$

Since we are working in an integral domain, this implies $X^{a-1}f' + g + Y^{b-1}h' = 0$ and therefore $g = -X^{a-1}f' - Y^{b-1}h'$. Hence we conclude our argument as follows.

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} Yf' \\ -X^{a-1}f' - Y^{b-1}h' \\ Xh' \end{pmatrix} = f' \begin{pmatrix} Y \\ -X^{a-1} \\ 0 \end{pmatrix} + h' \begin{pmatrix} 0 \\ -Y^{b-1} \\ X \end{pmatrix}$$

To see that the resolution of R/J is exact in the degree 2 position (i.e., that ∂_2^J is injective), let $(c\ d)^T \in \operatorname{Ker} \partial_2^J$ and observe that

$$0 = c \begin{pmatrix} -Y \\ X^{a-1} \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ -Y^{b-1} \\ X \end{pmatrix}$$

implies $dX = 0$ and $cY = 0$, so $c = 0 = d$.

As an aside, we point out that if one deletes one row of the matrix representing ∂_2^J and takes the determinant of the resulting matrix, then one obtains the entries of $(X^a\ XY\ Y^b)$, up to a sign. This is a special case of the Hilbert-Burch Theorem, which we discuss in Chapter III.B.

rmk190903e REMARK III.A.3.6. Notice the resolutions in Example III.A.3.5 are all of the form

$$0 \longrightarrow R^{b_2} \longrightarrow R^{b_1} \longrightarrow R^{b_0} \longrightarrow 0$$

and their exponents all satisfy $b_0 - b_1 + b_2 = 0$. This leads us to the following exercise and theorem.

exr190903f EXERCISE III.A.3.7. Let K be a field and consider the following exact sequence of K-vector spaces.

$$0 \longrightarrow K^{\beta_d} \longrightarrow \cdots \longrightarrow K^{\beta_0} \longrightarrow 0$$

Then

$$\sum_{i=0}^{d} (-1)^i \beta_i = 0.$$

thm190903g THEOREM III.A.3.8. *Let $I \leq R$ be a non-zero ideal and let*

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_0} \longrightarrow R/I \longrightarrow 0$$

be an exact sequence. Then

$$\sum_{i=0}^{d} (-1)^i \beta_i = 0.$$

PROOF. Let $0 = \mathfrak{p} \lneq R$, which is a prime ideal and let $K = R_{\mathfrak{p}} = k(X_1, \ldots, X_d)$ be the field of fractions of R (i.e., localize at $\mathfrak{p}$). Then for any $0 \neq s \in I$ and any $\overline{r}/t \in (R/I)_{\mathfrak{p}}$, we have

$$\frac{\overline{r}}{t} = \frac{\overline{sr}}{st} = \frac{0}{st} = 0,$$

so $(R/I)_{\mathfrak{p}} = 0$. We can localize the given resolution to obtain the following resolution.

$$\begin{array}{ccccccccc} 0 \longrightarrow & R_{\mathfrak{p}}^{\beta_d} & \longrightarrow \cdots \longrightarrow & R_{\mathfrak{p}}^{\beta_0} & \longrightarrow & (R/I)_{\mathfrak{p}} & \longrightarrow 0 \\ & \| & & \| & & \| & \\ & K^{\beta_d} & & K^{\beta_0} & & 0 & \end{array}$$

Thus the desired conclusion follows from Exercise III.A.3.7. □

notn190905a NOTATION III.A.3.9. Let $n \in \mathbb{N}$ and set

$$R_n = \{\text{homogeneous polynomials in } R \text{ of degree } n\} \cup \{0\}.$$

rmk190905b REMARK III.A.3.10. $R_n \subset R$ is a k-subspace, but is not an ideal unless $d = 0$.

housemyrules NOTATION III.A.3.11. Let $m \in \mathbb{Z}$. We say $R(-m)$ is a "shifted" or "twisted" copy of R. It has $R(-m) = R$ as an R-module, but if $f \in R$ is homogeneous, then

$$\deg_{R(-m)}(f) = \deg_R(f) + m,$$

i.e.,

$$R(-m)_n = R_{n-m}.$$

For instance, $1 \in k = R_0 = R(-m)_m$. It follows that $R(-m)$ is a free R-module with basis $\{1\}$ such that $\deg_{R(-m)}(1) = m$. More generally we have that

$$F = \bigoplus_{i=1}^{r} R(-m_i)$$

is a graded free R-module of rank r for $m_1, \ldots, m_r \in \mathbb{Z}$ and

$$F_n = \left(\bigoplus_{i=1}^{r} R(-m_i)\right)_n = \bigoplus_{i=1}^{r} R(-m_i)_n = \bigoplus_{i=1}^{r} R_{n-m_i}.$$

For instance, if $e_1, \ldots, e_r \in F$ is the standard basis, then $\deg_F(e_i) = m_i$. The homogeneous elements of F of degree n are of the form

$$\sum_{i=1}^{r} s_i e_i$$

where each s_i is homogeneous in R with $\deg_R s_i = n - m_i$, because we need $\deg_F(s_i e_i) = \deg_R s_i + m_i$.

ex190905c EXAMPLE III.A.3.12. In the R-module

$$\begin{matrix} R(-a) \\ \oplus \\ R(-b) \end{matrix}$$

we have the element

$$\begin{pmatrix} -Y^b \\ X^a \end{pmatrix} \in \begin{pmatrix} R(-a) \\ \oplus \\ R(-b) \end{pmatrix}_{a+b}$$

because this element can be written as

$$-Y^b e_1 + X^a e_2,$$

where $-Y^b$ and e_2 each have degree b, and X^a and e_1 have degree a.

def190905d DEFINITION III.A.3.13. Let F and G be free graded R-modules of finite rank. A homomorphism $\phi \colon F \to G$ is graded (or homogeneous) if $\phi(F_n) \subseteq G_n$ for all $n \in \mathbb{Z}$.

fact190905e FACT III.A.3.14. *A homomorphism $\phi \colon F \to G$ between graded free modules of finite rank is graded if and only if $\phi(e_i) \in G_{m_i}$ for all $i = 1, \ldots, r$, where $F = \oplus_{i=1}^{n} R(-m_i)$.*

ex190905f EXAMPLE III.A.3.15. Let $R = k[X, Y]$ and let $I = \langle X^a, Y^b \rangle \leq R$ be an ideal where $a, b \geq 2$. Then we have the following (augmented) free resolution of R/I.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R(-a-b) & \xrightarrow{\begin{pmatrix} -Y^b \\ X^a \end{pmatrix}} & \begin{matrix} R(-a) \\ \oplus \\ R(-b) \end{matrix} & \xrightarrow{\begin{pmatrix} X^a & Y^b \end{pmatrix}} & R & \longrightarrow & R/I & \longrightarrow & 0 \\
 & & \varepsilon & \longmapsto & \begin{pmatrix} -Y^b \\ X^a \end{pmatrix} & & & & & & \\
 & & & & e_1 & \longmapsto & X^a & & & & \\
 & & & & e_2 & \longmapsto & Y^b & & & &
\end{array}$$

This is graded because, for instance, the elements $\varepsilon \in R(-a-b)$ and $-Y^b \in R(-a)$ and $X^a \in R(-b)$ all have degree $a + b$.

fact190905g FACT III.A.3.16. *With notation as in Fact III.A.3.14, if ϕ is graded, then*

$$\operatorname{Im} \phi = \langle \phi(e_1), \ldots, \phi(e_r) \rangle$$

is generated by finitely many homogeneous elements. One can also show that $\operatorname{Ker} \phi$ *is generated by finitely many homogeneous elements of F.*

ex190905h EXAMPLE III.A.3.17. The graded homomorphism

$$\phi \colon \begin{matrix} R(-a) \\ \oplus \\ R(-b) \end{matrix} \xrightarrow{\begin{pmatrix} X^a & Y^b \end{pmatrix}} R$$

has kernel generated by the vector $(-Y^b \; X^a)^T$, a homogeneous element of degree $a + b$.

We now give a sketch of the proof of Hilbert's Syzygy Theorem (graded version).

Proof. By assumption $I = \langle f_1, \ldots, f_{\beta_1} \rangle$ and we let $\deg_R f_i = m_i$. We begin computing the resolution in the usual manner, surjecting onto R/I from R in the natural way and then onto $\operatorname{Ker} \tau = I$ from a free module.

$$\bigoplus_{i=1}^{\beta_1} R(-m_i) \xrightarrow[\partial_1]{\begin{pmatrix} f_1 & \cdots & f_{\beta_1} \end{pmatrix}} R \xrightarrow{\tau} R/I \longrightarrow 0$$

By construction $\operatorname{Im} \partial_1 = I = \operatorname{Ker} \tau$ and we consider that by Fact III.A.3.16, we know $\operatorname{Ker} \partial_1$ is free and generated by finitely many homogeneous elements of $\bigoplus_{i=1}^{\beta_1} R(-m_i)$. So there exists a non-negative integer β_2 and homogeneous column vectors $f_{1,i}, \ldots, f_{1,\beta_2} \in \bigoplus_{i=1}^{\beta_1} R(-m_i)$ such that $\operatorname{Ker} \partial_1 = \langle f_{1,i}, \ldots, f_{1,\beta_2} \rangle$. For each $i = 1, \ldots, \beta_2$ let $m_{1,i}$ denote the degree of $f_{1,i}$ and we may surject onto $\operatorname{Ker} \partial_1$ from the free module $\bigoplus_{i=1}^{\beta_2} R(-m_{1,i})$. Call this map τ_1. If $\iota_1 \colon \operatorname{Ker} \partial_1 \to \bigoplus_{i=1}^{\beta_1} R(-m_i)$ is the natural injection, then we define $\partial_2 = \iota_1 \circ \tau_1$ to produce the following commutative diagram .

$$\begin{tikzcd}
\bigoplus_{i=1}^{\beta_2} R(-m_{1,i}) \arrow[rr, "\begin{pmatrix} f_{1,1} & \cdots & f_{1,\beta_2} \end{pmatrix}", "\partial_2"'] \arrow[dr, "\tau_1"'] & & \bigoplus_{i=1}^{\beta_1} R(-m_i) \arrow[r, "\begin{pmatrix} f_1 & \cdots & f_{\beta_1} \end{pmatrix}", "\partial_1"'] & R \arrow[r, "\tau"] & R/I \arrow[r] & 0 \\
& \operatorname{Ker} \partial_1 \arrow[ur, "\iota_1"] \arrow[dr] & & & & \\
0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

Note that ∂_2 is given by a $\beta_1 \times \beta_2$ matrix. Note also that for each fixed i, each entry of $f_{1,i}$ is also homogeneous by the notation given in III.A.3.11. One can continue this procedure to produce the desired diagram. $\square$

ection063021e

III.A.4. Chain Complexes

Throughout this chapter, assume only that R is a commutative ring with identity.

fact190905i Fact III.A.4.1. *Given R-module homomorphisms* $L \xrightarrow{f} M \xrightarrow{g} N$ *, we have* $\operatorname{Im} f \subseteq \operatorname{Ker} g$ *if and only if* $g \circ f = 0$.

def190905j Definition III.A.4.2. A <u>chain complex over R</u> is a sequence of R-module homomorphisms

$$A = \qquad \cdots \xrightarrow{\partial^A_{i+2}} A_{i+1} \xrightarrow{\partial^A_{i+1}} A_i \xrightarrow{\partial^A_i} A_{i-1} \xrightarrow{\partial^A_{i-1}} \cdots$$

such that $\partial^A_i \circ \partial^A_{i+1} = 0$ for all $i \in \mathbb{Z}$. If A is an R-complex, then elements $a \in A_n$ have <u>homological degree</u> $|a| = n$.

note190905k Note III.A.4.3. We give a few remarks about the relationship between exact sequences and chain complexes.

ote190905k.a (a) An exact sequence of R-module homomorphisms is an R-complex.

ote190905k.b (b) R-complexes are not necessarily exact. For example, the R-complex $0 \longrightarrow M \longrightarrow 0$ is exact if and only if $M = 0$.

ote190905k.c (c) Given an augmented free resolution

$$P^+ = \qquad \cdots \xrightarrow{\partial^P_3} P_2 \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \xrightarrow{\tau} M \longrightarrow 0,$$

the sequence

$$P^+ = \qquad \cdots \xrightarrow{\partial^P_3} P_2 \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \longrightarrow 0$$

is not exact in general, but is an R-complex called a <u>(truncated) free resolution of M</u>.

def190905l Definition III.A.4.4. Let A be an R-complex as in Definition III.A.4.2. For all $n \in \mathbb{Z}$, denote by $Z_n(A) = Z_n$ the set of <u>cycles</u> of homological degree n and denote by $B_n(A) = B_n$ the set of <u>boundaries</u> of homological degree n, i.e.,

$$Z_n(A) = Z_n = \operatorname{Ker} \partial^A_n \subseteq A_n$$
$$B_n(A) = B_n = \operatorname{Im} \partial^A_{n+1} \subseteq Z_n,$$

where the containments on the right are as submodules. The <u>n^{th} homology module</u> of A is

$$H_n(A) = \frac{Z_n(A)}{B_n(A)}.$$

note190905m

NOTE III.A.4.5. Let A be an R-complex.

(a) A is exact if and only if $Z_n = B_n$ for all $n \in \mathbb{Z}$ if and only if $H_n(A) = 0$ for all $n \in \mathbb{Z}$.

(b) Given an augmented free resolution P^+ as in Note III.A.4.3(c), we have

$$H_n(P) \cong \begin{cases} 0 & n \neq 0 \\ M & n = 0, \end{cases}$$

because of the following.

$$M = \operatorname{Im} \tau \cong \frac{P_0}{\operatorname{Ker} \tau} = \frac{P_0}{\operatorname{Im} \partial_1^P} = \frac{\operatorname{Ker} \partial_0^P}{\operatorname{Im} \partial_1^P} = H_0(P)$$

(c) Given an R-complex as in Definition III.A.4.2, if $\partial_{i+1}^A = 0$ and $\partial_i^A = 0$, then we have

$$H_i(A) = \frac{\operatorname{Ker} \partial_i^A}{\operatorname{Im} \partial_{i+1}^A} = \frac{A_i}{0} \cong A_i.$$

(d) Given an R-complex as in Definition III.A.4.2, if $\partial_{i+1}^A = 0$, then we have

$$H_i(A) = \frac{\operatorname{Ker} \partial_i^A}{0} \cong \operatorname{Ker} \partial_i^A.$$

(e) Given an R-complex as in Definition III.A.4.2, if $\partial_i^A = 0$, then we have

$$H_i(A) = \frac{A_i}{\operatorname{Im} \partial_{i+1}^A} = \operatorname{Coker}\left(\partial_{i+1}^A\right).$$

def190910a

DEFINITION III.A.4.6. Let A and Y be R-complexes.

(a) The <u>shift</u> or <u>suspension</u> of A is an R-complex denoted ΣA where $(\Sigma A)_i = A_{i-1}$ and $\partial_i^{\Sigma A} = -\partial_{i-1}^A$.

(b) The <u>direct sum</u> of A and Y is the R-complex $A \oplus Y$ where $(A \oplus Y)_i = A_i \oplus Y_i$ and $\partial_i^{A\oplus Y}(a, y) = \left(\partial_i^A(a), \partial_i^Y(y)\right)$.

rmk190910b

REMARK III.A.4.7. The homology modules of ΣA are the homology modules of the original complex A:

$$H_i(\Sigma A) = H_{i-1}(A).$$

To see this, we observe

$$A = \qquad \cdots \xrightarrow{\partial_{i+2}^A} A_{i+1} \xrightarrow{\partial_{i+1}^A} A_i \xrightarrow{\partial_i^A} A_{i-1} \xrightarrow{\partial_{i-1}^A} \cdots$$

$$\Sigma A = \qquad \cdots \xrightarrow{-\partial_{i+1}^A} A_i \xrightarrow{-\partial_i^A} A_{i-1} \xrightarrow{-\partial_{i-1}^A} A_{i-2} \xrightarrow{-\partial_{i-2}^A} \cdots$$

and compute

$$H_i(\Sigma A) = \frac{\operatorname{Ker} \partial_i^{\Sigma A}}{\operatorname{Im} \partial_{i+1}^{\Sigma A}} = \frac{\operatorname{Ker} -\partial_{i-1}^A}{\operatorname{Im} -\partial_i^A} = \frac{\operatorname{Ker} \partial_{i-1}^A}{\operatorname{Im} \partial_i^A} = H_{i-1}(A).$$

The homology modules of $A \oplus Y$ are exactly what one might want them to be:

$$H_i(A \oplus Y) \cong H_i(A) \oplus H_i(Y).$$

This follows from the definition of $A \oplus Y$ and the first isomorphism theorem, which we show below.

$$\begin{matrix} A \\ \oplus \\ Y \end{matrix} \qquad \cdots \longrightarrow \begin{matrix} A_{i+1} \\ \oplus \\ Y_{i+1} \end{matrix} \xrightarrow{\begin{pmatrix} \partial_{i+1}^A & 0 \\ 0 & \partial_{i+1}^Y \end{pmatrix}} \begin{matrix} A_i \\ \oplus \\ Y_i \end{matrix} \xrightarrow{\begin{pmatrix} \partial_i^A & 0 \\ 0 & \partial_i^Y \end{pmatrix}} \begin{matrix} A_{i-1} \\ \oplus \\ Y_{i-1} \end{matrix} \longrightarrow \cdots$$

$$H_i(A \oplus Y) = \frac{\operatorname{Ker} \begin{matrix} \partial_i^A & 0 \\ 0 & \partial_i^Y \end{matrix}}{\operatorname{Im} \begin{matrix} \partial_{i+1}^A & 0 \\ 0 & \partial_{i+1}^Y \end{matrix}} = \frac{\begin{pmatrix} \operatorname{Ker} \partial_i^A \\ \oplus \\ \operatorname{Ker} \partial_i^Y \end{pmatrix}}{\begin{pmatrix} \operatorname{Im} \partial_{i+1}^A \\ \oplus \\ \operatorname{Im} \partial_{i+1}^Y \end{pmatrix}} \cong \begin{matrix} \big(\operatorname{Ker} \partial_i^A / \operatorname{Im} \partial_{i+1}^A\big) \\ \oplus \\ \big(\operatorname{Ker} \partial_i^Y / \operatorname{Im} \partial_{i+1}^Y\big) \end{matrix}$$

def190905n DEFINITION III.A.4.8. A chain map between R-complexes A and Y is a commutative ladder diagram.

$$\begin{tikzcd} A \arrow[d, "\phi"'] & \cdots \arrow[r, "\partial_{i+2}^A"] & A_{i+1} \arrow[r, "\partial_{i+1}^A"] \arrow[d, "\phi_{i+1}"] & A_i \arrow[r, "\partial_i^A"] \arrow[d, "\phi_i"] & A_{i-1} \arrow[r, "\partial_{i-1}^A"] \arrow[d, "\phi_{i-1}"] & \cdots \\ Y & \cdots \arrow[r, "\partial_{i+2}^Y"'] & Y_{i+1} \arrow[r, "\partial_{i+1}^Y"'] & Y_i \arrow[r, "\partial_i^Y"'] & Y_{i-1} \arrow[r, "\partial_{i-1}^Y"'] & \cdots \end{tikzcd}$$

In other words, $\phi = \{\phi_i\}$ is a sequence of R-module homomorphisms $\phi_i \colon A_i \to Y_i$ such that the above diagram commutes, i.e., such that $\partial_i^Y \circ \phi_i = \phi_{i-1} \circ \partial_i^A$ for all $i \in \mathbb{Z}$. We say the ϕ_i's are "compatible with the differentials" of the complexes. (For those familiar with the language of categories, chain maps are the "morphisms in the category of R-complexes".) The chain map ϕ is an isomorphism if it has a two-sided inverse, i.e., if there exists a chain map $\psi \colon Y \to A$ such that $\psi_i \circ \phi_i = \mathrm{id}_{A_i}$ and $\phi_i \circ \psi_i = \mathrm{id}_{Y_i}$ for all $i \in \mathbb{Z}$.

ex190910c EXAMPLE III.A.4.9. Let A and Y be R-complexes.

(a) The zero map $A \xrightarrow{0} Y$ is a chain map, since the following diagram commutes.

$$\begin{tikzcd} \cdots \arrow[r] & A_i \arrow[r] \arrow[d, "0"'] & A_{i-1} \arrow[r] \arrow[d, "0"] & \cdots \\ \cdots \arrow[r] & Y_i \arrow[r] & Y_{i-1} \arrow[r] & \cdots \end{tikzcd}$$

(b) For any $x \in R$, the "homothety" map $A \xrightarrow{x} A$ is a chain map, because the differentials ∂_i^A are R-linear, i.e., we have $\partial_i^A(xa) = x \cdot \partial_i^A(a)$ for all $a \in A_i$, so the following diagram commutes.

$$\begin{tikzcd} \cdots \arrow[r] & A_i \arrow[r, "\partial_i^A"] \arrow[d, "x"'] & A_{i-1} \arrow[r] \arrow[d, "x"] & \cdots \\ \cdots \arrow[r] & A_i \arrow[r, "\partial_i^A"] & A_{i-1} \arrow[r] & \cdots \end{tikzcd}$$

(c) Let the following be a free resolution of an R-module M.

$$P^+ = \qquad \cdots \xrightarrow{\partial_2^P} R^{\beta_1} \xrightarrow{\partial_1^P} R^{\beta_0} \xrightarrow{\tau} M \longrightarrow 0$$

Then the surjection τ determines the following chain map.

$$\begin{tikzcd} P \arrow[d, "\tau"'] & \cdots \arrow[r, "\partial_2^P"] & R^{\beta_1} \arrow[r, "\partial_1^P"] \arrow[d] & R^{\beta_0} \arrow[r] \arrow[d, "\tau"] & 0 \arrow[d] \\ M & \cdots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

(d) If $\phi \colon A \to Y$ is a chain map, then ϕ is an isomorphism if and only if each ϕ_i is an isomorphism, i.e., if and only if each ϕ_i is 1-1 and onto. (One proves this with a relatively standard diagram chase.)

The next result says that chain maps induce maps on homology modules.

thm190910d THEOREM III.A.4.10. *Let $\phi \colon A \to Y$ be a chain map.*

thm190910d.a (a) *We have $\phi_i(Z_i(A)) \subseteq Z_i(Y)$ and $\phi_i(B_i(A)) \subseteq B_i(Y)$ for all $i \in \mathbb{Z}$.*

thm190910d.b (b) *There exists a well-defined R-module homomorphism $H_i(\phi) \colon H_i(A) \to H_i(Y)$ given by*

$$H_i(\phi)(\overline{a}) = \overline{\phi(a)}.$$

Proof. (a) First let $a \in Z_i(A)$. Then we have

$$\left(\phi_{i-1} \circ \partial_i^A\right)(a) = \phi_{i-1}(0) = 0$$

by definition of $Z_i(A)$. Therefore since ϕ is chain map we have

$$\left(\partial_i^Y \circ \phi_i\right)(a) = \left(\phi_{i-1} \circ \partial_i^A\right)(a) = 0$$

which implies $\phi_i(a) \in Z_i(Y)$.

Second, if we let $b \in B_i(A)$, then there exists an element $c \in A_{i+1}$ such that $\partial_{i+1}^A(c) = b$. Since ϕ is a chain map we have

$$\phi(b) = \left(\phi_i \circ \partial_{i+1}^A\right)(c) = \left(\partial_{i+1}^Y \circ \phi_{i+1}\right)(c) \in B_i(Y).$$

(b) Let $Z_i(\phi)$ and $B_i(\phi)$ each be given by the same rule as ϕ_i with the appropriate restricted domain and codomain. By part (a) we have the following commutative diagram.

$$\begin{tikzcd}
B_i(A) \arrow[r, "\subseteq"] \arrow[d, "B_i(\phi)"'] & Z_i(A) \arrow[r, "\subseteq"] \arrow[d, "Z_i(\phi)"'] & A_i \arrow[d, "\phi_i"'] \\
B_i(Y) \arrow[r, "\subseteq"'] & Z_i(Y) \arrow[r, "\subseteq"'] & Y_i
\end{tikzcd}$$

We claim the following is also a commutative diagram, where τ_A and τ_Y are the natural surjections.

$$\begin{tikzcd}
B_i(A) \arrow[r, "\subseteq"] \arrow[d, "B_i(\phi)"'] & Z_i(A) \arrow[r, "\tau_A"] \arrow[d, "Z_i(\phi)"'] & H_i(A) \ni \overline{a} \arrow[d, dashed, "H_i(\phi)"'] \arrow[d, mapsto, shift left=6] \\
B_i(Y) \arrow[r, "\subseteq"'] & Z_i(Y) \arrow[r, "\tau_Y"'] & H_i(Y) \ni \overline{\phi(a)}
\end{tikzcd}$$

Note it suffices to show that $H_i(\phi)$ is well-defined and R-linear, since the commutivity of the diagram is by construction. We know $H_i(\phi)$ lands well by the first equality in part (a). To show $H_i(\phi)$ preserves equality (i.e., is independent of our choice of representative), let $a, a' \in Z_i(A)$ such that $\overline{a} = \overline{a'}$ in $H_i(A)$. This implies $a - a' \in B_i(A)$ and therefore it is now straightforward to show that $H_i(\phi)$ is R-linear. By the second equality in part (a) we have $\phi(a) - \phi(a') = \phi(a - a') \in B_i(Y)$, i.e., $\overline{\phi(a)} - \overline{\phi(a')} = 0 \in H_i(Y)$. $\square$

def190910e Definition III.A.4.11. A <u>quasiisomorphism</u> is a chain map $\phi \colon A \to Y$ such that $H_i(\phi) \colon H_i(A) \to H_i(Y)$ is an isomorphism for all $i \in \mathbb{Z}$.

ex190910f Example III.A.4.12. Let A and Y be R-complexes.

ex190910f.a (a) The zero map $A \xrightarrow{0} Y$ induces the zero map on homology since

$$H_i(0)(\overline{a}) = \overline{0(a)} = \overline{0} = 0.$$

ex190910f.b (b) For a fixed $x \in R$, the homothety map $A \xrightarrow{x} A$ induces a homothety map on homology, since

$$H_i(x)(\overline{a}) = \overline{x \cdot a} = x \cdot \overline{a}.$$

One might also use the more cumbersome, yet more transparent notation $\mu^{A,x}$ to denote the homothety map on A by the element x. With this notation, the above display says that $H_i(\mu^{A,x}) = \mu^{H_i(A),x}$.

As a for instance, if $x \in R$ is a unit, then $\mu^{A,x}$ is an isomorphism and the preceding paragraph implies $H_i(\mu^{A,x})$ is an isomorphism for all $i \in \mathbb{Z}$. Hence $\mu^{A,x}$ is a quasiisomorphism. In general, we will see in Proposition III.A.4.13 that if ϕ is an isomorphism, then it is also a quasiisomorphism.

ex190910f.c (c) Recall from Example III.A.4.9 that the augmented free resolution P^+ of an R-module M determines a chain map $\tau \colon P \to M$. We claim τ is a quasiisomorphism. To see that $H_i(\tau)$ is an isomorphism for every $i \in \mathbb{Z}$, first note that for all $i \neq 0$ one has $H_i(P) = 0 = H_i(M)$ since P is exact and $C_i = 0$ for all $i \neq 0$. Hence $H_i(\tau)$ is the identity map on the zero module and is therefore an isomorphism for all $i \neq 0$. When $i = 0$, we have the following commutative diagram.

$$\begin{tikzcd}
H_0(P) \arrow[r, "H_0(\tau)", "\therefore\cong"'] \arrow[d, "\alpha"', "\cong"] & H_0(M) \arrow[d, equal] \\
M \arrow[r, "\cong", "\gamma"'] & \dfrac{M}{0}
\end{tikzcd}$$

The map γ is the natural surjection and is an isomorphism by the first isomorphism theorem. Observe also that

$$H_0(P) = \frac{R^{\beta_0}}{\operatorname{Im}\partial_1^P} = \frac{R^{\beta_0}}{\operatorname{Ker}\tau}$$

so the surjection α induced by τ is an isomorphism also by the first isomorphism theorem. Hence $H_0(\tau)$ is a composition of isomorphisms and is therefore itself an isomorphism.

prop190910g

Proposition III.A.4.13. *Let $\phi\colon A \to Y$ and $\psi\colon C \to A$ be chain maps.*

prop190910g.a (a) *The composition $\phi\circ\psi\colon C \to Y$ is a chain map.*

prop190910g.b (b) *$H_i(-)$ respects compositions, i.e., $H_i(\phi\circ\psi) = H_i(\phi)\circ H_i(\psi)$ for all $i \in \mathbb{Z}$.*

prop190910g.c (c) *If ϕ is an isomorphism, then ϕ is a quasiisomorphism.*

Proof. (a) This is proved using a standard diagram chase on the following section of a ladder diagram.

$$\begin{tikzcd} C_i \arrow[r, "\partial_i^C"] \arrow[d, "\psi_i"'] & C_{i-1} \arrow[d, "\psi_{i-1}"] \\ A_i \arrow[r, "\partial_i^A"] \arrow[d, "\phi_i"'] & A_{i-1} \arrow[d, "\phi_{i-1}"] \\ Y_i \arrow[r, "\partial_i^Y"] & Y_{i-1} \end{tikzcd}$$

(b) Note that we are trying to prove that the commutative diagram

$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[dr, "\phi\circ\psi"'] & A \arrow[d, "\phi"] \\ & Y \end{tikzcd}$$

given in (a) induces the following commutative diagram.

$$\begin{tikzcd} H_i(C) \arrow[r, "H_i(\psi)"] \arrow[dr, "H_i(\phi\circ\psi)"'] & H_i(A) \arrow[d, "H_i(\phi)"] \\ & H_i(Y) \end{tikzcd}$$

To show this, for any $c \in C_i$ we have

$$H_i(\phi\circ\psi)(\overline{c}) = \overline{(\phi\circ\psi)(c)} = \overline{\phi(\psi(c))} = H_i(\phi)\left(\overline{\psi(c)}\right) = H_i(\phi)\left(H_i(\psi)(\overline{c})\right) = \left(H_i(\phi)\circ H_i(\psi)\right)(\overline{c}).$$

(c) Assume ϕ is an isomorphism and let $\zeta\colon Y \to A$ be its two-sided inverse. Then the composition $\phi\circ\zeta$ is equal to the homothety map $\mu^{Y,1}$. Moreover, by Example III.A.4.12(b) and by part (b) we have

$$H_i(\phi)\circ H_i(\zeta) = H_i(\phi\circ\zeta) = H_i(\mu^{Y,1}) = \mu^{H_i(Y),1} = \mathrm{id}_{H_i(Y)}.$$

Similarly we have

$$H_i(\zeta)\circ H_i(\phi) = H_i(\zeta\circ\phi) = H_i(\mu^{A,1}) = \mu^{H_i(A),1} = \mathrm{id}_{H_i(A)}.$$

Hence $H_i(\phi)$ is an isomorphism with the two-sided inverse $H_i(\zeta)$, i.e.,

$$\left(H_i(\phi)\right)^{-1} = H_i(\phi^{-1}).$$

□

ex190910h

Example III.A.4.14. The converse of Proposition III.A.4.13 fails in general. The chain map $\tau\colon P \to M$ from Example III.A.4.12 is a quasiisomorphism, but is almost never an isomorphism. For instance, if $M \not\cong R^{\beta_0}$, then it is not an isomorphism.

def190910i

Definition III.A.4.15. A short exact sequence of chain maps is a sequence

$$0 \longrightarrow A \xrightarrow{\phi} C \xrightarrow{\psi} D \longrightarrow 0$$

of chain maps such that each "level" is a short exact sequence

$$0 \longrightarrow A_i \xrightarrow{\phi_i} C_i \xrightarrow{\psi_i} D_i \longrightarrow 0.$$

For diagram chases it may also be convenient to display a short exact sequence of chain maps as follows.

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial^A_{i+2}"'] & \vdots \arrow[d, "\partial^C_{i+2}"'] & \vdots \arrow[d, "\partial^D_{i+2}"'] & \\
0 \arrow[r] & A_{i+1} \arrow[r, "\phi_{i+1}"] \arrow[d, "\partial^A_{i+1}"'] & C_{i+1} \arrow[r, "\psi_{i+1}"] \arrow[d, "\partial^C_{i+1}"'] & D_{i+1} \arrow[r] \arrow[d, "\partial^D_{i+1}"'] & 0 \\
0 \arrow[r] & A_i \arrow[r, "\phi_i"] \arrow[d, "\partial^A_i"'] & C_i \arrow[r, "\psi_i"] \arrow[d, "\partial^C_i"'] & D_i \arrow[r] \arrow[d, "\partial^D_i"'] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r, "\phi_{i-1}"] \arrow[d, "\partial^A_{i-1}"'] & C_{i-1} \arrow[r, "\psi_{i-1}"] \arrow[d, "\partial^C_{i-1}"'] & D_{i-1} \arrow[r] \arrow[d, "\partial^D_{i-1}"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Below we present Theorem III.A.4.16 along with two different proofs. The first requires the Snake Lemma, which we present as an unnumbered result before giving the theorem.

lem190912a

Lemma (Snake Lemma). *Given a commutative diagram of R-modules with exact rows*

$$\begin{tikzcd}
 & U \arrow[r, "f"] \arrow[d, "u"] & V \arrow[r, "g"] \arrow[d, "v"] & W \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & U' \arrow[r, "f'"'] & V' \arrow[r, "g'"'] & W' &
\end{tikzcd}$$

there exists an exact sequence

$$\operatorname{Ker} u \xrightarrow{\tilde{f}} \operatorname{Ker} v \xrightarrow{\tilde{g}} \operatorname{Ker} w \xrightarrow{\sigma} \operatorname{Coker}(u) \xrightarrow{\overline{f'}} \operatorname{Coker}(v) \xrightarrow{\overline{g'}} \operatorname{Coker}(w)$$

$$x \longmapsto f(x) \qquad \overline{x'} \longmapsto \overline{f'(x')}$$

$$y \longmapsto g(y) \qquad \overline{y'} \longmapsto \overline{g'(y')}$$

where $\operatorname{Coker}(u) = U'/\operatorname{Im} u$, *and the other cokernels are defined similarly. The map σ is defined as follows. Let $z \in \operatorname{Ker} w$. Then $w(z) = 0$ and since g is surjective let $y \in V$ such that $g(y) = z$. By the commutivity of the diagram we have*

$$g'(v(y)) = w(g(y)) = w(z) = 0,$$

so $v(y) \in \operatorname{Ker} g' = \operatorname{Im} f'$. Let $x' \in U'$ such that $f'(x') = v(y)$. Then $\sigma(z)$ is defined as

$$\sigma(z) = \overline{x'}.$$

Now we present the theorem promised. The first proof uses the Snake Lemma and the second is a more "manual" proof.

thm190910j

Theorem III.A.4.16. *Given a short exact sequence of chain maps as in Definition III.A.4.15, there exists the following long exact sequence on homology.*

$$\begin{tikzcd} \cdots \arrow[r, "\eth_{i+1}"] & H_i(A) \arrow[r, "H_i(\phi)"] & H_i(C) \arrow[r, "H_i(\psi)"] & H_i(D) \arrow[dll, rounded corners, "\eth_i"] \\ & H_{i-1}(A) \arrow[r, "H_{i-1}(\phi)"] & \cdots & \end{tikzcd}$$

We call $\eth_i$ *a* <u>*connecting homomorphism*</u>.

Proof. First let us construct $\eth_i$. It will be helpful to have the following section of ladder diagram in view for this part.

$$\begin{tikzcd} & & C_i \arrow[r, "\psi_i"] \arrow[d, "\partial_i^C"'] & D_i \arrow[r] \arrow[d, "\partial_i^D"] & 0 \\ 0 \arrow[r] & A_{i-1} \arrow[r, "\phi_{i-1}"'] & C_{i-1} \arrow[r, "\psi_{i-1}"'] & D_{i-1} \arrow[r] & 0 \end{tikzcd}$$

Let $\overline{d} \in H_i(D)$ and we want to define $\eth_i(\overline{d})$. Since $d \in Z_i(D)$, we know $\partial_i^D(d) = 0$ and since ψ_i is surjective let $c \in C_i$ such that $\psi_i(c) = d$. Since ψ is a chain map we have

$$\psi_{i-1}\left(\partial_i^C(c)\right) = \partial_i^D\left(\psi_i(c)\right) = \partial_i^D(d) = 0,$$

so $\partial_i^C(c) \in \operatorname{Ker} \psi_{i-1} = \operatorname{Im} \phi_{i-1}$. Therefore let $a \in A_{i-1}$ such that $\phi_{i-1}(a) = \partial_i(c)$. We define

$$\eth_i\left(\overline{d}\right) = \overline{a} \in H_{i-1}(A).$$

We claim the following is a commutative diagram R-module homomorphisms with exact rows.

$$\begin{tikzcd} & \operatorname{Coker}\left(\partial_{i+2}^A\right) \arrow[r, "\overline{\phi_{i+1}}"] \arrow[d, "\widehat{\partial_{i+1}^A}"'] & \operatorname{Coker}\left(\partial_{i+2}^C\right) \arrow[r, "\overline{\psi_{i+1}}"] \arrow[d, "\widehat{\partial_{i+1}^C}"'] & \operatorname{Coker}\left(\partial_{i+2}^D\right) \arrow[r] \arrow[d, "\widehat{\partial_{i+1}^D}"'] & 0 \\ 0 \arrow[r] & \operatorname{Ker}\partial_i^A \arrow[r, "\widetilde{\phi_i}"'] & \operatorname{Ker}\partial_i^C \arrow[r, "\widetilde{\psi_i}"'] & \operatorname{Ker}\partial_i^D & \end{tikzcd} \tag{III.A.4.16.1}$$

eqn190912a

Step 1: We show that

$$\widehat{\partial_{i+1}^A}\colon \operatorname{Coker}\left(\partial_{i+2}^A\right) \longmapsto \operatorname{Ker}\partial_i^A$$
$$\overline{a} \longmapsto \partial_{i+1}^A(a)$$

is a well-defined R-module homomorphism. Since $\partial_{i+1}^A(A_{i+1}) = B_i(A) \subseteq Z_i(A)$, we may restrict the codomain of ∂_{i+1}^A to get the well-defined R-module homomorphism $\zeta\colon A_{i+1} \to Z_i(A)$. Since $B_{i+1}(A) \subseteq Z_{i+1}(A)$ we also have $\zeta(B_{i+1}(A)) = \partial_{i+1}^A(B_{i+1}(A)) = 0$. Therefore we have the commutative diagram

$$\begin{tikzcd} A_{i+1} \arrow[r, two heads] \arrow[d, "\zeta"'] & A_{i+1}/B_{i+1}(A) \arrow[dl, dashed, "\exists!\widehat{\partial_{i+1}^A}"] \\ Z_i(A) & \end{tikzcd}$$

where $\widehat{\partial_{i+1}^A}(\overline{x}) = \partial_{i+1}^A(x)$. Moreover, we have

$$\operatorname{Im}\widehat{\partial_{i+1}^A} = \operatorname{Im}\zeta = \operatorname{Im}\partial_{i+1}^A$$

and

$$\operatorname{Ker}\widehat{\partial_{i+1}^A} = \frac{\operatorname{Ker}\zeta}{B_{i+1}(A)} = \frac{\operatorname{Ker}\partial_{i+1}^A}{\operatorname{Im}\partial_{i+2}^A} = H_{i+1}(A).$$

Identical arguments can be used to show the well-definedness of $\widehat{\partial^C_{i+1}}$ and $\widehat{\partial^D_{i+1}}$ as well. One also finds that $\operatorname{Im}\widehat{\partial^C_{i+1}} = \operatorname{Im}\partial^C_{i+1}$, $\operatorname{Im}\widehat{\partial^D_{i+1}} = \operatorname{Im}\partial^D_{i+1}$, $\operatorname{Ker}\widehat{\partial^C_{i+1}} = H_{i+1}(C)$, and $\operatorname{Ker}\widehat{\partial^D_{i+1}} = H_{i+1}(D)$ as well.

Step 2: We apply the Snake Lemma to

$$\begin{tikzcd}
0 \arrow[r] & A_{i+2} \arrow[r, "\phi_{i+2}"] \arrow[d, "\partial^A_{i+2}"'] & C_{i+2} \arrow[r, "\psi_{i+2}"] \arrow[d, "\partial^C_{i+2}"'] & D_{i+2} \arrow[r] \arrow[d, "\partial^D_{i+2}"'] & 0 \\
0 \arrow[r] & A_{i+1} \arrow[r, "\phi_{i+1}"'] & C_{i+1} \arrow[r, "\psi_{i+1}"'] & D_{i+1} \arrow[r] & 0
\end{tikzcd}$$

to get the exact sequence

$$\begin{array}{ccccccc}
\cdots \longrightarrow \operatorname{Coker}\left(\partial^A_{i+2}\right) & \xrightarrow{\overline{\phi_{i+1}}} & \operatorname{Coker}\left(\partial^C_{i+2}\right) & \xrightarrow{\overline{\psi_{i+1}}} & \operatorname{Coker}\left(\partial^D_{i+2}\right) \longrightarrow 0 \\
\overline{a} \longmapsto \overline{\phi_{i+1}(a)} & & \overline{c} \longmapsto \overline{\psi_{i+1}(c)} & &
\end{array}$$

where the exactness on the right follows from the surjectivity of ψ_{i+1}.

Step 3: We apply the Snake Lemma to

$$\begin{tikzcd}
0 \arrow[r] & A_{i} \arrow[r, "\phi_{i}"] \arrow[d, "\partial^A_{i}"'] & C_{i} \arrow[r, "\psi_{i}"] \arrow[d, "\partial^C_{i}"'] & D_{i} \arrow[r] \arrow[d, "\partial^D_{i}"'] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r, "\phi_{i-1}"'] & C_{i-1} \arrow[r, "\psi_{i-1}"'] & D_{i-1} \arrow[r] & 0
\end{tikzcd}$$

to get the exact sequence

$$\begin{array}{ccccc}
0 \longrightarrow \operatorname{Ker}\partial^A_i & \xrightarrow{\widetilde{\phi}_i} & \operatorname{Ker}\partial^C_i & \xrightarrow{\widetilde{\psi}_i} & \operatorname{Ker}\partial^D_i \longrightarrow \cdots \\
a \longmapsto \phi_i(a) & & c \longmapsto \psi_i(c) & &
\end{array}$$

where the exactness on the left follows from the injectivity of ϕ_i.

Step 4: We show that (III.A.4.16.1) commutes. For any $\overline{c} \in \operatorname{Coker}\left(\partial^C_{i+2}\right)$ we have

$$\widehat{\partial^D_{i+1}}\left(\overline{\psi_{i+1}}(\overline{c})\right) = \widehat{\partial^D_{i+1}}\left(\overline{\psi_{i+1}(c)}\right) = \partial^D_{i+1}\left(\psi_{i+1}(c)\right)$$

and

$$\widetilde{\psi}_i\left(\widehat{\partial^C_{i+1}}(\overline{c})\right) = \widetilde{\psi}_i\left(\partial^C_{i+1}(c)\right) = \psi_i\left(\partial^C_{i+1}(c)\right),$$

which are equivalent since ψ is a chain map. One can similarly show that the left square commutes using the fact that ϕ is a chain map.

Step 5: From the conclusion of Step 1 we have

$$\operatorname{Coker}\left(\widehat{\partial^A_{i+1}}\right) = \frac{\operatorname{Ker}\partial^A_i}{\operatorname{Im}\widehat{\partial^A_{i+1}}} = \frac{\operatorname{Ker}\partial^A_i}{\operatorname{Im}\partial^A_{i+1}} = H_i(A),$$

as well as $\operatorname{Coker}\left(\widehat{\partial^C_{i+1}}\right) = H_i(C)$ and $\operatorname{Coker}\left(\widehat{\partial^D_{i+1}}\right) = H_i(D)$.

Step 6: Having established our claim that (III.A.4.16.1) is a commutative diagram, we apply the Snake Lemma once more to obtain the following exact sequence.

$$\begin{array}{cccccc}
\operatorname{Ker}\widehat{\partial^A_{i+1}} \xrightarrow[H_{i+1}(\phi)]{\widetilde{\overline{\phi_{i+1}}}} & \operatorname{Ker}\widehat{\partial^C_{i+1}} \xrightarrow[H_{i+1}(\psi)]{\widetilde{\overline{\psi_{i+1}}}} & \operatorname{Ker}\widehat{\partial^D_{i+1}} \xrightarrow[\eth_{i+1}]{\sigma} & \operatorname{Coker}\left(\widehat{\partial^A_{i+1}}\right) \xrightarrow[H_i(\phi)]{\overline{\overline{\widetilde{\phi}_i}}} & \operatorname{Coker}\left(\widehat{\partial^C_{i+1}}\right) \xrightarrow[H_i(\psi)]{\overline{\overline{\widetilde{\psi}_i}}} & \operatorname{Coker}\left(\widehat{\partial^D_{i+1}}\right) \\
\| & \| & \| & \| & \| & \| \\
H_{i+1}(A) & H_{i+1}(C) & H_{i+1}(D) & H_i(A) & H_i(C) & H_i(D) \\
\\
\overline{a} \longmapsto & \overline{\phi_{i+1}(a)} & & & \overline{c} \longmapsto & \overline{\psi_i(c)}
\end{array}$$

It remains only to justify that the map σ given by the Snake Lemma is the same map $\eth_{i+1}$ that we constructed. To do so we perform a diagram chase on (III.A.4.16.1). For any $\overline{d} \in \operatorname{Ker}\widehat{\partial^D_{i+2}}$ we have $\partial^D_{i+2}(d) = \widehat{\partial^D_{i+2}}\left(\overline{d}\right) = 0$ and by the exactness of the top row we let $\overline{c} \in \operatorname{Coker}\left(\partial^C_{i+2}\right)$ such that $\overline{\psi_{i+1}(c)} = \overline{\psi_{i+1}}(\overline{c}) = \overline{d}$. Since the right square commutes we have

$$\psi_i(\partial^C_{i+1}(c)) = \widetilde{\psi}_i\left(\widehat{\partial^C_{i+1}}(\overline{c})\right) = \widehat{\partial^D_{i+1}}\left(\overline{\psi_{i+1}}(\overline{c})\right) = \widehat{\partial^D_{i+1}}\left(\overline{d}\right) = 0,$$

so $\widehat{\partial^C_{i+1}}(\overline{c}) \in \operatorname{Ker}\widetilde{\psi}_i = \operatorname{Im}\widetilde{\phi}_i$ and $\partial^C_{i+1}(c) \in \ker(\psi_i) = \operatorname{Im}\phi_i$. Let $a \in \operatorname{Ker}\partial^A_i$ such that $\phi_i(a) = \widetilde{\phi}_i(a) = \widehat{\partial^C_{i+1}}(\overline{c}) = \partial^C_{i+1}(c)$. Comparing the rules defining σ and $\eth_i$, we conclude that

$$\sigma\left(\overline{d}\right) = \overline{a} = \eth_i\left(\overline{d}\right).$$

□

Alternate proof. As with the previous proof, the first step is to construct $\eth$. Since this argument is the same as that in the previous proof, we begin with the second step.

Step 2: We show $\eth_i$ is well-defined. First we have

$$\phi_{i-2}\left(\partial^A_{i-1}(a)\right) = \partial^C_{i-1}\left(\phi_{i-1}(a)\right)$$

since ϕ is a chain map. Then

$$\partial^C_{i-1}\left(\phi_{i-1}(a)\right) = \partial^C_{i-1}\left(\partial^C_i(c)\right) = 0$$

using the definition of a and that C is an R-complex. Since ϕ_{i-2} is injective, this implies $\partial^A_{i-1}(a) = 0$, i.e., $a \in \operatorname{Ker}\partial^A_{i-1}$, as desired.

Second we will show $\overline{a} \in H_{i-1}(A)$ is independent of any choices made in Step 1. Let $d, d' \in \operatorname{Ker}\partial^D_i$ such that $\overline{d} = \xi = \overline{d'}$, let $c, c' \in C_i$ such that $\psi_i(c) = d$ and $\psi_i(c') = d'$, and let $a, a' \in A_{i-1}$ such that $\phi_{i-1}(a) = \partial^C_i(c)$ and $\phi_{i-1}(a') = \partial^C_i(c')$. We need to show $\overline{a} = \overline{a'}$ in $H_{i-1}(A) = \operatorname{Ker}\partial^A_{i-1}/\operatorname{Im}\partial^A_i$, or in other words, we need to show $a - a' \in \operatorname{Im}\partial^A_i$.

By assumption $\overline{d} = \overline{d'} \in H_i(A) = \operatorname{Ker}\partial^A_i/\operatorname{Im}\partial^A_{i+1}$, so $d - d' \in \operatorname{Im}\partial^A_{i+1}$ and we let $\eta \in D_{i+1}$ such that $\partial^D_{i+1}(\eta) = d - d'$. Since ψ_{i+1} is surjective, we may let $\nu \in C_{i+1}$ such that $\psi_{i+1}(\nu) = \eta$ and we compute the following.

$$\psi_i(c - c' - \partial^C_{i+1}(\nu)) = \psi_i(c) - \psi_i(c') - (\psi_i \circ \partial^C_{i+1})(\nu) = d - d' - (d - d') = 0$$

In the above calculation we rely only on the definitions of our elements and the linearity of ψ_i. By this calculation we know $c - c' - \partial^C_{i+1}(\nu) \in \ker(\psi_i) = \operatorname{Im}\phi_i$ so let $\omega \in A_i$ such that $\phi_i(\omega) = c - c' - \partial^C_{i+1}(\nu)$. Since $a, a', \partial^A_i(\omega) \in A_{i-1}$, we use the linearity of ϕ_{i-1} to get

$$\phi_{i-1}\left(\partial^A_i(\omega) - (a - a')\right) = \left(\phi_{i-1} \circ \partial^A_i\right)(\omega) - \phi_{i-1}(a) + \phi_{i-1}(a').$$

Since ϕ is a chain map, then

$$\left(\phi_{i-1} \circ \partial^A_i\right)(\omega) - \phi_{i-1}(a) + \phi_{i-1}(a') = \left(\partial^C_i \circ \phi_i\right)(\omega) - \partial^C_i(c) + \partial^C_i(c').$$

The definition of ω gives a similar argument as for ϕ_{i-2} above.

$$\begin{aligned}
\left(\partial^C_i \circ \phi_i\right)(\omega) - \partial^C_i(c) + \partial^C_i(c') &= \partial^C_i\left(c - c' - \partial^C_{i+1}(\nu)\right) - \partial^C_i(c) + \partial^C_i(c') \\
&= \partial^C_i\left(c - c' - \partial^C_{i+1}(\nu) - c + c'\right) \\
&= -\left(\partial^C_i \circ \partial^C_{i+1}\right)(\nu) \\
&= 0.
\end{aligned}$$

Since ϕ_{i-1} is injective, this implies $\partial_i^A(\omega) - (a - a') = 0$ or equivalently

$$a - a' = \partial_i^A(\omega) \in \operatorname{Im} \partial_i^A$$

completing this step.

Step 3: Here we prove $\eth_i$ is an R-module homomorphism. Let $\xi, \xi' \in H_i(D)$ and $r \in R$. Also let $d, d' \in \operatorname{Ker} \partial_i^D$ such that $\overline{d} = \xi$ and $\overline{d'} = \xi'$, let $c, c' \in C_i$ such that $\psi_i(c) = d$ and $\psi_i(c') = d'$, and let $a, a' \in A_{i-1}$ such that $\phi_{i-1}(a) = \partial_i^C(c)$ and $\phi_{i-1}(a') = \partial_i^C(c')$.

Notice that $rd + d' \in \operatorname{Ker} \partial_i^D$ and hence it makes sense to write $\overline{rd + d'} = r\xi + \xi'$. Notice also that $rc + c' \in C_i$ so we have

$$\psi_i(rc + c') = \psi_i(rc) + \psi_i(c') = r \cdot \psi_i(c) + \psi_i(c') = rd + d'.$$

Finally note that $ra + a' \in A_{i-1}$ for which we have

$$\begin{aligned}
\phi_{i-1}(ra + a') &= \phi_{i-1}(ra) + \phi_{i-1}(a') = r \cdot \phi_{i-1}(a) + \phi_{i-1}(a') \\
&= r \cdot \partial_i^C(c) + \partial_i^C(c') = \partial_i^C(rc) + \partial_i^C(c') = \partial_i^C(rc + c').
\end{aligned}$$

Therefore we have an element satisfying the definition of $\eth_i$ described in Step 1 of the previous proof so we conclude this step in the following display.

$$\eth_i(r\xi + \xi') = \overline{ra + a'} = r \cdot \overline{a} + \overline{a} = r \cdot \eth_i(\xi) + \eth_i(\xi)$$

Step 4: We tackle the first of several questions of exactness. Here we show $\operatorname{Im} H_i(\phi) \subseteq \operatorname{Ker} H_i(\psi)$. Let $\delta \in H_i(A)$ and let $\rho \in \operatorname{Ker} \partial_i^A$ such that $\overline{\rho} = \delta$. Therefore we have

$$H_i(\psi)\left(H_i(\phi)(\delta)\right) = H_i(\psi)\left(\overline{\phi_i(\rho)}\right) = \overline{(\psi_i \circ \phi_i)(\rho)} = \overline{0} = 0$$

where the third equality comes from the exactness of the original sequence of chain maps.

Step 5: We now show $\operatorname{Im} H_i(\phi) \supseteq \operatorname{Ker} H_i(\psi)$. Let $\delta \in \operatorname{Ker} H_i(\psi)$ and let $\rho \in \operatorname{Ker} \partial_i^C$ such that $\overline{\rho} = \delta$. This gives

$$0 = H_i(\psi)(\overline{\rho}) = \overline{\psi_i(\rho)} \in H_i(D) = \frac{\operatorname{Ker} \partial_i^D}{\operatorname{Im} \partial_{i+1}^D}.$$

Therefore $\psi_i(\rho) \in \operatorname{Im} \partial_{i+1}^D$ so we lift to some $\mu \in D_{i+1}$ such that $\partial_{i+1}^D(\mu) = \psi_i(\rho)$ and lift again to some $\sigma \in C_{i+1}$ such that $\psi_{i+1}(\sigma) = \mu$ (since ψ_{i+1} is surjective). Since $\rho, \partial_{i+1}^C(\sigma) \in C_i$, we consider the element $\rho - \partial_{i+1}^C(\sigma) \in C_i$. Using linearity and the fact that ψ is a chain map we compute

$$\psi_i(\rho - \partial_{i+1}^C(\sigma)) = \psi_i(\rho) - (\psi_i \circ \partial_{i+1}^C)(\sigma) = \psi_i(\rho) - (\partial_{i+1}^D \circ \psi_{i+1})(\sigma) = \psi_i(\rho) - \partial_{i+1}^D(\mu) = 0.$$

Hence $\rho - \partial_{i+1}^C(\sigma) \in \ker(\psi_i) = \operatorname{Im} \phi_i$ and we let $\tau \in A_i$ such that $\phi_i(\tau) = \rho - \partial_{i+1}^C(\sigma)$. We claim $\tau \in \operatorname{Ker} \partial_i^A$ and point out it suffices to show $\left(\phi_{i-1} \circ \partial_i^A\right)(\tau) = 0$ since ϕ_{i-1} is injective. We compute

$$\left(\phi_{i-1} \circ \partial_i^A\right)(\tau) = \partial_i^C(\phi_i(\tau)) = \partial_i^C(\rho - \partial_{i+1}^C(\sigma)) = \partial_i^C(\rho) - \left(\partial_i^C \circ \partial_{i+1}^C\right)(\sigma) = 0$$

where the last equality holds by definition of ρ and because C is a chain complex.

We consider $\rho, \partial_{i+1}^C(\sigma) \in \operatorname{Ker} \partial_i^C$ and $\tau \in \operatorname{Ker} \partial_i^A$, which represent the cosets $\overline{\rho}, \overline{\partial_{i+1}^C(\sigma)}$ $\in H_i(C)$ and $\overline{\tau} \in H_i(A)$. Therefore it makes sense to compute

$$H_i(\phi)(\overline{\tau}) = \overline{\phi_i(\tau)} = \overline{\rho - \partial_{i+1}^C(\sigma)} = \overline{\rho} - \overline{\partial_{i+1}^C(\sigma)} = \overline{\rho} - \overline{0} = \overline{\rho} = \delta.$$

Hence $\delta \in \operatorname{Im} H_i(\phi)$, completing this step.

Step 6: Continuing our proof of exactness, we show here that $\operatorname{Im} H_i(\psi) \subseteq \operatorname{Ker} \eth_i$. Let $\zeta \in H_i(C)$ and let $c \in \operatorname{Ker} \partial_i^C$ such that $\overline{c} = \zeta$. We want to show that $\left(\eth_i \circ H_i(\psi)\right)(\overline{c}) = 0$. Define $d = \psi_i(c)$ and we have

$$H_i(\psi)(\overline{c}) = \overline{\psi_i(c)} = \overline{d}.$$

Computing $\eth_i(H_i(\psi)(\overline{c})) = \eth_i(\overline{d})$ requires some $a \in \operatorname{Ker} \partial_{i-1}^A$ such that $\phi_{i-1}(a) = \partial_i^C(c)$. Since $c \in \operatorname{Ker} \partial_i^C$ by assumption, $\partial_i^C(c) = 0 = \phi_{i-1}(0)$, so setting $a = 0$ we get

$$\eth_i(\overline{d}) = \overline{a} = \overline{0} = 0.$$

Step 7: We now show $\operatorname{Im} H_i(\psi) \supseteq \operatorname{Ker} \eth_i$. Let $\xi \in \operatorname{Ker} \eth_i \subseteq H_i(D)$ and let $d \in \operatorname{Ker} \partial_i^D$ such that $\xi = \overline{d}$. Fix some $c \in C_i$ such that $\psi_i(c) = d$ and some $a \in A_{i-1}$ such that $\phi_{i-1}(a) = \partial_i^C(c) \in \ker(\psi_{i-1}) = \operatorname{Im} \phi_{i-1}$. Our construction in Step 1 implies $\eth_i(\xi) = \overline{a}$ so we have

$$0 = \eth_i(\xi) = \overline{a} \in H_{i-1}(A) = \frac{\operatorname{Ker} \partial_{i-1}^A}{\operatorname{Im} \partial_i^A}.$$

Hence $a \in \operatorname{Im} \partial_i^A$ and we let $\omega \in A_i$ such that $\partial_i^A(\omega) = a$. Moreover, $\phi_i(\omega)$, $c \in C_i$ so we compute the following.

$$\begin{aligned} \partial_i^C(c - \phi_i(\omega)) &= \partial_i^C(c) - \left(\partial_i^C \circ \phi_i\right)(\omega) \\ &= \partial_i^C(c) - \left(\phi_{i-1} \circ \partial_i^A\right)(\omega) \\ &= \partial_i^C(c) - \phi_{i-1}(a) \\ &= \partial_i^C(c) - \partial_i^C(c) \\ &= 0 \end{aligned}$$

Therefore $c - \phi_i(\omega) \in \operatorname{Ker} \partial_i^C$ and hence $\overline{c - \phi_i(\omega)} \in H_i(C)$. We may also compute

$$H_i(\psi)(\overline{c - \phi_i(\omega)}) = \overline{\psi_i(c - \phi_i(\omega))} = \overline{\psi_i(c) - (\psi_i \circ \phi_i)(\omega)} = \overline{\psi_i(c)} = \overline{d} = \xi$$

where the third equality holds by the exactness of the i^{th} row of the given diagram. Hence $\xi \in \operatorname{Im} H_i(\psi)$, which completes this step.

Step 8: Here we show $\operatorname{Im} \eth_i \subseteq \operatorname{Ker} H_{i-1}(\phi)$. Let $\xi \in H_i(D)$ and let $d \in \operatorname{Ker} \partial_i^D$ such that $\xi = \overline{d}$. We want to show that $H_{i-1}(\phi)\left(\eth_i(\overline{d})\right) = 0$. Since ψ_i is surjective, let $c \in C_i$ such that $\psi_i(c) = d$ and since $\partial_i^C(c) \in \operatorname{Ker} \psi_{i-1} = \operatorname{Im} \phi_{i-1}$, let $a \in A_{i-1}$ such that $\phi_{i-1}(a) = \partial_i^C(c)$. We therefore have

$$H_{i-1}(\phi)(\eth_i(\overline{d})) = H_{i-1}(\phi)(\overline{a}) = \overline{\phi_{i-1}(a)} = \overline{\partial_i^C(c)} = 0$$

which completes this step.

Step 9: We finally show that $\operatorname{Im} \eth_i \supseteq \operatorname{Ker} H_{i-1}(\phi)$. Let $\lambda \in \operatorname{Ker} H_{i-1}(\phi)$ and fix some element $a \in \operatorname{Ker} \partial_{i-1}^A$ such that $\lambda = \overline{a} \in H_{i-1}(A)$. By assumption we have

$$0 = H_{i-1}(\phi)(\lambda) = H_{i-1}(\phi)(\overline{a}) = \overline{\phi_{i-1}(a)} \in H_{i-1}(C) = \frac{\operatorname{Ker} \partial_{i-1}^C}{\operatorname{Im} \partial_i^C}.$$

It follows that $\phi_{i-1}(a) \in \operatorname{Im} \partial_i^C$, so we may let $c \in C_i$ such that $\partial_i^C(c) = \phi_{i-1}(a)$. Denote $\psi_i(c) = d$ and notice by our construction in Step 1, this element is a good candidate on which to apply $\eth_i$. Observe that

$$\partial_i^D(d) = \partial_i^D(\psi_i(c)) = \psi_{i-1}(\partial_i^C(c)) = (\psi_{i-1} \circ \phi_{i-1})(a) = 0$$

so $d \in \operatorname{Ker} \partial_i^D$. Therefore $\overline{d} \in H_i(D)$ and

$$\eth_i(\overline{d}) = \overline{a} = \lambda.$$

This completes this proof of the theorem. □

def190912b DEFINITION III.A.4.17. Let $\alpha\colon A \to Y$ be a chain map. The mapping cone of α is an R-complex $\operatorname{Cone}(\alpha)$ defined by

$$\operatorname{Cone}(\alpha)_i = \begin{matrix} A_{i-1} \\ \oplus \\ Y_i \end{matrix} \xrightarrow[\partial_i^{\operatorname{Cone}(\alpha)}]{\begin{pmatrix} -\partial_{i-1}^A & 0 \\ \alpha_{i-1} & \partial_i^Y \end{pmatrix}} \begin{matrix} A_{i-2} \\ \oplus \\ Y_{i-1} \end{matrix} = \operatorname{Cone}(\alpha)_{i-1}.$$

ex190912c EXAMPLE III.A.4.18. Recall again Examples III.A.4.9 and III.A.4.12.

(a) The mapping cone of the zero map $A \xrightarrow{0} Y$ is given by

$$\begin{matrix} A_{i-1} \\ \oplus \\ Y_i \end{matrix} \xrightarrow{\begin{pmatrix} -\partial_{i-1}^A & 0 \\ 0 & \partial_i^Y \end{pmatrix}} \begin{matrix} A_{i-2} \\ \oplus \\ Y_{i-1} \end{matrix}.$$

Hence $\operatorname{Cone}(0) = \Sigma A \oplus Y$.

(b) The mapping cone of the homothety map $\mu^{A,x}\colon A \to A$ where $x \in R$ is given by

$$\begin{tikzcd} \begin{matrix} A_{i-1} \\ \oplus \\ A_i \end{matrix} \arrow[rr, "\begin{pmatrix} -\partial_i^A & 0 \\ x & \partial_{i-1}^A \end{pmatrix}"] & & \begin{matrix} A_{i-2} \\ \oplus \\ A_{i-1} \end{matrix} \end{tikzcd}$$

where the entry x denotes the map $x \cdot \mathrm{id}_{A_{i-1}}$.

(c) Finally, the mapping cone of $\tau\colon P \to M$ is

$$\begin{tikzcd} & \mathrm{Cone}(\tau)_2 \arrow[d, equal] & & \mathrm{Cone}(\tau)_1 \arrow[d, equal] & & \mathrm{Cone}(\tau)_0 \arrow[d, equal] & \mathrm{Cone}(\tau)_{-1} \arrow[d, equal] \\ \cdots \arrow[r] & \begin{matrix} R^{\beta_1} \\ \oplus \\ 0 \end{matrix} \arrow[rr, "\begin{pmatrix} -\partial_1^P & 0 \\ 0 & 0 \end{pmatrix}"] & & \begin{matrix} R^{\beta_0} \\ \oplus \\ 0 \end{matrix} \arrow[rr, "\begin{pmatrix} 0 & 0 \\ \tau & 0 \end{pmatrix}"] & & \begin{matrix} 0 \\ \oplus \\ M \end{matrix} \arrow[r] & 0 \end{tikzcd}$$

and is isomorphic to

$$\cdots \longrightarrow R^{\beta_1} \xrightarrow{-\partial_1^P} R^{\beta_0} \xrightarrow{\tau} M \longrightarrow 0$$

where M is of homological degree 0. Note also that

$$\Sigma^{-1}\,\mathrm{Cone}(\tau) = \qquad \cdots \longrightarrow R^{\beta_1} \xrightarrow{\partial_1^P} R^{\beta_0} \xrightarrow{-\tau} M \longrightarrow 0 \qquad \cong P^+$$

where M is of homological degree -1.

(d) If $\alpha\colon A \to Y$ is a chain map, then $\mathrm{Cone}(\alpha)_{i+1}$ is the direct sum of the two R-modules indicated below.

$$\begin{tikzcd} A \arrow[d, "\alpha"] & \cdots \arrow[r] & A_{i+1} \arrow[r] \arrow[d] & A_i \arrow[r] \arrow[d] & A_{i-1} \arrow[r] \arrow[d] & \cdots \\ Y & \cdots \arrow[r] & Y_{i+1} \arrow[r] & Y_i \arrow[r] & Y_{i-1} \arrow[r] & \cdots \end{tikzcd}$$

`prop190912d` **Proposition III.A.4.19.** *If $\alpha\colon A \to Y$ is a chain map, then $\mathrm{Cone}(\alpha)$ is an R-complex.*

Proof. It suffices to show $\partial_{i-1}^{\mathrm{Cone}(\alpha)} \circ \partial_i^{\mathrm{Cone}(\alpha)} = 0$. To this end we compute

$$\begin{pmatrix} -\partial_{i-2}^A & 0 \\ \alpha_{i-2} & \partial_{i-1}^Y \end{pmatrix} \begin{pmatrix} -\partial_{i-1}^A & 0 \\ \alpha_{i-1} & \partial_i^Y \end{pmatrix} = \begin{pmatrix} \partial_{i-2}^A \circ \partial_{i-1}^A & 0 \\ \partial_{i-1}^Y \circ \alpha_{i-1} - \alpha_{i-2} \circ \partial_{i-1}^A & \partial_{i-1}^Y \circ \partial_i^Y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that the $2,2$ and $2,1$ and $1,1$-entries of the third matrix here are zero since Y is an R-complex, α is a chain map, and A is an R-complex, respectively. □

`thm190912e` **Theorem III.A.4.20.** *Let $\alpha\colon A \to Y$ be a chain map.*

`thm190912e.a` (a) *There exists a short exact sequence of chain maps*

$$0 \longrightarrow Y \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} \mathrm{Cone}(\alpha) \xrightarrow{(0\ 1)} \Sigma A \longrightarrow 0.$$

`thm190912e.b` (b) *The connecting homomorphism $\eth_i$ given by Theorem III.A.4.16 is actually the map induced on homology modules given by Theorem III.A.4.10, i.e.,*

$$\begin{tikzcd} \eth_i \arrow[d, equal] & : & H_i(\Sigma A) \arrow[r] \arrow[d, equal] & H_{i-1}(Y). \\ H_{i-1}(\alpha) & & H_{i-1}(A) & \end{tikzcd}$$

PROOF. (a) Consider the following diagram with split exact rows:

$$\begin{tikzcd}
0 \arrow[r] & Y_i \arrow[r, "\binom{1}{0}"] \arrow[d, "\partial_i^Y"'] & \begin{matrix} A_{i-1} \\ \oplus \\ Y_i \end{matrix} \arrow[r, "(0\ 1)"] \arrow[d, "\begin{pmatrix} -\partial_{i-1}^A & 0 \\ \alpha_{i-1} & \partial_i^Y \end{pmatrix}"'] & A_{i-1} \arrow[r] \arrow[d, "-\partial_{i-1}^A"'] & 0 \\
0 \arrow[r] & Y_{i-1} \arrow[r, "\binom{1}{0}"'] & \begin{matrix} A_{i-2} \\ \oplus \\ Y_{i-1} \end{matrix} \arrow[r, "(0\ 1)"'] & A_{i-2} \arrow[r] & 0.
\end{tikzcd}$$

A diagram chase shows that this diagram commutes, so it is an exact sequence of chain maps.

(b) Let $\overline{a} \in H_{i-1}(A)$. Then we can chase this element from $H_i(\Sigma A)$ to $H_{i-1}(Y)$, following the diagram from the proof of (a).

$$\begin{tikzcd}
 & \begin{pmatrix} a \\ 0 \end{pmatrix} \arrow[r, mapsto] \arrow[d, mapsto] & a \arrow[d, mapsto] \\
\alpha_{i-1}(a) \arrow[r, mapsto] & \begin{pmatrix} 0 \\ \alpha_{i-1}(a) \end{pmatrix} & 0.
\end{tikzcd}$$

Therefore, we have $\eth_i(\overline{a}) = \overline{\alpha_{i-1}(a)} = H_{i-1}(\alpha)(\overline{a})$.

$\square$

cor190917a

COROLLARY III.A.4.21. *Let $\phi\colon A \to C$ be a chain map. Then ϕ is a quasiisomorphism if and only if* $\mathrm{Cone}(\phi)$ *is exact.*

PROOF. Consider the following long exact sequence from Theorems III.A.4.16 and III.A.4.20:

$$\cdots \longrightarrow H_i(\mathrm{Cone}(\phi)) \longrightarrow H_{i-1}(A) \xrightarrow{H_{i-1}(\phi)} H_{i-1}(C) \longrightarrow H_{i-1}(\mathrm{Cone}(\phi)) \longrightarrow \cdots.$$

($\Leftarrow$) Suppose $\mathrm{Cone}(\phi)$ is exact. This implies that $H_i(\mathrm{Cone}(\phi)) = 0$ for all i, so the above long exact sequence looks like

$$0 \longrightarrow H_{i-1}(A) \xrightarrow[H_{i-1}(\phi)]{\cong} H_{i-1}(C) \longrightarrow 0.$$

The isomorphism here is from Fact III.A.2.2(d). Therefore ϕ is a quasiisomorphism by definition.

($\Rightarrow$) Suppose ϕ is a quasiisomorphism. Then a different piece of the above long exact sequence looks like

$$\cdots \longrightarrow H_i(A) \xrightarrow[H_i(\phi)]{\cong} H_i(C) \longrightarrow H_i(\mathrm{Cone}(\phi)) \longrightarrow H_{i-1}(A) \xrightarrow[H_{i-1}(\phi)]{\cong} H_{i-1}(C) \longrightarrow \cdots.$$

This implies the unlabeled middle two maps are both 0, and it follows that $H_i(\mathrm{Cone}(\phi)) = 0$ for all i. Therefore, $\mathrm{Cone}(\phi)$ is exact. $\square$

ection063021f

III.A.5. Application: Long Exact Sequences in Ext and Tor

fact190917b

FACT III.A.5.1. *Let N be an R-module and let*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be exact. Then Hom_R *is left exact and* $\otimes$ *is right exact.*

In other words, the following sequences are exact:

(a) $\mathrm{Hom}_R(N,-) = \quad 0 \longrightarrow \mathrm{Hom}_R(N,M') \xrightarrow[\mathrm{Hom}_R(N,\alpha)]{\alpha_*} \mathrm{Hom}_R(N,M) \xrightarrow[\mathrm{Hom}_R(N,\beta)]{\beta_*} \mathrm{Hom}_R(N,M'') \longrightarrow ?$

(b) $\mathrm{Hom}_R(-,N) = \quad 0 \longrightarrow \mathrm{Hom}_R(M'',N) \xrightarrow[\mathrm{Hom}_R(\alpha,N)]{\alpha^*} \mathrm{Hom}_R(M,N) \xrightarrow[\mathrm{Hom}_R(\beta,N)]{\beta^*} \mathrm{Hom}_R(M',N) \longrightarrow ?$

(c) $- \otimes_R N =$ $$? \longrightarrow M' \otimes_R N \xrightarrow{\alpha\otimes N} M \otimes_R N \xrightarrow{\beta\otimes N} M'' \otimes_R N \longrightarrow 0$$

ques190917c

QUESTION III.A.5.2. What goes in the ?'s ?

defn190917d

DEFINITION III.A.5.3. Let P be a free resolution of M. Then

(a) $\mathrm{Ext}^i_R(M,N) = H_{-i}(\mathrm{Hom}_R(P,N))$, and
(b) $\mathrm{Tor}_i^R(M,N) = H_i(P \otimes_R N)$.

prop190917e

PROPOSITION III.A.5.4. *Let A be an R-complex and N and R-module. Then the following are R-complexes:*

(a) $\mathrm{Hom}_R(A,N) =$ $$\cdots \longrightarrow \mathrm{Hom}_R(A_{i-1},N) \xrightarrow{(\partial_i^A)^*} \mathrm{Hom}_R(A_i,N) \xrightarrow{(\partial_{i+1}^A)^*} \cdots$$

(b) $A_i \otimes_R N =$ $$\cdots \longrightarrow A_i \otimes_R N \xrightarrow{\partial_i^A\otimes N} A_{i-1} \otimes_R N \xrightarrow{\partial_{i-1}^A\otimes N} \cdots$$

PROOF. (a) Let $\gamma \in \mathrm{Hom}_R(A_{i-1},N)$, and consider the diagram:

$$\begin{array}{llcl} \mathrm{Hom}_R(A,N) = & \cdots \longrightarrow \mathrm{Hom}_R(A_{i-1},N) & \xrightarrow{(\partial_i^A)^*} \mathrm{Hom}_R(A_i,N) & \xrightarrow{(\partial_{i+1}^A)^*} \cdots \\ & \gamma & \longmapsto \gamma \circ \partial_i^A & \longmapsto \gamma \circ \partial_i^A \circ_{i+1}^A = 0 \end{array}$$

(b) Let $a \otimes n \in A_i \otimes_R N$, and consider the diagram:

$$\begin{array}{llcl} A_i \otimes_R N = & \cdots \longrightarrow A_i \otimes_R N & \xrightarrow{\partial_i^A\otimes N} A_{i-1} \otimes_R N & \xrightarrow{\partial_{i-1}^A\otimes N} \cdots \\ & a \otimes n & \longmapsto \partial_i^A(a) \otimes n & \longmapsto \partial_{i-1}^A \partial_i^A(a) \otimes n = 0 \otimes n = 0 \end{array}$$

By definition, this shows that both of the above are R-complexes.

□

fact190917f

FACT III.A.5.5. Ext^i_R *and* Tor_i^R *are independent of choice of* P.

thm190917g

THEOREM III.A.5.6. *Let N be an R-module and let*

$$0 \longrightarrow M' \longrightarrow M \xrightarrow{\alpha} M'' \xrightarrow{\beta} 0$$

be exact. Then there exist the following long exact sequences:

thm190917g.a

(a)

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_R(N,M') \xrightarrow{\alpha_*} \mathrm{Hom}_R(N,M) \xrightarrow{\beta_*} \mathrm{Hom}_R(N,M'') \longrightarrow \\ \longrightarrow \mathrm{Ext}^1_R(N,M') \longrightarrow \mathrm{Ext}^1_R(N,M) \longrightarrow \mathrm{Ext}^1_R(N,M'') \longrightarrow \\ \longrightarrow \mathrm{Ext}^2_R(N,M') \longrightarrow \cdots \end{array}$$

thm190917g.b

(b)

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_R(M'',N) \xrightarrow{\alpha^*} \mathrm{Hom}_R(M,N) \xrightarrow{\beta^*} \mathrm{Hom}_R(M',N) \longrightarrow \\ \longrightarrow \mathrm{Ext}^1_R(M'',N) \longrightarrow \mathrm{Ext}^1_R(M,N) \longrightarrow \mathrm{Ext}^1_R(M',N) \longrightarrow \\ \longrightarrow \mathrm{Ext}^2_R(M'',N) \longrightarrow \cdots \end{array}$$

thm190917g.c (c)

$$\begin{tikzcd}
 & \cdots \arrow[r] & \operatorname{Tor}_2^R(M'', N) \arrow[dll, rounded corners] \\
\operatorname{Tor}_1^R(M', N) \arrow[r] & \operatorname{Tor}_1^R(M, N) \arrow[r] & \operatorname{Tor}_1^R(M'', N) \arrow[dll, rounded corners] \\
M' \otimes_R N \arrow[r, "\alpha\otimes N"] & M \otimes_R N \arrow[r, "\beta\otimes N"] & M'' \otimes_R N \arrow[r] & 0
\end{tikzcd}$$

Proof. We will prove part (a). Let F be a free resolution of N, and construct the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(F, M') \arrow[r, "\alpha_*"] & \operatorname{Hom}_R(F, M) \arrow[r, "\beta_*"] & \operatorname{Hom}_R(F, M'') \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Hom}_R(F_0, M') \arrow[r, "\alpha_*"] \arrow[d, "(\partial_1^F)^*"] & \operatorname{Hom}_R(F_0, M) \arrow[r, "\beta_*"] \arrow[d, "(\partial_1^F)^*"] & \operatorname{Hom}_R(F_0, M'') \arrow[r] \arrow[d, "(\partial_1^F)^*"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_R(F_1, M') \arrow[r, "\alpha_*"] \arrow[d, "(\partial_2^F)^*"] & \operatorname{Hom}_R(F_1, M) \arrow[r, "\beta_*"] \arrow[d, "(\partial_2^F)^*"] & \operatorname{Hom}_R(F_1, M'') \arrow[r] \arrow[d, "(\partial_2^F)^*"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Here, we have β_* defined so that for $\delta \in \operatorname{Hom}_R(F_i, M)$, we have $\beta_*(\delta) = \beta \circ \delta$. Since the F_i's are projective modules, this implies exactness at $\operatorname{Hom}_R(F_i, M'')$ in each row of the above diagram. Then by Theorem III.A.4.16, we can form the following long exact sequence from each of the above short exact sequences of chain maps:

$$\begin{tikzcd}
 & & 0 \arrow[dll, rounded corners] \\
H_0(\operatorname{Hom}_R(F, M')) \arrow[r, "H_0(\alpha_*)"] & H_0(\operatorname{Hom}_R(F, M)) \arrow[r, "H_0(\beta_*)"] & H_0(\operatorname{Hom}_R(F, M'')) \arrow[dll, rounded corners] \\
H_{-1}(\operatorname{Hom}_R(F, M')) \arrow[r] & H_{-1}(\operatorname{Hom}_R(F, M)) \arrow[r] & H_{-1}(\operatorname{Hom}_R(F, M'')) \arrow[dll, rounded corners] \\
H_{-2}(\operatorname{Hom}_R(F, M')) \arrow[r] & \cdots .
\end{tikzcd}$$

Each of the homology groups of degree $-i$ for $i \geq 1$ are exactly the corresponding Ext_R^i from the statement of the result. Therefore, we only have to show the first row corresponds to the stated theorem. First, we check that $\operatorname{Hom}_R(N, M') \cong H_0(\operatorname{Hom}_R(F, M'))$. Notice that

$$F^+ = \qquad \cdots \xrightarrow{\partial_2^F} F_1 \xrightarrow{\partial_1^F} F_0 \xrightarrow{\tau} N \longrightarrow 0$$

is exact, so the following sequence

$$(F^+)^* = \qquad 0 \longrightarrow \operatorname{Hom}_R(N, M') \xrightarrow{\tau^*} \operatorname{Hom}_R(F_0, M') \xrightarrow{(\partial_1^F)^*} \operatorname{Hom}_R(F_1, M')$$

is exact. We can use the injectivity of τ^* and the exactness at $\operatorname{Hom}_R(F_0, M')$ to build the following sequence of isomorphisms and equalities:

$$\mathrm{Hom}_R(N,M') \cong \mathrm{Im}\,\tau^* = \mathrm{Ker}\,(\partial_1^F)^* = H_0(\mathrm{Hom}_R(F,M')).$$

The argument is similar for the sequences containing M and M''. Next, one can check the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{Hom}_R(N,M') \arrow[r, "\cong"] \arrow[d, "\alpha_*"] & H_0(\mathrm{Hom}_R(F,M')) \arrow[d, "H_0(\alpha_*)"] \\
\mathrm{Hom}_R(N,M) \arrow[r, "\cong"] & H_0(\mathrm{Hom}_R(F,M))
\end{tikzcd}$$

One obtains an analogous diagram for β, and these show how the long exact sequence given in the proof above matches up with the one in the statement of the result. This proves part (a). In the interest of time, we omit the remainder of the proof. $\square$

We end this chapter with some computations of long exact sequences.

ex190919a

Example III.A.5.7. Let $R = k[X,Y]$ and $N = R/\langle X,Y\rangle$. Consider the following short exact sequence

$$0 \longrightarrow R \xrightarrow{X} R \longrightarrow R/(X) \longrightarrow 0.$$

To compute the associated long exact sequence for $\mathrm{Ext}_R(N,-)$, we use the following projective resolutions of N (augmented and truncated), then we dualize.

$$\begin{aligned}
P^+ &= \quad 0 \longrightarrow R \xrightarrow{\begin{pmatrix}-Y\\ X\end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix}X & Y\end{pmatrix}} R \longrightarrow N \longrightarrow 0\\
P &= \quad 0 \longrightarrow R \xrightarrow{\begin{pmatrix}-Y\\ X\end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix}X & Y\end{pmatrix}} R \longrightarrow 0
\end{aligned}$$

$$\begin{tikzcd}
P^* = \mathrm{Hom}_R(P,R) = \arrow[d, "\cong"] & 0 \arrow[r] & \mathrm{Hom}_R(R,R) \arrow[r, "\partial_1^*"] \arrow[d, "\cong"] & \mathrm{Hom}_R(R^2,R) \arrow[r, "\partial_2^*"] \arrow[d, "\cong"] & \mathrm{Hom}_R(R,R) \arrow[r] \arrow[d, "\cong"] & 0 \\
\Sigma^{-2}P = & 0 \arrow[r] & R \arrow[r, swap, "\begin{pmatrix}-Y\\ X\end{pmatrix}"] & R^2 \arrow[r, swap, "\begin{pmatrix}X & Y\end{pmatrix}"] & R \arrow[r] & 0
\end{tikzcd}$$

The isomorphism above is straightforward to verify. We will discus this "self-duality" isomorphism in more detail later in the course. Furthermore, from the way that $\Sigma^{-2}P$ is defined, we have

$$\mathrm{Ext}^i_R(N,R) \cong H_{-i}(P^*) \cong H_{-i}(\Sigma^{-2}P) = H_{2-i}(P) \cong \begin{cases} R/(X,Y) = N & \text{if } i = 2\\ 0 & \text{if } i \neq 2\end{cases}$$

Next, we need to find $\mathrm{Ext}^i_R(N,R/(X))$. To do so, we consider the isomorphic sequences

$$\begin{tikzcd}
\mathrm{Hom}_R(P,R/(X)) = \arrow[d, swap, "\cong"] & 0 \arrow[r] & \mathrm{Hom}_R(R,R/(X)) \arrow[r, "\partial_1^*"] & \mathrm{Hom}_R(R^2,R/(X)) \arrow[r, "\partial_2^*"] & \mathrm{Hom}_R(R,R/(X)) \arrow[r] & 0 \\
A = \arrow[d, swap, "\cong"] & 0 \arrow[r] & R/(X) \arrow[r, "\left(\begin{smallmatrix}X\\ Y\end{smallmatrix}\right)=\left(\begin{smallmatrix}0\\ Y\end{smallmatrix}\right)"] & (R/(X))^2 \arrow[r, "\left(\begin{smallmatrix}-Y & X\end{smallmatrix}\right)=\left(\begin{smallmatrix}-Y & 0\end{smallmatrix}\right)"] & R/(X) \arrow[r] & 0 \\
C = & 0 \arrow[r] & k[Y] \arrow[r, "\left(\begin{smallmatrix}0\\ Y\end{smallmatrix}\right)"] & (k[Y])^2 \arrow[r, "\left(\begin{smallmatrix}-Y & 0\end{smallmatrix}\right)"] & k[Y] \arrow[r] & 0
\end{tikzcd}$$

To compute $H_i(C)$, first consider the kernel of the map $\begin{pmatrix} 0 \\ Y \end{pmatrix} : k[Y] \to (k[Y])^2$. This map is injective since Y is a non-zero-divisor on $k[Y]$. Therefore the kernel is 0, so for homological degree 0 we get

$$\mathrm{Ext}^0_R(N, R/(X)) = H_0(\mathrm{Hom}_R(P, R/(X))) = 0.$$

Next consider the image of the map $\begin{pmatrix} -Y & 0 \end{pmatrix} : (k[Y])^2 \to k[Y]$. The image of this map is $\langle Y \rangle$, so for homological degree -2 we get

$$\mathrm{Ext}^2_R(N, R/(X)) = H_{-2}(\mathrm{Hom}_R(P, R/(X))) \cong k[Y]/\langle Y \rangle \cong N.$$

Finally consider the image of the first map and the kernel of the second map. We have $\mathrm{Im} \begin{pmatrix} 0 \\ Y \end{pmatrix} = \left\langle \begin{pmatrix} 0 \\ Y \end{pmatrix} \right\rangle$. Let $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathrm{Ker} \begin{pmatrix} -Y & 0 \end{pmatrix}$, so $-Yf = 0$. Since Y is a non-zero-divisor of $(k[Y])^2$, then $f = 0$. Therefore,

$$\mathrm{Ker} \begin{pmatrix} -Y & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ g \end{pmatrix} \mid g \in k[Y] \right\} = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \text{ so}$$

$$\mathrm{Ext}^1_R(N, R/(X)) = \frac{\mathrm{Ker} \begin{pmatrix} -Y & 0 \end{pmatrix}}{\mathrm{Im} \begin{pmatrix} 0 \\ Y \end{pmatrix}} = \frac{\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 0 \\ Y \end{pmatrix} \right\rangle} \cong \frac{k[Y]}{\langle Y \rangle} \cong \frac{k[X,Y]}{\langle X, Y \rangle} = N.$$

Now we have the information needed to construct the long exact sequence in $\mathrm{Ext}_R(N, -)$ following the notation from Theorem III.A.5.6(a):

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^0_R(N,R) = 0 \arrow[r] & \mathrm{Ext}^0_R(N,R) = 0 \arrow[r] & \mathrm{Ext}^0_R(N,R/(X)) \arrow[dll, rounded corners] \\
& \mathrm{Ext}^1_R(N,R) = 0 \arrow[r] & \mathrm{Ext}^1_R(N,R) = 0 \arrow[r] & \mathrm{Ext}^1_R(N,R/(X)) \cong N \arrow[dll, rounded corners] \\
& \mathrm{Ext}^2_R(N,R) \cong N \arrow[r, "0", "X"'] & \mathrm{Ext}^2_R(N,R) \cong N \arrow[r] & \mathrm{Ext}^2_R(N,R/(X)) \cong N \arrow[dll, rounded corners] \\
& \mathrm{Ext}^3_R(N,R) = 0 \arrow[r] & \cdots &
\end{tikzcd}$$

Since most of the terms are 0, the only interesting part simplifies to the following exact sequence, where the labelled properties can be determined using the exactness.

$$0 \longrightarrow N \xrightarrow{\cong} N \xrightarrow{0} N \xrightarrow{\cong} N \longrightarrow 0.$$

Additionally, it is possible to show that

$$\mathrm{Ext}^i_R(N, N) \cong \begin{cases} N^2 & \text{if } i = 1 \\ N & \text{if } i = 0, 2 \\ 0 & \text{otherwise} \end{cases}.$$

We leave it as an exercise here to verify this and to compute the long exact sequence in $\mathrm{Ext}_R(N, -)$ associated to the following short exact sequence:

$$0 \longrightarrow R/(X) \xrightarrow{Y} R/(X) \longrightarrow N \longrightarrow 0.$$

Exercises

Let R be a non-zero commutative ring with identity. For the following four exercises, consider a sequence of R-modules and R-module homomorphisms

$$A = \cdots \xrightarrow{\partial^A_{i+1}} A_i \xrightarrow{\partial^A_i} \cdots.$$

Assume that each R-module A_i is free with finite basis B_i.

exer190822a EXERCISE III.A.5.8. Fix an integer i. Assume that for all $b \in B_i$ we have $\partial^A_{i-1}(\partial^A_i(b)) = 0$. Prove that $\partial^A_{i-1}\partial^A_i = 0$.

exer190822b0 EXERCISE III.A.5.9. Fix integers i and j, and let $f\colon B_i \times B_j \to A_{i+j}$ be a function. Prove that there is a unique well-defined R-bilinear map $\mu_{i,j}\colon A_i \times A_j \to A_{i+j}$ such that $\mu_{i,j}(b, b') = f(b, b')$ for all $b \in B_i$ and $b' \in B_j$.

exer190822b EXERCISE III.A.5.10. Fix integers i and j, and let $\mu_{i,j}\colon A_i \times A_j \to A_{i+j}$ be an R-bilinear map. For all $a \in A_i$ and $A' \in A_j$, set $aa' = \mu_{i,j}(a, a')$.

(a) Assume that $i = 0$ and there is an element $1 \in A_0$ such that $1b' = b'$ for all $b' \in B_j$. Prove that $1a' = a'$ for all $a' \in A_j$.
(b) Assume that for all $b \in B_i$ and $b' \in B_j$ we have $bb' = (-1)^{ij}b'b$. Prove that for all $a \in A_i$ and $a' \in A_j$ we have $aa' = (-1)^{ij}a'a$.
(c) Assume that for all $b \in B_i$ and $b', b'' \in B_j$ we have $b(b' + b'') = bb' + bb''$. (Use the standard order of operations here.) Prove that for all $a \in A_i$ and $a', a'' \in A_j$ we have $a(a' + a'') = aa' + aa''$.

exer190822c EXERCISE III.A.5.11. For all integers i and j, let $\mu_{i,j}\colon A_i \times A_j \to A_{i+j}$ be an R-bilinear map. For all $a \in A_i$ and $A' \in A_j$, set $aa' = \mu_{i,j}(a, a')$. Fix integers i, j, and k; and assume that for all $b \in B_i$ and $b' \in B_j$ and $b'' \in B_k$ we have $b(b'b'') = (bb')b''$. Prove that for all $a \in A_i$ and $a' \in A_j$ and $a'' \in A_k$ we have $a(a'a'') = (aa')a''$.

exr210722a EXERCISE III.A.5.12. Let K be a field and consider an exact sequence

$$0 \to K^{\beta_d} \to \cdots \to K^{\beta_0} \to 0.$$

Prove that $\sum_{i=0}^{d}(-1)^i\beta_i = 0$.

Hint: induct on d. The base cases are $d = 0, 1, 2$. In the inductive step, show that $\beta_d \leq \beta_{d-1}$ and filet the given sequence as follows.

$$\begin{tikzcd}
 & & & 0 \arrow[dr] & & 0 & & & \\
0 \arrow[dr] & & & & K^{(\beta_{d-1}-\beta_d)} \arrow[ur] \arrow[dr] & & & & \\
0 \arrow[r] & K^{\beta_d} \arrow[rr] \arrow[dr] & & K^{\beta_{d-1}} \arrow[ur] \arrow[rr] & & K^{\beta_{d-2}} \arrow[r] & \cdots \arrow[r] & K^{\beta_0} \arrow[r] & 0 \\
 & & K^{\beta_d} \arrow[ur] \arrow[dr] & & & & & & \\
 & 0 \arrow[ur] & & 0 & & & & &
\end{tikzcd}$$

exr210722b EXERCISE III.A.5.13. Let k be a field, and set $R = k[X, Y, Z]$ with $I = \langle XY, XZ, YZ\rangle$. Find an augmented free resolution of R/I over R, and prove that your resolution is exact.

exr210722c EXERCISE III.A.5.14. Let R be a commutative ring with identity, and consider the following diagram of R-module homomorphisms where the rows are exact.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] & F \arrow[r, "\tau"] & C \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\tau'"] & C' \arrow[r] & 0
\end{tikzcd}$$

Assume that F is a finite rank free R-module, and show that there are R-module homomorphisms g and h making the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "h"] & F \arrow[r, "\tau"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\tau'"] & C' \arrow[r] & 0
\end{tikzcd}$$

CHAPTER III.B

Examples of Free Resolutions

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ection063021g

III.B.1. Resolutions of Mapping Cones

Our first goal of this section is, given an R-module homomorphism $f : M \to N$ and resolutions of M and N, to build a resolution of $N/f(M)$.

lem190919b LEMMA III.B.1.1 (Lifting Lemma). *Let $f : M \to N$ be an R-module homomorphism, let P^+ be a free resolution of M, and let Q^+ be a free resolution of N. Then there exist chain maps $F : P \to Q$ and $F^+ : P^+ \to Q^+$ such that $F^+_{-1} = f$ and such that the following diagram commutes:*

$$\begin{tikzcd} H_0(P) \arrow[r, "H_0(F)"] \arrow[d, "\overline{\tau}", "\cong"'] & H_0(Q) \arrow[d, "\cong", "\overline{\pi}"'] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

Graphically, the following diagram commutes:

$$\begin{tikzcd} P^+ = \arrow[d, "F^+"'] & \cdots \arrow[r, "\partial_2^P"] & P_1 \arrow[r, "\partial_1^P"] \arrow[d, dashed, "F_1"] & P_0 \arrow[r, "\tau"] \arrow[d, dashed, "F_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\ Q^+ = & \cdots \arrow[r, "\partial_2^Q"] & Q_1 \arrow[r, "\partial_1^Q"] & Q_0 \arrow[r, "\pi"] & N \arrow[r] & 0. \end{tikzcd}$$

PROOF. Consider the commutative diagram III.B.1.3.1 on the following page, which is obtained via repeated application of Exercise III.A.5.14. We can see the diagram below commutes, since the equality in the bottom right comes from the commutativity of diagram III.B.1.3.1. Furthermore, the diagram shows that $F^+_{-1} = f$.

$$\begin{tikzcd} \overline{p} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \tau(p) \arrow[ddd, mapsto] \\ & H_0(P) \arrow[r, "\overline{\tau}"] \arrow[d, "H_0(F)"] & M \arrow[d, "f"] & \\ & H_0(Q) \arrow[r, "\overline{\pi}"] & N & \\ \overline{F_0(p)} \arrow[rr, mapsto] & & \pi(F_0(p)) = f(\tau(p)) & \end{tikzcd}$$

$\square$

thm190919d THEOREM III.B.1.2. *Keep the same notation from Lemma III.B.1.1. If f is one-to-one, then* $\operatorname{Cone}(F)$ *is a free resolution of $N/f(M)$.*

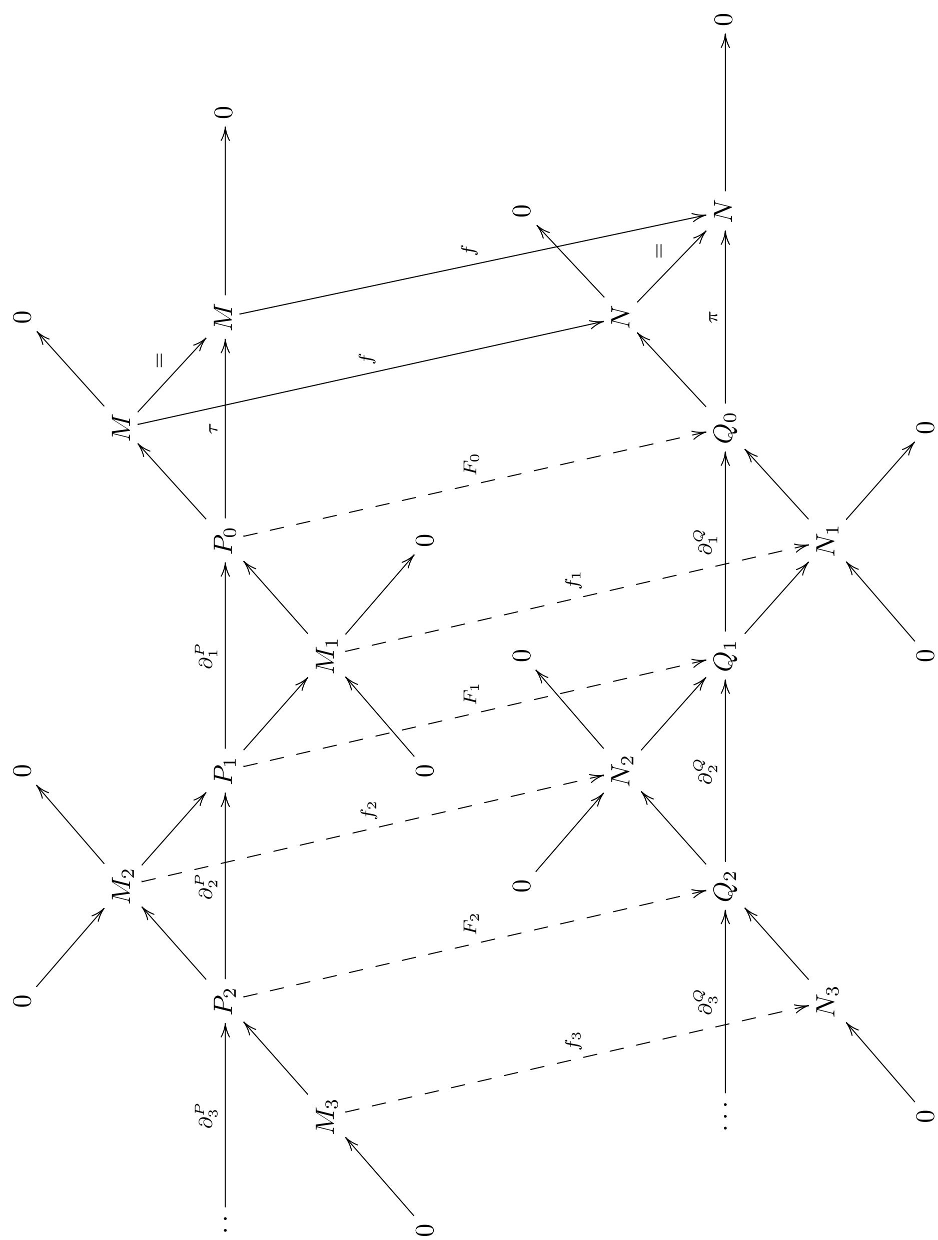
To prove this theorem, we require the following lemma.

lem190919e LEMMA III.B.1.3. *Let $F = (\cdots \xrightarrow{\partial_2^F} F_1 \xrightarrow{\partial_1^F} F_0 \longrightarrow 0)$ be an R-complex such that each F_i is free and $H_i(F) = 0$ for all $i \neq 0$. Then F is a free resolution of $H_0(F)$.*

PROOF. Define $\tau\colon F_0 \to H_0(F)$ to be the natural projection:

$$F_0 \xrightarrow{\tau} \frac{F_0}{\operatorname{Im} F_1 \to F_0} = \frac{\operatorname{Ker} F_0 \to 0}{\operatorname{Im} F_1 \to F_0} = H_0(F).$$

Then $F^+ = (\cdots \xrightarrow{\partial_2^F} F_1 \xrightarrow{\partial_1^F} F_0 \xrightarrow{\tau} H_0(F) \longrightarrow 0)$ is exact, since τ is onto and $\operatorname{Ker}\tau = \operatorname{Im}\partial_1^F$ by construction. $\square$



(III.B.1.3.1) eqn190919c

Now we can prove the theorem and thus acheive our first goal for this section.

Proof of III.B.1.2. First, we know $\mathrm{Cone}(F)_i = \begin{matrix} P_{i-1} \\ \oplus \\ Q_i \end{matrix}$ is free because P_{i-1} and Q_i are free modules. Then, compute $H_i(\mathrm{Cone}(F))$ using the following short exact sequence:

$$0 \longrightarrow Q \longrightarrow \mathrm{Cone}(F) \longrightarrow \Sigma P \longrightarrow 0.$$

Then the corresponding long exact sequence of homology modules is

$$\begin{tikzcd} H_i(Q) \arrow[r] & H_i(\mathrm{Cone}(F)) \arrow[r] & H_{i-1}(P) \arrow[r, "\eth_{i-1}"] & H_{i-1}(Q). \end{tikzcd}$$

Here, $\eth_{i-1} = H_{i-1}(F)$. If $i \geq 2$, then $H_i(Q) = 0 = H_{i-1}(P)$, so the section of this sequence becomes

$$0 \longrightarrow H_i(\mathrm{Cone}(F)) \longrightarrow 0,$$

so $H_i(\mathrm{Cone}(F)) = 0$ as well. If $i = 1$, then $\eth_0 = H_0(F)$, which is related to f in Lemma III.B.1.1, so if f is one-to-one, then $H_0(F)$ is one-to-one as well. Then we have

$$\begin{tikzcd} 0 \arrow[r] & H_1(\mathrm{Cone}(F)) \arrow[r] & H_0(P) \arrow[r, "H_0(F)"] & H_0(Q). \end{tikzcd}$$

A diagram chase shows that $H_1(\mathrm{Cone}(F)) = 0$. If $i = 0$, then we have

$$\begin{tikzcd} 0 \arrow[r] & H_0(P) \arrow[r, "H_0(F)"] \arrow[d, "\cong"', "\overline{\tau}"] & H_0(Q) \arrow[r] \arrow[d, "\cong"', "\overline{\pi}"] & H_0(\mathrm{Cone}(F)) \arrow[r] \arrow[d, dashed, "\exists!\rho"] & H_{-1}(P) = 0 \\ 0 \arrow[r] & M \arrow[r, "f"] & N \arrow[r] & N/f(M) \arrow[r] & 0. \end{tikzcd}$$

By the Snake Lemma, ρ is an isomorphism. Finally, we look at the structure of $\mathrm{Cone}(F)$.

$$\begin{tikzcd} P = \arrow[d, "F"] & (\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "F_1"] & P_0 \arrow[r] \arrow[d, "F_0"] & 0 \arrow[r] & 0) \\ Q = & (\cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & 0) \end{tikzcd}$$

$$\mathrm{Cone}(F) = \qquad \cdots \longrightarrow \begin{matrix} P_0 \\ \oplus \\ Q_1 \end{matrix} \longrightarrow \begin{matrix} 0 \\ \oplus \\ Q_0 \end{matrix} \longrightarrow 0$$

By Lemma III.B.1.3, $\mathrm{Cone}(F)$ is a free resolution of $N/f(M)$.

□

Our next goal is to build a free resolution for R-modules of the form $R/\langle f_1, \ldots, f_n \rangle$.

def190924a DEFINITION III.B.1.4. For any ideal $J \leq R$ and any element $r \in R$, the colon ideal is

$$(J :\, r) = \{s \in R \mid sr \in J\}.$$

ex190924b EXAMPLE III.B.1.5.

(a) In the ring $\mathbb{Z}$, for the ideal $36\mathbb{Z}$ and the element 15 we have the colon ideal $(36\mathbb{Z} :\, 15) = 12\mathbb{Z} \leq \mathbb{Z}$.

(b) More generally, let R be a unique factorization domain and let $f, g \in R$ be elements with respective prime factorizations $f = up_1^{e_1} \cdots p_n^{e_n}$ and $g = vp_1^{d_1} \cdots p_n^{d_n}$. Then we have the colon ideal $(J :\, g) = p_1^{c_1} \cdots p_n^{c_n} R$ where we set

$$c_i = (e_i - d_i)_+ = \begin{cases} e_i - d_i & e_i \geq d_i \\ 0 & e_i < d_i. \end{cases}$$

ex190924b.c (c) Let $R = k[X_1, \ldots, X_d]$ be a polynomial ring and for any vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^d$ set

$$\mathbf{X}^{\mathbf{a}} = X_1^{a_1} \cdots X_d^{a_d}.$$

If $\mathbf{b_1}, \ldots, \mathbf{b_n} \in \mathbb{N}^d$, then for the ideal $J = \langle \mathbf{X}^{\mathbf{b_1}}, \ldots, \mathbf{X}^{\mathbf{b_n}} \rangle$ one can show

$$(J :\, \mathbf{X}^{\mathbf{a}}) = \left\langle \mathbf{X}^{(\mathbf{b_1} - \mathbf{a})_+}, \ldots, \mathbf{X}^{(\mathbf{b_n} - \mathbf{a})_+} \right\rangle.$$

For instance, in the ring $R = k[X,Y]$ we have

$$\left(\left\langle X^3, X^2Y^2, Y^4\right\rangle : XY^2\right) = \left\langle X^2, X, Y^2\right\rangle = \left\langle X, Y^2\right\rangle.$$

prop190924c PROPOSITION III.B.1.6. *Let $J \leq R$ be an ideal and let $r \in R$.*

(a) $J \subseteq (J : r) \leq R$
(b) $(J : r) = R$ *if and only if* $r \in J$.
(c) *The sequence*

$$0 \longrightarrow \frac{R}{(J:r)} \xrightarrow{r\cdot} \frac{R}{J} \xrightarrow{\psi} \frac{R}{(J+rR)} \longrightarrow 0$$

is exact where ψ is the natural surjection.

PROOF. (a) The inclusion follows from the definition of an ideal. Hence $(J : r)$ is non-empty. For any $s_1, s_2 \in (J : r)$ and any $t \in R$ we have

$$(ts_1 + s_2)r = ts_1r + s_2r \in J$$

since J is an ideal. This proves colon ideals are indeed ideals.

(b) We observe that $(J : r) = R$ if and only if $1 \in (J : r)$, if and only if $r = 1 \cdot r \in J$.

(c) Since $J \subseteq J + rR$, the map ψ above is a well-defined surjective R-module homomorphism. By the second isomorphism theorem

$$\frac{J+rR}{J} \cong \frac{rR}{J \cap rR}.$$

Since $rR/(J \cap rR)$ is cyclic with generator $\overline{r} = r + J \cap rR$, the module $(J + rR)/J$ is cyclic with generator $\overline{r} = r + J \cap rR$. Hence by the third isomorphism theorem we have

$$\frac{R/J}{\langle\overline{r}\rangle\,(R/J)} = \frac{R/J}{(J+rR)/J} \cong \frac{R}{J+rR}. \qquad \text{(III.B.1.6.1)}$$ eqn190924a

Therefore the sequence

$$R \xrightarrow{\overline{r}\cdot} \frac{R}{J} \xrightarrow{\psi} \frac{R}{J+rR} \longrightarrow 0$$

is exact since the display (III.B.1.6.1) above shows that

$$\operatorname{Ker}\psi = \langle\overline{r}\rangle\,(R/J) = \operatorname{Im} R \xrightarrow{\overline{r}\cdot} R/J.$$

Next we observe that

$$\operatorname{Ker} R \xrightarrow{\overline{r}\cdot} R/J = \{s \in R \mid s\overline{r} = 0 \in R/J\} = \{s \in R \mid sr \in J\} = (J : r)$$

and consider the commutative diagram

$$\begin{tikzcd} R \arrow[r, "\overline{r}\cdot"] \arrow[d, "\pi"'] & R/J \\ R/(J:r) \arrow[ur, dashed, "\exists\overline{\pi}"'] & \end{tikzcd}$$

where π is the natural surjection and $\overline{\pi}$ is a well-defined R-module monomorphism. Moreover a short diagram chase shows that $\overline{\pi}$ is the homothety map $r\cdot$ and $\operatorname{Im} r\cdot = \operatorname{Im}\overline{r}$. It follows that the desired short exact sequence exists. $\square$

thm190924d THEOREM III.B.1.7. *Let $f_1, \ldots, f_j, r \in R$, let $J = \langle f_1, \ldots, f_n\rangle \leq R$ be an ideal, let P be a free resolution of $R/(J : r)$, and let Q be a free resolution of R/J. Then there is a chain map $\Phi^+ \colon P^+ \to Q^+$ such that* $\operatorname{Cone}(\Phi)$ *is a free resolution of* $R/(J + rR)$.

PROOF. The existence of Φ^+ follows from Lemma III.B.1.1:

$$\begin{tikzcd}
P^+ = \arrow[d, dashed, "\exists\Phi^+"'] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, dashed, "\Phi_1^+"] & P_0 \arrow[r] \arrow[d, dashed, "\Phi_0^+"] & \frac{R}{(J\,:\,r)} \arrow[r] \arrow[d, hook, "r\cdot"] & 0 \\
Q^+ = & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & \frac{R}{J} \arrow[r] & 0.
\end{tikzcd}$$

By Theorem III.B.1.2 we know $\operatorname{Cone}(\Phi)$ is a free resolution of $(R/J)/\operatorname{Im} r\cdot$ where $r\cdot$ is the vertical homothety map in the above ladder diagram. Moreover, by the proof of Proposition III.B.1.6 we have

$$\frac{R/J}{\operatorname{Im} r\cdot} = \frac{R/J}{\langle \overline{r}\rangle (R/J)} \cong \frac{R}{J + rR} = \frac{R}{\langle f_1, \ldots, f_n\rangle + \langle r\rangle} = \frac{R}{\langle f_1, \ldots, f_n, r\rangle}$$

which completes the proof. $\square$

ection063021h

III.B.2. The Koszul Complex

ex190924e EXAMPLE III.B.2.1. Set $R = k[X,Y]$. A free resolution of $R/\langle Y\rangle$ is

$$P^+ = \left(0 \longrightarrow R \xrightarrow{Y\cdot} R \xrightarrow{\tau} R/\langle Y\rangle \longrightarrow 0 \right)$$

and we can use this to build a free resolution of $R/\langle X,Y\rangle$. We consider the homothety chain map

$$\begin{tikzcd}
P^+ = \arrow[d, "X\cdot"', "\Phi^+"] & 0 \arrow[r] & R \arrow[r, "Y\cdot"] \arrow[d, "X\cdot"] & R \arrow[r, "\tau"] \arrow[d, "X\cdot"] & R/\langle Y\rangle \arrow[r] \arrow[d, hook, "X\cdot"] & 0 \\
P^+ = & 0 \arrow[r] & R \arrow[r, "Y\cdot"'] & R \arrow[r, "\tau"'] & R/\langle Y\rangle \arrow[r] & 0
\end{tikzcd}$$

where the homothety map $X\cdot\colon R/\langle Y\rangle \to R/\langle Y\rangle$ is injective since X is a non-zero-divisor on $R/\langle Y\rangle \cong k[X]$. Truncating we get

$$\begin{tikzcd}
P^+ = \arrow[d, "X\cdot"', "\Phi"] & 0 \arrow[r] & R \arrow[r, "Y\cdot"] \arrow[d, "X\cdot"] & R \arrow[r] \arrow[d, "X\cdot"] & 0 \\
P^+ = & 0 \arrow[r] & R \arrow[r, "Y\cdot"'] & R \arrow[r] & 0.
\end{tikzcd}$$

Then $\operatorname{Cone}(\Phi)$ is a free resolution of $R/\langle X,Y\rangle$ by Theorem III.B.1.7.

$$\begin{aligned}
\operatorname{Cone}(\Phi) &= \left(0 \longrightarrow \begin{matrix} R \\ \oplus \\ 0 \end{matrix} \xrightarrow{\left(\begin{smallmatrix} -Y & 0 \\ X & 0 \end{smallmatrix}\right)} \begin{matrix} R \\ \oplus \\ R \end{matrix} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ X & Y \end{smallmatrix}\right)} \begin{matrix} 0 \\ \oplus \\ R \end{matrix} \longrightarrow 0 \right) \\
&\cong \left(0 \longrightarrow R \xrightarrow{\left(\begin{smallmatrix} -Y \\ X \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} X & Y \end{smallmatrix}\right)} R \longrightarrow 0 \right)
\end{aligned}$$

def190924f DEFINITION III.B.2.2. Let $x, y, x_1, \ldots, x_n \in R$ be given. The Koszul complex is defined inductively on n.

$$\begin{aligned}
n = 1: &\qquad K^R(x) = \ 0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0 \\
n = 2: &\qquad K^R(x,y) = \operatorname{Cone}\left(K^R(y) \xrightarrow{x} K^R(y) \right) \\
&\qquad\qquad \cong \ 0 \longrightarrow R \xrightarrow{\left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & y \end{smallmatrix}\right)} R \longrightarrow 0 \\
n \geq 2: &\qquad K^R(x_1, \ldots, x_n) = \operatorname{Cone}\left(K^R(x_2, \ldots, x_n) \xrightarrow{x_1} K^R(x_2, \ldots, x_n) \right)
\end{aligned}$$

ex190924g

EXAMPLE III.B.2.3. Having already explicitly written the Koszul complex for one and two elements, we compute $K^R(x,y,z) = \mathrm{Cone}\Big(K^R(y,z) \xrightarrow{x} K^R(y,z)\Big)$. First we display the appropriate homothety chain map.

$$\begin{tikzcd}
K^R(y,z) \arrow[d, "x"'] & 0 \arrow[r] & R \arrow[r, "\left(\begin{smallmatrix} -z \\ y \end{smallmatrix}\right)"] \arrow[d, "x"'] & R^2 \arrow[r, "\left(\begin{smallmatrix} y & z \end{smallmatrix}\right)"] \arrow[d, "x\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)"] & R \arrow[r] \arrow[d, "x"'] & 0 \\
K^R(y,z) & 0 \arrow[r] & R \arrow[r, "\left(\begin{smallmatrix} -z \\ y \end{smallmatrix}\right)"'] & R^2 \arrow[r, "\left(\begin{smallmatrix} y & z \end{smallmatrix}\right)"'] & R \arrow[r] & 0
\end{tikzcd}$$

Now we can write down the appropriate cone. $K^R(x,y,z)$ is equal to

$$0 \longrightarrow \begin{matrix} R \\ \oplus \\ 0 \end{matrix} \xrightarrow{\begin{pmatrix} -\left(\begin{smallmatrix} z \\ -y \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \\ x & 0 \end{pmatrix}} \begin{matrix} R^2 \\ \oplus \\ R \end{matrix} \xrightarrow{\begin{pmatrix} -\left(\begin{smallmatrix} y & z \end{smallmatrix}\right) & 0 \\ \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) & \left(\begin{smallmatrix} -z \\ y \end{smallmatrix}\right) \end{pmatrix}} \begin{matrix} R \\ \oplus \\ R^2 \end{matrix} \xrightarrow{\begin{pmatrix} 0 & \left(\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right) \\ x & \left(\begin{smallmatrix} y & z \end{smallmatrix}\right) \end{pmatrix}} \begin{matrix} 0 \\ \oplus \\ R \end{matrix} \longrightarrow 0$$

and thus isomorphic to

$$0 \longrightarrow R \xrightarrow{\left(\begin{smallmatrix} z \\ -y \\ x \end{smallmatrix}\right)} R^3 \xrightarrow{\left(\begin{smallmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{smallmatrix}\right)} R^3 \xrightarrow{\left(\begin{smallmatrix} x & y & z \end{smallmatrix}\right)} R \longrightarrow 0.$$

Notice the presence of binomial coefficients from Pascal's triangle in the exponents of the previous display. These are also present in the $n=1$ and $n=2$ cases of Definition III.B.2.2. This leads us to the following proposition.

prop190926a

PROPOSITION III.B.2.4. *Let* $\mathbf{x} = x_1, \ldots, x_n \in R$ *and* $K = K^R(\mathbf{x})$. *Then:*

prop190926a.a (a) $K^R(\mathbf{x})_i \cong R^{\binom{n}{i}}$.

prop190926a.b (b) $\partial_1^K = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} : R^n \to R$.

prop190926a.c (c) $\partial_n^K = \begin{pmatrix} (-1)^{n-1}x_n \\ \vdots \\ -x_2 \\ x_1 \end{pmatrix} : R \to R^n$.

prop190926a.d (d) *The matrix representing* ∂_i^K *consists of* 0*'s and* $\pm x_k$*'s.*

PROOF. We will use induction on n.
Base case: The case for $n=1$ is covered in Definition III.B.2.2.
Inductive case: Set $\mathbf{x}' = x_2, \ldots, x_n$ and $K' = K^R(\mathbf{x}')$. Then we consider each part of the result:

(a) From the inductive hypothesis, we have $K'_i \cong R^{\binom{n-1}{i}}$. Therefore

$$K_i = \begin{matrix} K'_{i-1} \\ \oplus \\ K'_i \end{matrix} \cong \begin{matrix} R^{\binom{n-1}{i-1}} \\ \oplus \\ R^{\binom{n-1}{i}} \end{matrix} \cong R^{\binom{n-1}{i-1}+\binom{n-1}{i}} = R^{\binom{n}{i}}.$$

(b) We use Definition III.B.2.2 to construct the ∂_1^K map from the inductive hypothesis.

$$\partial_1^K\colon \begin{matrix} K'_0 \\ \oplus \\ K'_1 \end{matrix} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ x_1 & \partial_1^{K'} \end{smallmatrix}\right)} \begin{matrix} 0 \\ \oplus \\ K'_0 \end{matrix} \Rightarrow \partial_1^K\colon \begin{matrix} R \\ \oplus \\ R^{n-1} \end{matrix} \xrightarrow{\left(\begin{smallmatrix} x_1 & x_2 & \cdots & x_n \end{smallmatrix}\right)} \begin{matrix} 0 \\ \oplus \\ R \end{matrix} \Rightarrow \partial_1^K\colon\ R^n \xrightarrow{\left(\begin{smallmatrix} x_1 & x_2 & \cdots & x_n \end{smallmatrix}\right)} R.$$

The first implication comes from the result in part (a) of the proposition, while the second implication uses the isomorphism from $R \oplus R^{n-1}$ to R^n.

(c) We construct ∂_n^K in a similar way as above.

$$\partial_n^K:\ \begin{matrix} K'_{n-1} \\ \oplus \\ 0 \end{matrix} \xrightarrow{\begin{pmatrix} -\partial^{K'}_{n-1} & 0 \\ x_1 & 0 \end{pmatrix}} \begin{matrix} K'_{n-2} \\ \oplus \\ K'_{n-1} \end{matrix} \Rightarrow \partial_n^K:\ \begin{matrix} R \\ \oplus \\ 0 \end{matrix} \xrightarrow{\begin{pmatrix} -(-1)^{n-2}x_n \\ \vdots \\ -x_2 \\ x_1 \end{pmatrix}} \begin{matrix} R^{n-1} \\ \oplus \\ R \end{matrix} \Rightarrow \partial_n^K:\ R \xrightarrow{\begin{pmatrix} (-1)^{n-1}x_n \\ \vdots \\ -x_2 \\ x_1 \end{pmatrix}} R^n.$$

(d) For each i, we have

$$\partial_i^K = \begin{pmatrix} -\partial^{K'}_{i-1} & 0 \\ x_1 \cdot \mathrm{id} & \partial_i^{K'} \end{pmatrix}.$$

Then the inductive hypothesis tells us that $-\partial^{K'}_{i-1}$ and $\partial_i^{K'}$ consist only of 0's and $\pm x_2, \pm x_3, \ldots, \pm x_n$. Furthermore, $x_1 \cdot \mathrm{id}$ only consists of 0's and x_1's. Therefore, ∂_i^K only consists of 0's and $\pm x_i$'s for $i \in [n]$. $\square$

Now we consider the question: when is K a resolution? To answer this question, we introduce the following definition.

`def190926b` DEFINITION III.B.2.5. A sequence $x_n, x_{n-1}, \ldots, x_1 \in R$ is <u>weakly R-regular</u> if:

(1) x_n is a non-zero-divisor on R.
(2) x_{n-1} is a non-zero-divisor on $R/\langle x_n \rangle$.
(3) x_{n-2} is a non-zero-divisor on $R/\langle x_n, x_{n-1} \rangle$.
$\vdots$
(i) x_{n-i+1} is a non-zero-divisor on $R/\langle x_n, \ldots, x_{n-i+2} \rangle$.
$\vdots$
(n) x_1 is a non-zero-divisor on $R/\langle x_n, \ldots, x_2 \rangle$.

`ex190926c` EXAMPLE III.B.2.6. Let A be a commutative ring with identity, and let $R = A[X_1, \ldots, X_d]$. Then $\mathbf{X} = X_d, \ldots, X_1$ is weakly R-regular. We check the conditions of Definition III.B.2.5:

(1) X_d is a non-zero-divisor on R.
(2) $R/\langle X_d \rangle \cong A[X_1, \ldots, X_{d-1}]$, so X_{d-1} is a non-zero-divisor on $R/\langle X_d \rangle$.

Continuing in this fashion, we see that $\mathbf{X}$ is weakly R-regular. More generally, if $X_{i_1}, \ldots, X_{i_n}$ are distinct variables in $R = A[X_1, \ldots, X_d]$, then $X_{i_1}, \ldots, X_{i_n}$ is also weakly R-regular.

The following theorem says a free resolution of a ring mod a weakly-regular sequence is the Koszul complex applied to that sequence.

`thm190926d` THEOREM III.B.2.7. *If $\mathbf{x} = x_n, \ldots, x_1 \in R$ is weakly R-regular, then $K^R(\mathbf{x})$ is a free resolution of $R/\langle \mathbf{x} \rangle$.*

`cor190926e` COROLLARY III.B.2.8. *If $R = A[X_1, \ldots, X_d]$, then $K^R(X_1, \ldots, X_d)$ is a free resolution of $R/\langle X_1, \ldots, X_d \rangle \cong A$.*

PROOF OF III.B.2.7. We will use induction on n.

Base case: Let $n = 1$. Then $K^R(x_1) = 0 \longrightarrow R \xrightarrow{x_1} R \longrightarrow 0$. Since x_1 is a non-zero-divisor on R, we then have

$$H_1(K^R(x_1) \cong \operatorname{Ker}\ R \xrightarrow{x_1} R\ = 0.$$

Then $K^R(x_1)$ is a free resolution of $H_0(K^R(x_1)) \cong R/\langle x_1 \rangle$ by Lemma III.B.1.3.

Inductive Case: Let $\mathbf{x}' = x_n, \ldots, x_2$. By definition, $\mathbf{x}'$ is weakly R-regular. The inductive hypothesis tells us that $K' = K^R(\mathbf{x}')$ is a free resolution of $R/\langle \mathbf{x}' \rangle$. Then we claim that $(\langle \mathbf{x}' \rangle : x_1) = \langle \mathbf{x}' \rangle$.

Proof of claim. By Proposition III.B.1.6, we have $(\langle \mathbf{x}' \rangle : x_1) \supseteq \langle \mathbf{x}' \rangle$. Then we want to show $(\langle \mathbf{x}' \rangle : x_1) \subseteq \langle \mathbf{x}' \rangle$. Let $\alpha \in (\langle \mathbf{x}' \rangle : x_1)$, so $x_1 \cdot \alpha \in \langle \mathbf{x}' \rangle$. Then in $R/\langle \mathbf{x}' \rangle$, $x_1 \overline{\alpha} = 0$. But x_1 is a non-zero-divisor on $R/\langle \mathbf{x}' \rangle$ by condition (n) of Definition III.B.2.5, so $\overline{\alpha} = 0$ in $R/\langle \mathbf{x}' \rangle$. Therefore, $\alpha \in \langle \mathbf{x}' \rangle$.

Now consider the following free resolutions given by the inductive hypothesis:

$$\begin{tikzcd}
(K')^+ = \arrow[d, "x_1"'] & 0 \arrow[r] & R \arrow[r] \arrow[d, "x_1"'] & R^{n-1} \arrow[r] \arrow[d, "x_1"'] & \cdots \arrow[r] & R^{n-1} \arrow[r] \arrow[d, "x_1"'] & R \arrow[r] \arrow[d, "x_1"'] & R/\langle \mathbf{x}' \rangle = R/(\langle \mathbf{x}' \rangle : x_1) \arrow[r] \arrow[d, hook, "x_1"'] & 0 \\
(K')^+ = & 0 \arrow[r] & R \arrow[r] & R^{n-1} \arrow[r] & \cdots \arrow[r] & R^{n-1} \arrow[r] & R \arrow[r] & R/\langle \mathbf{x}' \rangle \arrow[r] & 0
\end{tikzcd}$$

By Theorem III.B.1.7, $K = \operatorname{Cone}(\ K' \xrightarrow{x_1} K'\ )$ is a free resolution of $R/\langle \mathbf{x}' \rangle$. $\square$

rmk190926f

REMARK III.B.2.9. If $\mathbf{x} = x_1, \ldots, x_n \in R$ is weakly R-regular, then $K^R(x_n, \ldots, x_1)$ is a free resolution of $R/\langle \mathbf{x} \rangle$ by Theorem III.B.2.7 and Exercise III.B.5.38.

rmk190926g

REMARK III.B.2.10. The ranks of the modules in $K^R(\mathbf{x})$ are symmetric because of the symmetry in Pascal's triangle. This leads us to the next property, called the self-duality of the Koszul complex. Note that it generalizes the self-duality we observed in Example III.A.5.7.

thm190926h

THEOREM III.B.2.11 (Self-duality of the Koszul complex). *The Koszul complex is self-dual, i.e.,* $K^R(\mathbf{x}) \cong \Sigma^n \operatorname{Hom}_R(K^R(\mathbf{x}), R)$.

PROOF. We will use induction on n.

Base Case: Let $n = 1$. Then we can directly write down an isomorphism between the complexes $K^R(x_1)$ and $\Sigma \operatorname{Hom}_R(K^R(x_1), R)$ as follows:

$$\begin{tikzcd}
K = K^R(x_1) = & 0 \arrow[r] & R \arrow[r, "x_1"] & R \arrow[r] & 0 & \\
K^* = \operatorname{Hom}_R(K, R) = & & 0 \arrow[r] & R \arrow[r, "x_1"] & R \arrow[r] & 0 \\
\Sigma K^* = \arrow[d, "\cong"'] & 0 \arrow[r] & R \arrow[r, "-x_1"] \arrow[d, "-1"'] & R \arrow[r] \arrow[d, "1"'] & 0 & \\
K = & 0 \arrow[r] & R \arrow[r, "x_1"] & R \arrow[r] & 0 &
\end{tikzcd}$$

Base Case: Let $n = 2$. Then we can directly write down an isomorphism between the complexes $K^R(x_1, x_2)$ and $\Sigma^2 \operatorname{Hom}_R(K^R(x_1, x_2), R)$ as follows:

$$\begin{tikzcd}
K = K^R(x_1, x_2) = & 0 \arrow[r] & R \arrow[r, "\left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)"] & R^2 \arrow[r, "\left(\begin{smallmatrix} x & y \end{smallmatrix}\right)"] & R \arrow[r] & 0 & \\
K^* = \operatorname{Hom}_R(K, R) = & & 0 \arrow[r] & R \arrow[r, "\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)"] & R^2 \arrow[r, "\left(\begin{smallmatrix} -y & x \end{smallmatrix}\right)"] & R \arrow[r] & 0 \\
\Sigma^2 K^* = \arrow[d, "\cong"'] & 0 \arrow[r] & R \arrow[r, "\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)"] \arrow[d, "-1"'] & R^2 \arrow[r, "\left(\begin{smallmatrix} -y & x \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)"'] & R \arrow[r] \arrow[d, "1"'] & 0 & \\
K = & 0 \arrow[r] & R \arrow[r, "\left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)"'] & R^2 \arrow[r, "\left(\begin{smallmatrix} x & y \end{smallmatrix}\right)"'] & R \arrow[r] & 0 &
\end{tikzcd}$$

For both of the above base cases, a diagram chase can check that the diagrams are commutative.

Inductive Case: Suppose that $\Phi \colon K' = K^R(x_2, \ldots, x_n) \to \Sigma^{n-1}(K')^*$ is an isomorphism. Notice that the following diagram commutes:

$$\begin{tikzcd}
K' \arrow[r, "x_1"] \arrow[d, "\cong"', "\Phi"] & K' \arrow[d, "\cong"', "\Phi"] \\
\Sigma^{n-1}(K')^* \arrow[r, "x_1"] & \Sigma^{n-1}(K')^*.
\end{tikzcd}$$

Then by using Exercise III.B.5.36 and the previous diagram, we have.

$$\begin{aligned}
K &= \operatorname{Cone}(\ K' \xrightarrow{x_1} K'\ ) \\
&\cong \operatorname{Cone}(\ \Sigma^{n-1}(K')^* \xrightarrow{x_1} \Sigma^{n-1}(K')^*\ )
\end{aligned}$$

By Lemma III.B.2.12, we then get

$$\begin{aligned}\operatorname{Cone}(\ \Sigma^{n-1}(K')^* \xrightarrow{x_1} \Sigma^{n-1}(K')^*\ ) &\cong \Sigma^{n-1}\operatorname{Cone}(\ (K')^* \xrightarrow{x_1} (K')^*\ )\\ &\cong \Sigma^{n-1}\left(\Sigma(\operatorname{Cone}(\ K' \xrightarrow{x_1} K'\ ))^*\right)\\ &\cong \Sigma^n K^*.\end{aligned}$$

An alternate proof is given later in the chapter (in Theorem III.B.2.17) which does not utilize Lemma III.B.2.12. □

lem190926i

Lemma III.B.2.12. *Let $\Psi\colon A \to C$ be a chain map.*

lem190926i.a (a) *Define $\Sigma\Psi\colon \Sigma A \to \Sigma C$, using the same rule as Ψ. Then $\Sigma\Psi$ is a chain map and*

$$\operatorname{Cone}(\Sigma\Psi) \cong \Sigma\operatorname{Cone}(\Psi).$$

Moreover, inductively applying the result for positive integers n gives

$$\operatorname{Cone}(\Sigma^n\Psi) \cong \Sigma^n\operatorname{Cone}(\Psi).$$

lem190926i.b (b) *The map $\Psi^*\colon C^* \to A^*$ where $(-)^* = \operatorname{Hom}_R(-,R)$ is a chain map and*

$$\operatorname{Cone}(\Psi^*) \cong \Sigma\operatorname{Cone}(\Psi)^*.$$

Sketch of Proof. Consider the following commutative diagram.

$$\begin{array}{ccccc} C^* = & & (C_{-n})^* & \xrightarrow{(\partial^C_{1-n})^*} & (C_{1-n})^* \\ \Big\downarrow{\scriptstyle \Psi^*} & & \Big\downarrow{\scriptstyle \Psi^*_{-n}} & & \Big\downarrow{\scriptstyle \Psi^*_{1-n}} \\ A^* = & & (A_{-n})^* & \xrightarrow{(\partial^A_{1-n})^*} & (A_{1-n})^* \end{array}$$

This shows that $\operatorname{Cone}(\Psi^*)_n = (C_{1-n})^* \oplus (A_{-n})^*$. A similar computation shows that this is $(\Sigma\operatorname{Cone}(\Psi)^*)_n$. In the interest of time, we omit the remaining details of the proof. □

The Exterior Basis for the Koszul Complex.

def191001a

Definition III.B.2.13. Let $\mathbf{x} = x_1, \ldots, x_n \in R$ be a weakly R-regular sequence in R. Then $L = L^R(\mathbf{x})$ is the following sequence of R-module homomorphisms, where we label each module with its homological degree.

$$\begin{array}{ccccccccccccccccc} 0 & \longrightarrow & R & \longrightarrow & R^n & \longrightarrow & \cdots & \longrightarrow & R^{\binom{n}{i}} & \longrightarrow & \cdots & \longrightarrow & R^n & \longrightarrow & R & \longrightarrow & 0 \\ & & n+1 & & n & & & & i & & & & 1 & & 0 & & \end{array}$$

A basis for $R^{\binom{n}{i}} = L_i$ is $e_\Lambda = e_{\lambda_1,\ldots,\lambda_i}$ where $\Lambda = \{\lambda_1, \ldots, \lambda_i\}$ and $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$. We may also write $\Lambda = \{\lambda_1 < \cdots < \lambda_i\} \subseteq [n] := \{1, \ldots, n\}$. The differentials of the sequence are given by

$$\partial^L_i(e_{\lambda_1,\ldots,\lambda_i}) = \sum_{j=1}^{i}(-1)^{j-1}x_j e_{\lambda_1,\ldots,\widehat{\lambda}_j,\ldots,\lambda_i} \in R^{\binom{n}{i-1}}$$

where e_Γ with $|\Gamma| = i-1$ is a basis vector in $R^{\binom{n}{i-1}}$ and $\lambda_1, \ldots, \widehat{\lambda}_j, \ldots, \lambda_i = \lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_i$. For instance,

$$\begin{aligned}\partial^L_2(e_{pq}) &= x_p e_q - x_q e_p,\\ \partial^L_3(e_{pqr}) &= x_p e_{qr} - x_q e_{pr} + x_r e_{pq}, \text{ and}\\ \partial^L_1(e_p) &= x_p e_\emptyset = x_p \cdot 1_R.\end{aligned}$$

thm191001b

Theorem III.B.2.14. *The sequence of R-module homomorphisms L is an R-complex and $L \cong K = K^R(\mathbf{x})$.*

ex191001c

EXAMPLE III.B.2.15. We give explicitly the R-complex L for sequences of sizes two and three before proving L is an R-complex.

$$L^R(x,y) = \qquad 0 \longrightarrow R \xrightarrow{\left(\begin{smallmatrix}-y\\ x\end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix}x & y\end{smallmatrix}\right)} R \longrightarrow 0$$

$$e_{12} \longmapsto xe_2 - ye_1$$

$$e_1 \longmapsto x$$

$$e_2 \longmapsto y$$

$$L^R(x,y,z) = \qquad 0 \longrightarrow R \xrightarrow{\left(\begin{smallmatrix}z\\ -y\\ x\end{smallmatrix}\right)} R^3 \xrightarrow{\left(\begin{smallmatrix}-y & -z & 0\\ x & 0 & -z\\ 0 & x & y\end{smallmatrix}\right)} R^3 \xrightarrow{\left(\begin{smallmatrix}x & y & z\end{smallmatrix}\right)} R \longrightarrow 0$$

$$e_{123} \longmapsto xe_{23} - ye_{13} + ze_{12}$$

$$e_1 \longmapsto x$$

$$e_2 \longmapsto y$$

$$e_3 \longmapsto z$$

$$e_{12} \longmapsto xe_2 - ye_1$$

$$e_{13} \longmapsto xe_3 - ze_1$$

$$e_{23} \longmapsto ye_3 - ze_2$$

PROOF OF THEOREM III.B.2.14. To prove L is and R-complex, it suffices to fix an arbitrary i and show that we have $\left(\partial^L_{i-1} \circ \partial^L_i\right)(e_\Lambda) = 0$ for all subsets Λ satisfying $|\Lambda| = i$. To see the argument more concretely, we first observe

$$\begin{aligned}\left(\partial^L_2 \circ \partial^L_3\right)(e_{pqr}) &= \partial^L_2\left(x_pe_{qr} - x_qe_{pr} + x_re_{pq}\right)\\ &= x_p\partial^L_2(e_{qr}) - x_q\partial^L_2(e_{pr}) + x_r\partial^L_2(e_{pq})\\ &= x_p(x_qe_r - x_re_q) - x_q(x_pe_r - x_re_p) + x_r(x_pe_q - x_qe_p)\\ &= 0.\end{aligned}$$

In general we have

$$\begin{aligned}\left(\partial^L_{i-1} \circ \partial^L_i\right)(e_\Lambda) &= \partial^L_{i-1}\left(\sum_{j=1}^{i}(-1)^{j-1}x_{\lambda_j}e_{\lambda_1,\dots,\widehat{\lambda}_j,\dots,\lambda_i}\right)\\ &= \sum_{j=1}^{i}(-1)^{j-1}x_{\lambda_j}\partial^L_{i-1}(e_{\lambda_1,\dots,\widehat{\lambda}_j,\dots,\lambda_i})\\ &= \sum_{j=1}^{i}(-1)^{j-1}x_{\lambda_j}\left[\left(\sum_{\ell=1}^{j-1}(-1)^{\ell-1}x_{\lambda_\ell}e_{\lambda_1,\dots,\widehat{\lambda}_\ell,\dots,\widehat{\lambda}_j,\dots,\lambda_i}\right) + \left(\sum_{\ell=j+1}^{i}(-1)^{\ell-2}x_{\lambda_\ell}e_{\lambda_1,\dots,\widehat{\lambda}_j,\dots,\widehat{\lambda}_\ell,\dots,\lambda_i}\right)\right]\\ &= 0\end{aligned}$$

The last equality holds since every basis vector $e_{\lambda_1,\dots,\widehat{\lambda}_p,\dots,\widehat{\lambda}_q,\dots,\lambda_i}$ occurs twice in the sum and of opposite signs. In the case when λ_p is removed first, the coefficient is $(-1)^{p-1+q-2}x_{\lambda_p}x_{\lambda_q}$, and in the case when λ_q is removed first, the coefficient is $(-1)^{q-1+p-1}x_{\lambda_q}x_{\lambda_p}$.

We prove $L \cong K$ by induction on n. The base cases $n = 2, 3$ are done by Example III.B.2.15 and the case $n = 1$ is routine. For the inductive step set $\mathbf{x}' = x_2, \dots, x_n$, and $K' = K^R(\mathbf{x}')$ and $L' = L^R(\mathbf{x}')$. By the induction hypothesis $L' \cong K'$ and we let $\Psi\colon L' \to K'$ be an isomorphism, which automatically makes the following diagram commute.

$$\begin{tikzcd} L' \arrow[r, "\Psi", "\cong"'] \arrow[d, "x_1"'] & K' \arrow[d, "x_1"] \\ L' \arrow[r, "\cong", "\Psi"'] & K' \end{tikzcd}$$

By Exercise III.B.5.36(b) this diagram shows that $K = \mathrm{Cone}(K' \xrightarrow{x_1} K') \cong \mathrm{Cone}(L' \xrightarrow{x_1} L')$. It therefore suffices to show $L \cong \mathrm{Cone}(L' \xrightarrow{x_1} L')$. We claim the chain map ϕ in the diagram

$$\begin{tikzcd} \mathrm{Cone}(L' \xrightarrow{x_1} L')_i & = & \begin{matrix} L'_{i-1} \\ \oplus \\ L'_i \end{matrix} \arrow[r, "\begin{pmatrix} -\partial^{L'}_{i-1} & 0 \\ x_1 & \partial^{L'}_i \end{pmatrix}"] \arrow[d, "\phi_i"'] & \begin{matrix} L'_{i-2} \\ \oplus \\ L'_{i-1} \end{matrix} \arrow[d, "\phi_{i-1}"] & = & \mathrm{Cone}(L' \xrightarrow{x_1} L')_{i-1} \\ & & L_i \arrow[r] & L_{i-1} & & \end{tikzcd}$$

is the desired isomorphism, where ϕ_i is defined on basis vectors as follows. Basis vectors for $L'_{i-1} \oplus L'_i$ are of the form $\begin{pmatrix} 0 & e_{\lambda_1,\ldots,\lambda_i} \end{pmatrix}^T$ and $\begin{pmatrix} e_{\gamma_2,\ldots,\gamma_i} & 0 \end{pmatrix}^T$ where $2 \leq \lambda_1 < \cdots < \lambda_i \leq n$ and $2 \leq \gamma_2 < \cdots < \gamma_i \leq n$. We bound below by 2 since $L' = L^R(\mathbf{x}')$ and we begin our index for $\Gamma = \{\gamma_2 < \cdots < \gamma_i\}$ with 2 since L'_{i-1} has basis vectors of size $i-1$. It suffices to show the diagram above commutes with respect to these basis vectors. Define ϕ by the following:

$$\phi_i \begin{pmatrix} 0 \\ e_{\lambda_1,\ldots,\lambda_i} \end{pmatrix} = e_{\lambda_1,\ldots,\lambda_i} \qquad \text{and} \qquad \phi_i \begin{pmatrix} e_{\gamma_2,\ldots,\gamma_i} \\ 0 \end{pmatrix} = e_{1,\gamma_2,\ldots,\gamma_i}.$$

We compute

$$\begin{aligned} \phi_{i-1}\left(\partial_i^{\mathrm{Cone}(x_1)} \begin{pmatrix} 0 \\ e_{\lambda_1,\ldots,\lambda_i} \end{pmatrix}\right) &= \phi_{i-1}\begin{pmatrix} 0 \\ \partial_i^{L'}(e_{\lambda_1,\ldots,\lambda_i}) \end{pmatrix} \\ &= \phi_{i-1}\begin{pmatrix} 0 \\ \sum_{j=1}^i (-1)^{j-1} x_{\lambda_j} e_{\lambda_1,\ldots,\widehat{\lambda}_j,\ldots,\lambda_i}) \end{pmatrix} \\ &= \sum_{j=1}^i (-1)^{j-1} x_{\lambda_j} \phi_{i-1}\begin{pmatrix} 0 \\ e_{\lambda_1,\ldots,\widehat{\lambda}_j,\ldots,\lambda_i} \end{pmatrix} \\ &= \sum_{j=2}^i (-1)^{j-1} x_{\lambda_j} e_{\lambda_1,\ldots,\widehat{\lambda}_j,\ldots,\lambda_i} \\ &= \partial_i^L(e_{\lambda_1,\ldots,\lambda_i}) \\ &= \partial_i^L\left(\phi_i\begin{pmatrix} 0 \\ e_{\lambda_1,\ldots,\lambda_i} \end{pmatrix}\right) \end{aligned}$$

and similarly

$$\begin{aligned} \phi_{i-1}\left(\partial_i^{\mathrm{Cone}(x_1)} \begin{pmatrix} e_{\gamma_2,\ldots,\gamma_i} \\ 0 \end{pmatrix}\right) &= \phi_{i-1}\begin{pmatrix} -\partial^{L'}_{i-1}(e_{\gamma_2,\ldots,\gamma_i}) \\ x_1 e_{\gamma_2,\ldots,\gamma_i} \end{pmatrix} \\ &= \phi_{i-1}\begin{pmatrix} -\sum_{j=2}^i (-1)^j x_{\gamma_j} e_{\gamma_2,\ldots,\widehat{\gamma}_j,\ldots,\gamma_i} \\ x_1 e_{\gamma_2,\ldots,\gamma_i} \end{pmatrix} \\ &= \sum_{j=2}^i (-1)^{j-1} x_{\gamma_j} e_{1,\gamma_2,\ldots,\widehat{\gamma}_j,\ldots,\gamma_i} + x_1 e_{\gamma_2,\ldots,\gamma_i} \\ &= \partial_i^{\mathrm{Cone}(x_1)}(e_{1,\gamma_2,\ldots,\gamma_i}) \\ &= \partial_i^{\mathrm{Cone}(x_1)}\left(\phi_i\begin{pmatrix} e_{\gamma_2,\ldots,\gamma_i} \\ 0 \end{pmatrix}\right), \end{aligned}$$

so ϕ is a chain map and by construction it bijectively maps basis vectors to basis vectors. (The basis vector assignment is 1-1, so it is bijective by the pigeon hole principle.) Therefore ϕ_i is bijective for all $i \in \mathbb{Z}$ and ϕ is therefore an isomorphism of R-complexes. $\square$

fact191001d

FACT III.B.2.16. *If A is an R-complex, then $\Sigma^n A$ is isomorphic to*

$$\cdots \xrightarrow{-\partial^A_{i-n+1}} A_{i-n} \xrightarrow{-\partial^A_{i-n}} A_{i-n-1} \xrightarrow{-\partial^A_{i-n-1}} \cdots .$$

The following result was already presented as Theorem III.B.2.11. We give it again for convenience and present another proof in which we use the exterior basis for K and the dual basis for K^*.

thm191001e

THEOREM III.B.2.17 (Self-duality of the Koszul complex). *Let $\mathbf{x} = x_1, \ldots, x_n \in R$ be given, and set $K = K^R(\mathbf{x})$ and $K^* = \operatorname{Hom}_R(K, R)$. Then $\Sigma^n K^* \cong K$.*

ALTERNATE PROOF. Recall that

$$\begin{array}{ccc} \operatorname{Hom}_R(R^t, R) & \cong & R^t \\ e_1^* & & e_1 \\ \vdots & \longleftarrow\!\shortmid & \vdots \\ e_t^* & & e_t \end{array}$$

with $e_i^*(e_j) = \delta_{ij}$ and $e_i^*(\sum_j \alpha_j e_j) = \alpha_i$. Then we have $(K^*)_{-i} = \left(R^{\binom{n}{i}}\right)^* \cong R^{\binom{n}{i}}$ with dual basis e_Λ^* and

$$e_\Lambda^*(e_\Gamma) = \begin{cases} 0 & \text{if } \Lambda \neq \Gamma \\ 1 & \text{if } \Lambda = \Gamma, \end{cases}$$

where $|\Lambda| = i = |\Gamma|$. Note also that

$$\partial^{K^*}_{-i}(e_\Lambda^*) = \left(\partial^K_{i+1}\right)^* (e_\Lambda^*) = e_\Lambda^* \circ \partial^K_{i+1} \qquad \text{(III.B.2.17.1)}$$ eqn191001b

for $\Lambda \subseteq [n]$ satisfying $|\Lambda| = i$. Let $\gamma \in [n] \setminus \Lambda$, and define $s(\gamma, \Lambda)$ to be the number of swaps needed to put the list $\gamma, \lambda_1, \ldots, \lambda_i$ in order. In other words, if $\lambda_{j-1} < \gamma < \lambda_j$, then $s(\gamma, \Lambda) = j - 1$.

For example, if $n = 9$ and $\Lambda = \{3 < 5 < 7\}$ and $\gamma = 4$, then $s(\gamma, \Lambda) = 1$.

Continuing the proof, we claim that $\partial^{K^*}_{-i}(e_\Lambda^*) = \sum_{\gamma \in [n]\setminus\Lambda} (-1)^{s(\gamma,\Lambda)} x_\gamma e^*_{\Lambda \cup \{\gamma\}}$ in $K^*_{i+1} = \operatorname{Hom}_R(K_{i+1}, R)$ with basis e_Γ^* for $|\Gamma| = i + 1$. It suffices to show that

$$\partial^{K^*}_{-i}(e_\Lambda^*)(e_\Gamma) = \sum_{\gamma \in [n]\setminus\Lambda} (-1)^{s(\gamma,\Lambda)} x_\gamma e^*_{\Lambda \cup \{\gamma\}}(e_\Gamma). \qquad \text{(III.B.2.17.2)}$$ eqn191003a

First consider the left hand side. By display (III.B.2.17.1) for $\Gamma = \{\gamma_1 < \cdots < \gamma_{i+1}\}$, we have

$$\begin{aligned} \partial^{K^*}_{-i}(e_\Lambda^*)(e_\Gamma) &= e_\Lambda^*(\partial^K_{i+1}(e_\Gamma)) \\ &= e_\Lambda^* \left(\sum_{j=1}^{i+1} (-1)^{j-1} x_{\gamma_j} e_{\Gamma \setminus \{\gamma_j\}} \right) \\ &= \sum_{j=1}^{i+1} (-1)^{j-1} x_{\gamma_j} e_\Lambda^*(e_{\Gamma \setminus \{\gamma_j\}}) \\ &= \begin{cases} 0 & \text{if } \Lambda \not\subseteq \Gamma \\ (-1)^{j-1} x_{\gamma_j} & \text{if } \Gamma \setminus \{\gamma_j\} = \Lambda \text{ and } \Lambda \subseteq \Gamma \end{cases}. \end{aligned}$$

Next consider the right hand side of display (III.B.2.17.2). If $\Lambda \not\subseteq \Gamma$, then $\Lambda \cup \{\gamma\} \neq \Gamma$ for all $\gamma \in [n] \setminus \Lambda$, so $e^*_{\Lambda \cup \{\gamma\}}(e_\Gamma) = 0$. Therefore, the right hand side of (III.B.2.17.2) is 0 as well. If $\Lambda \subseteq \Gamma$, then there is a unique $\gamma_j \in [n] \setminus \Lambda$ such that $\Lambda \cup \{\gamma_j\} = \Gamma$. Then the right hand side of (III.B.2.17.2) is equal to $(-1)^{s(\gamma_j,\Lambda)} x_{\gamma_j}$. Notice that $\Lambda = \{\gamma_1 < \cdots < \gamma_{j-1} < \gamma_{j+1} < \cdots\}$, so $s(\gamma_j, \Lambda) = j - 1$.

Now we want to show that following diagram commutes:

$$\begin{tikzcd}
(K^*)_{-i} = (\Sigma^n K^*)_{n-i} \arrow[r, "\partial^{\Sigma^n K^*}_{n-i}", "-\partial^{K^*}_{-i}"'] \arrow[d, "\Phi_{n-i}"] & (\Sigma^n K^*)_{n-i-1} = (K^*)_{-(i+1)} \arrow[d, "\Phi_{n-i-1}"] \\
K_{n-i} \arrow[r, "\partial^K_{n-i}"] & K_{n-i-1}.
\end{tikzcd}$$

A diagram chase of the above diagram follows:

$$\begin{tikzcd}
e^*_\Lambda \arrow[r, mapsto] \arrow[dd, mapsto] & -\sum_{\gamma\in[n]\setminus\Lambda} (-1)^{s(\gamma,\Lambda)} x_\gamma e^*_{\Lambda\cup\{\gamma\}} \arrow[d, mapsto] \\
& -\sum_{\gamma\in[n]\setminus\Lambda} (-1)^{s(\gamma,\Lambda)+\sum_{\lambda\in\Lambda\setminus\{\gamma\}}\lambda} x_\gamma e_{[n]\setminus(\Lambda\cup\{\gamma\})} \\
(-1)^{\sum_{\lambda\in\Lambda}\lambda} e_{[n]\setminus\Lambda} \arrow[r, mapsto] & (-1)^{\sum_{\lambda\in\Lambda}\lambda} \sum_{\gamma\in[n]\setminus\Lambda} (-1)^{t(\gamma,\Lambda)-1} x_\gamma e_{([n]\setminus\Lambda)\setminus\{\gamma\}}
\end{tikzcd}$$

Here, $t(\gamma,\Lambda)$ is the position of γ in $[n]\setminus\Lambda$. Observe that $s(\gamma,\Lambda)+t(\gamma,\Lambda)=\gamma$. Consider the example from earlier in the proof. For instance, if $n=9$ and $\Lambda=\{3<5<7\}$ and $\gamma=4$, then we have $t(\gamma,\Lambda)=3$ because $[n]\setminus\Lambda=\{1<2<4<6<8<9\}$. Then we can see that $s(\gamma,\Lambda)+t(\gamma,\Lambda)=4=\gamma$.

Now we need to check that the two lines in the bottom right corner of the above diagram are equal. It suffices to show that the powers of (-1) are congruent modulo 2:

$$\begin{aligned}
s(\gamma,\Lambda)+\sum_{\lambda\in\Lambda\setminus\{\gamma\}}\lambda+1 &\overset{?}{\equiv} t(\gamma,\Lambda)-1+\sum_{\lambda\in\Lambda}\lambda \pmod 2\\
s(\gamma,\Lambda)+t(\gamma,\Lambda) &\overset{?}{\equiv} \sum_{\lambda\in\Lambda}\lambda-\sum_{\lambda\in\Lambda\setminus\{\gamma\}}\lambda \pmod 2\\
\gamma &\overset{\checkmark}{\equiv} \gamma \pmod 2.
\end{aligned}$$

Therefore, this is a chain map. To determine whether Φ is an isomorphism, it suffices to show that Φ is one-to-one and onto. Since Φ maps the dual basis of $\Sigma^n K^*$ to the exterior basis of K and the bases are the same size, Φ describes a basis bijection. Therefore, the induced map is an isomorphism. $\square$

III.B.3. Application: Depth Sensitivity of the Koszul Complex

ection063021i

In this chapter, assume that R is noetherian.

rmk191003a REMARK III.B.3.1. If $\mathbf{x}=x_1,\ldots,x_n\in R$ is weakly R-regular, then $K=K^R(\mathbf{x})$ is a resolution of $R/\langle\mathbf{x}\rangle$. Therefore, $H_i(K)=0$ for all $i\neq 0$ and $H_0(K)\cong R/(\mathbf{x})$.

The question that comes up now is what happens when $\mathbf{x}$ is not weakly R-regular? In other words, which homology modules still vanish when $\mathbf{x}$ in not weakly R-regular? The answer is not immediately obvious, but we will see that $H_i(K^R(x))$ may be non-zero for some $i>0$. For example, consider the Koszul complex on one element:

$$K^R(x) = \qquad 0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0.$$

Then $H_1(K^R(x))\cong \operatorname{Ker}\left(R \xrightarrow{x} R\right)=0$ if and only if x is a non-zero-divisor on R. This suggests that there is a connection between the existence of weakly R-regular sequences and vanishing of homology modules of $K^R(x)$. As another example, we consider the Koszul complex on two elements below.

ex191003b EXAMPLE III.B.3.2. Let $x,y\in R$ and set $K=K^R(x,y)$ and $K'=K^R(y)$. Assume that y is a non-zero-divisor on R, then $H_i(K')=0$ for all $i>0$. We consider what happens to the homology modules of K if x is a zero-divisor on $R/\langle y\rangle$. We have the following short exact sequence by Theorem III.A.4.20:

$$0 \longrightarrow K' \longrightarrow K \longrightarrow \Sigma K' \longrightarrow 0.$$

We consider the long exact sequence of homology modules that arises from the above short exact sequence. In particular, we consider the case for $i \geq 2$ and separately the case for $i = 1$. For $i \geq 2$, we have

$$\cdots \longrightarrow \underbrace{H_i(K')}_{=0} \longrightarrow H_i(K) \longrightarrow \underbrace{H_{i-1}(K')}_{=0} \xrightarrow{x} H_{i-1}(K') \longrightarrow \cdots .$$

Therefore, by Fact III.A.2.2c, we have $H_i(K) = 0$. For $i = 1$, we have

$$\begin{tikzcd}
\cdots \arrow[r] & \underbrace{H_1(K')}_{=0} \arrow[r] & H_1(K) \arrow[r] & H_0(K') \arrow[r, "x"] \arrow[d, "\cong"'] & H_0(K') \arrow[r] \arrow[d, "\cong"'] & \cdots \\
& & & R/\langle y\rangle \arrow[r, "x", "\text{not 1-1}"'] & R/\langle y\rangle .
\end{tikzcd}$$

Then $H_1(K) \cong \operatorname{Ker}\left(R/\langle y\rangle \xrightarrow{x} R/\langle y\rangle\right) \neq 0$, since x is a zero-divisor on $R/\langle y\rangle$.

rmk191003c

Remark III.B.3.3. The point of this chapter is that we can say exactly which $s \in \mathbb{N}$ satisfy $H_i(K) = 0$ for all $i > s$. The method of doing so is in terms of "depth". Conversely, if we know which s satisfies $H_i(K) = 0$ for all $i > s$, then we can calculate the depth.

To define depth, we need the following theorem which we state without proof.

thm191003d

Theorem III.B.3.4 (Rees). *Let $I \lneq R$.*

thm191003d.a (a) *The following are equivalent:*
  (i) *There exists a sequence $\mathbf{y} = y_1, \ldots, y_m \in I$ that is weakly R-regular, and*
  (ii) $\operatorname{Ext}^i_R(R/I, R) = 0$ *for all $i < m$.*

thm191003d.b (b) *There exists a maximal weakly R-regular sequence in I. In other words, there exists a weakly R-regular sequence $\mathbf{y} = y_1, \ldots, y_m \in I$ such that for all $x \in I$, the sequence $y_1, \ldots, y_m, x$ is not weakly R-regular.*

thm191003d.c (c) *Every maximal weakly R-regular sequence has the same length and the length is*

$$m = \min\left\{ i \geq 0 \;\middle|\; \operatorname{Ext}^i_R(R/I, R) \neq 0 \right\}.$$

defn191003e

Definition III.B.3.5. The <u>depth of I in R</u>, denoted $\operatorname{depth}(I, R)$, is the length of any maximal weakly R-regular sequence in $I \lneq R$. When the ring and the ideal are understood, we will denote the depth as δ.

ex191003f

Example III.B.3.6. We verify the conclusion of Theorem III.B.3.4(c) in the special case of $R = A[X_1, \ldots, X_d]$, where A is a nonzero noetherian commutative ring with identity, and $I = \langle X_1, \ldots, X_n\rangle$. Notice that I is weakly R-regular. Then $K = K^R(X_1, \ldots, X_n)$ is a free resolution of R/I. Therefore

$$\operatorname{Ext}^i_R(R/I, R) = H_{-i}(K^*) = H_{n-i}(\Sigma^n K^*) \cong H_{n-i}(K) = \begin{cases} 0 & \text{if } n > i \\ R/I & \text{if } n = i. \end{cases}$$

The isomorphism above comes from the self-duality property of K. Then we can see that

$$\min\left\{ i \geq 0 \;\middle|\; \operatorname{Ext}^i_R(R/I, R) \neq 0 \right\} = n,$$

which is also the length of the maximal weakly R-regular sequence $X_1, \ldots, X_n \in I$.

rmk191008a

Remark III.B.3.7. One can see δ visually as follows:

$$\underbrace{\operatorname{Ext}^0_R(R/I, R), \operatorname{Ext}^1_R(R/I, R), \ldots, \operatorname{Ext}^{\delta-1}_R(R/I, R)}_{=0}, \underbrace{\operatorname{Ext}^{\delta}_R(R/I, R)}_{\neq 0}.$$

So one may think of δ as the number of initial vanishing Ext modules.

thm191008b

Theorem III.B.3.8 (Auslander-Buchsbaum). *Let $\mathbf{x} = x_1, \ldots, x_n \in I$ such that $I = \langle \mathbf{x}\rangle \lneq R$ and assume $\mathbf{y} = y_1, \ldots, y_m \in I$ is weakly R-regular. Then $H_i\left(K^R(\mathbf{x})\right) = 0$ for all $i > n - m$ and*

$$H_{n-m}\left(K^R(\mathbf{x})\right) \cong \operatorname{Ext}^m_R(R/I, R).$$

cor191008c

Corollary III.B.3.9 (Depth-sensitivity of the Koszul complex). *In the context of Theorem III.B.3.8 we have $H_i\left(K^R(\mathbf{x})\right) = 0$ for all $i > n - \delta$ and $H_{n-\delta}\left(K^R(\mathbf{x})\right) \neq 0$, i.e.,*

$$\delta = n - \max\left\{ i \geq 0 \;\middle|\; H_i\left(K^R(\mathbf{x})\right) \neq 0 \right\}.$$

rmk191008d

REMARK III.B.3.10. One can once again see δ visually as follows:

$$\underbrace{H_n\left(K^R(\mathbf{x})\right), H_{n-1}\left(K^R(\mathbf{x})\right), \ldots, H_{n-\delta+1}\left(K^R(\mathbf{x})\right)}_{=0}, \underbrace{H_{n-\delta}\left(K^R(\mathbf{x})\right)}_{\neq 0}.$$

So one can think of δ as the number of "initial" vanishings of $H_i\left(K^R(\mathbf{x})\right)$ when counting from degree n.

ex191008e

EXAMPLE III.B.3.11. With this example we verify the conclusions of Theorem III.B.3.4 and Corollary III.B.3.9 for the ideal $I = \langle XY, XZ, YZ\rangle \lneq R$ where $R = k[X,Y,Z]$, i.e., we will show

$$\begin{aligned}&\delta = \operatorname{depth}(I,R) = 2,\\ &\operatorname{Ext}^0_R(R/I,R) = \operatorname{Ext}^1_R(R/I,R) = 0 \text{ and } \operatorname{Ext}^2_R(R/I,R) \neq 0, \text{ and}\\ &H_3(K) = H_2(K) = 0 \text{ and } H_1(K) \neq 0\end{aligned}$$

where we set $K = K^R(XY, XZ, YZ)$.

To show that $\delta = 2$ we build a weakly regular sequence. Begin with the non-zero-divisor $XY \in I$. Then

$$\frac{R}{\langle XY\rangle} \cong \frac{k[X,Y]}{\langle XY\rangle}[Z]$$

and since $Z, X+Y \in I$ are each non-zero-divisors in the above quotient and I is an ideal, we know $XZ+YZ = (X+Y)Z \in I$ is also a non-zero-divisor. One can then check that $XY, XZ+YZ$ is a maximal weakly R-regular sequence in I, so $\delta = 2$.

To compute Ext modules we first need a projective resolution of R/I:

$$P = \qquad 0 \longrightarrow R^2 \xrightarrow[\partial_2^P]{\left(\begin{smallmatrix}-Z & -Z\\ Y & 0\\ 0 & X\end{smallmatrix}\right)} R^3 \xrightarrow[\partial_1^P]{(XY\ XZ\ YZ)} R \longrightarrow 0.$$

Taking the dual we obtain

$$P^* \cong \qquad \underset{}{0} \longrightarrow \underset{0}{R} \xrightarrow[(\partial_1^P)^*]{\left(\begin{smallmatrix}XY\\ XZ\\ YZ\end{smallmatrix}\right)} \underset{-1}{R^3} \xrightarrow[(\partial_2^P)^*]{\left(\begin{smallmatrix}-Z & Y & 0\\ -Z & 0 & X\end{smallmatrix}\right)} \underset{-2}{R^2} \longrightarrow 0.$$

Since each entry in the matrix defining $\left(\partial_1^P\right)^*$ is a non-zero-divisor on R, we have

$$\operatorname{Ext}^0_R(R/I,R) \cong \operatorname{Ker} \begin{matrix}XY\\ XZ\\ YZ\end{matrix} = 0$$

so P^* is exact in degree 0. Alternatively, one can prove this using the isomorphisms

$$\operatorname{Ext}^0_R(R/I,R) \cong \operatorname{Hom}_R(R/I,R) \overset{(1)}{\cong} \{r \in R \mid rI = 0\} \overset{(2)}{=} 0,$$

where (2) holds since I contains a non-zero-divisor and the isomorphism in (1) is the evaluation map $\phi \mapsto \phi(1)$.

In degree -2 we observe that

$$\operatorname{Ext}^2_R(R/I,R) = \frac{R^2}{\left\langle \left(\begin{smallmatrix}-Z\\ -Z\end{smallmatrix}\right), \left(\begin{smallmatrix}Y\\ 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\ X\end{smallmatrix}\right)\right\rangle}.$$

We also note that

$$\left\langle \begin{pmatrix}-Z\\ -Z\end{pmatrix}, \begin{pmatrix}Y\\ 0\end{pmatrix}, \begin{pmatrix}0\\ X\end{pmatrix}\right\rangle \subseteq \mathfrak{m} \oplus \mathfrak{m} \subseteq R \oplus R = R^2$$

where $\mathfrak{m} = \langle X,Y,Z\rangle \lneq R$ is a maximal ideal. Thus there exists a surjection

$$\frac{R^2}{\left\langle \left(\begin{smallmatrix}-Z\\ -Z\end{smallmatrix}\right), \left(\begin{smallmatrix}Y\\ 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\ X\end{smallmatrix}\right)\right\rangle} \twoheadrightarrow \frac{R \oplus R}{\mathfrak{m} \oplus \mathfrak{m}} \cong \left(\frac{R}{\mathfrak{m}}\right)^2 \cong k^2 \neq 0$$

where the inequality holds because k is a field. Hence, P^* is not exact in this degree.

To show P^* is exact in degree -1 it suffices to show that $\operatorname{Ker}\left(\partial_2^P\right)^* \subseteq \operatorname{Im}\left(\partial_1^P\right)^*$. We observe that if $\begin{pmatrix}f & g & h\end{pmatrix}^T \in \operatorname{Ker}\left(\partial_2^P\right)^*$, then

$$Yg = Zf = Xh \qquad \text{(III.B.3.11.1)}$$ eqn191008a

which implies there exist $f_1, g_1, h_1 \in R$ such that

$$f = XYf_1,\ g = XZg_1,\ \text{and } h = YZh_1.$$

Substituting this into (III.B.3.11.1) we obtain

$$XYZg_1 = XYZf_1 = XYZh_1$$

which implies $f_1 = g_1 = h_1$. Hence

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = f_1 \begin{pmatrix} XY \\ XZ \\ YZ \end{pmatrix} \in \operatorname{Im}\left(\partial_1^P\right)^*$$

so P^* is exact in degree -1.

We now study the following Koszul complex.

$$0 \longrightarrow \underset{3}{R} \xrightarrow[\partial_3^K]{\left(\begin{smallmatrix} YZ \\ -XY \\ XY \end{smallmatrix}\right)} \underset{2}{R^3} \xrightarrow[\partial_2^K]{\left(\begin{smallmatrix} -XZ & -YZ & 0 \\ XY & 0 & -YZ \\ 0 & XY & XZ \end{smallmatrix}\right)} \underset{1}{R^3} \xrightarrow[\partial_1^K]{\left(\begin{smallmatrix} XY & XZ & YZ \end{smallmatrix}\right)} R \longrightarrow 0$$

For exactness in degree 3, argue as in the computation of $\operatorname{Ext}_R^0(R/I, R)$. To prove exactness in degree 2 it suffices to show $\operatorname{Ker}\partial_2^K \subseteq \operatorname{Im}\partial_3^K$ and this argument is analogous to the argument used to compute $\operatorname{Ext}_R^1(R/I, R)$.

To see that $H_1(K)$ is non-zero, we observe

$$\begin{aligned} \operatorname{Ker}\partial_1^K &= \operatorname{Ker} \begin{matrix} XY & XZ & YZ \end{matrix} \\ &= \operatorname{Im}\partial_2^P \\ &= \left\langle \begin{pmatrix} -Z \\ Y \\ 0 \end{pmatrix}, \begin{pmatrix} -Z \\ 0 \\ X \end{pmatrix} \right\rangle \\ &\supsetneq \left\langle \begin{pmatrix} -XZ \\ XY \\ 0 \end{pmatrix}, \begin{pmatrix} -YZ \\ 0 \\ XY \end{pmatrix}, \begin{pmatrix} 0 \\ -YZ \\ XZ \end{pmatrix} \right\rangle \\ &= \operatorname{Im}\partial_2^K. \end{aligned}$$

ex191008f EXAMPLE III.B.3.12. Set $R = k[X, Y]/\langle XY \rangle$ and define $I = \langle x, y \rangle \lneq R$ where $x = \overline{X} \in R$ and $y = \overline{Y} \in R$. We again will verify the conclusions of Theorem III.B.3.4 and Corollary III.B.3.9. One can conclude $\delta = 1$ by verifying that $x + y$ is a maximal weakly R-regular sequence in I. It remains to show that

$$\operatorname{Ext}_R^0(R/I, R) = 0 \text{ and } \operatorname{Ext}_R^1(R/I, R) \neq 0,$$

and

$$H_2(K) = 0 \text{ and } H_1(K) \neq 0,$$

where we set $K = K^R(x, y)$. Check that a (truncated) projective resolution of R/I is

$$P = \qquad \cdots \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & y \end{smallmatrix}\right)} R \longrightarrow 0$$

and applying $\operatorname{Hom}_R(-, R)$ we obtain

$$P^* = \qquad 0 \longrightarrow \underset{0}{R} \xrightarrow[\left(\partial_1^P\right)^*]{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)} \underset{-1}{R^2} \xrightarrow[\left(\partial_2^P\right)^*]{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)} \cdots.$$

Compare this with P to conclude that P^* is exact in every degree below -1. Since $x + y \in I$ is a non-zero-divisor we have

$$\operatorname{Ext}_R^0(R/I, R) = \operatorname{Hom}_R(R/I, R) = 0.$$

In degree 1 we have

$$\operatorname{Ker}\left(\partial_2^P\right)^* = \operatorname{Ker} \begin{matrix} y & 0 \\ 0 & x \end{matrix} = \operatorname{Im} \begin{matrix} x & 0 \\ 0 & y \end{matrix} = \left\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle \supsetneq \left\langle \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \operatorname{Im}\left(\partial_1^P\right)^*,$$

so $\operatorname{Ext}^1_R(R/I,R) \neq 0$.

The Koszul complex $K = K^R(x,y)$ is familiar.

$$0 \longrightarrow \underset{2}{R} \xrightarrow[\partial_2^K]{\left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)} \underset{1}{R^2} \xrightarrow[\partial_1^K]{\left(\begin{smallmatrix} x & y \end{smallmatrix}\right)} \underset{0}{R} \longrightarrow 0$$

One can show $H_2(K) = 0$ by showing that $\operatorname{Ker}\partial_2^K = 0$ as we did in the computation of $\operatorname{Ext}^0_R(R/I,R)$. We see that $H_1(K) \neq 0$ since

$$\operatorname{Ker}\partial_1^K = \operatorname{Im}\begin{matrix} y & 0 \\ 0 & x \end{matrix} = \left\langle \begin{pmatrix} y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix} \right\rangle \supsetneq \left\langle \begin{pmatrix} -y \\ x \end{pmatrix} \right\rangle = \operatorname{Im}\partial_2^K.$$

Sketch of Proof of Theorem III.B.3.8. Let $K = K^R(\mathbf{x})$. To prove the result, we would induct on m. In the interest of time, we will only show the base cases for $m = 0$ and $m = 1$; the inductive step follows like the $m = 1$ case.

First consider the base case for $m = 0$. Then we have

$$\begin{aligned} H_n(K) &\cong \operatorname{Ker}\ R \xrightarrow[\partial_n^K]{\begin{pmatrix} \pm x_n \\ \vdots \\ -x_2 \\ x_1 \end{pmatrix}} R^n \\ &= \{ r \in R \mid x_i r = 0 \ \forall\, i = 1, \ldots, n \} \\ &= \{ r \in R \mid Ir = 0 \} \\ &\cong \operatorname{Hom}_R(R/I,R) \\ &\cong \operatorname{Ext}^0_R(R/I,R). \end{aligned}$$

The isomorphism from $\operatorname{Hom}_R(R/I,R)$ to $\{r \in R \mid Ir = 0\}$ is given by sending $\phi \in \operatorname{Hom}_R(R/I,R)$ to $\phi(\overline{1})$. This gives us our result for $m = 0$.

Next consider the base case for $m = 1$. Notice that $H_n(K) \cong \operatorname{Hom}_R(R/I,R)$ by the $m = 0$ case. (In the inductive step, this observation would be replaced by the inductive hypothesis.) By Theorem III.B.3.4(a) and since $m = 1$, we get $\operatorname{Hom}_R(R/I,R) \cong \operatorname{Ext}^0_R(R/I,R) = 0$. Now consider the following short exact sequence with $\overline{R} = R/\langle y_1 \rangle$:

$$0 \longrightarrow R \xrightarrow{y_1} R \longrightarrow \overline{R} \longrightarrow 0. \qquad \text{(III.B.3.12.1)}$$ eqn191010a

This induces the following short exact sequence on Koszul complexes, with $\overline{K} = K^{\overline{R}}(\overline{\mathbf{x}})$:

$$0 \longrightarrow K \xrightarrow{y_1} K \longrightarrow \overline{K} \longrightarrow 0.$$

By Theorem III.A.4.16, this induces the following long exact sequence of homologies:

$$\begin{tikzcd} 0 \arrow[r] & \underbrace{H_n(K)}_{=0} \arrow[r, "y_1"] & \underbrace{H_n(K)}_{=0} \arrow[r] & H_n(\overline{K}) \arrow[dll, rounded corners] \\ & H_{n-1}(K) \arrow[r, "y_1 = 0"] & H_{n-1}(K) \arrow[r] & \cdots . \end{tikzcd}$$

By Exercise III.B.5.40, the assumption $y_1 \in \langle \mathbf{x} \rangle$ implies $y_1 \cdot H_{n-1}(K) = 0$, which tells us the map in the second row of the above sequence is the zero map. Therefore by Fact III.A.2.2(d), the above sequence simplifies to

$$0 \longrightarrow H_n(\overline{K}) \xrightarrow{\cong} H_{n-1}(K) \longrightarrow 0.$$

Then $H_{n-1}(K) \cong H_n(\overline{K}) \cong \operatorname{Hom}_R(\overline{R}/\overline{I}, \overline{R}) \cong \operatorname{Hom}_R(R/I, \overline{R})$. Now consider the long exact sequence in $\operatorname{Ext}^i_R(R/I, -)$ associated to equation III.B.3.12.1:

$$
\begin{aligned}
0 \longrightarrow \underbrace{\operatorname{Hom}_R(R/I, R)}_{=0} \xrightarrow{y_1} \underbrace{\operatorname{Hom}_R(R/I, R)}_{=0} \longrightarrow \operatorname{Hom}_R(R/I, \overline{R}) \\
\longrightarrow \operatorname{Ext}^1_R(R/I, R) \xrightarrow{y_1=0} \operatorname{Ext}^1_R(R/I, R) \longrightarrow \cdots.
\end{aligned}
$$

As in the previous long exact sequence, Fact III.A.2.2(d) implies that $\operatorname{Ext}^1_R(R/I, R) \cong \operatorname{Hom}_R(R/I, \overline{R})$, so $\operatorname{Ext}^1_R(R/I, R) \cong H_{n-1}(K)$. This gives us our result for $m = 1$. $\square$

rmk191010a REMARK III.B.3.13. Let $\delta = \operatorname{depth}(I, R)$. Then from Theorem III.B.3.4 and Theorem III.B.3.8, we have

$$
\begin{gathered}
\overbrace{\operatorname{Ext}^0_R(R/I, R), \ldots, \operatorname{Ext}^{\delta-1}_R(R/I, R)}^{=0}, \overbrace{\operatorname{Ext}^{\delta}_R(R/I, R)}^{\neq 0}, \overbrace{\operatorname{Ext}^{\delta+1}_R(R/I, R), \ldots}^{?} \\
\underbrace{H_n(K^R(\mathbf{x})), \ldots, H_{n-m+1}(K^R(\mathbf{x}))}_{=0}, \underbrace{H_{n-m}(K^R(\mathbf{x}))}_{\neq 0}, \underbrace{H_{n-m-1}(K^R(\mathbf{x})), \ldots}_{?}.
\end{gathered}
$$

We would like to know what happens to the end of each of the sequences in the above remark. The following fact tells us that all homologies past δ are nonzero. The proof of this fact requires some localization and Nakayama's Lemma, which are outside the scope of this course.

fact191010b FACT III.B.3.14 (Rigidity of Koszul homology). *Let $I = \langle \mathbf{x} \rangle \subsetneq R$ and $\delta = \operatorname{depth}(I, R)$. Then $H_i(K^R(\mathbf{x})) \neq 0$ for all $i = 0, \ldots, \delta$.*

However, there is not as clear-cut an answer for the sequence in $\operatorname{Ext}^i_R(R/I, R)$ for $i > \delta$. The following example shows two different cases.

ex191010c EXAMPLE III.B.3.15. Let $\mathbf{X} = X_1, \ldots, X_m$ and $\mathbf{Y} = Y_1, \ldots, Y_n$.

(a) Consider $R = k[\mathbf{X}, \mathbf{Y}]/\langle \mathbf{X} \rangle^2$. Set $x_i = \overline{X_i} \in R$ and $y_j = \overline{Y_j} \in R$ and $I = \langle \mathbf{x}, \mathbf{y} \rangle$. Notice that $\mathbf{y}$ is a maximal weakly R-regular sequence in I so $\delta = n$ because $R \cong \frac{k[\mathbf{X}]}{\langle \mathbf{X} \rangle^2}[\mathbf{Y}]$. Then:

$$
\begin{aligned}
\operatorname{Ext}^i_R(R/I, R) &: \overbrace{\operatorname{Ext}^0_R(R/I, R), \ldots, \operatorname{Ext}^{n-1}_R(R/I, R)}^{=0}, \overbrace{\operatorname{Ext}^{n}_R(R/I, R)}^{\neq 0}, \overbrace{\operatorname{Ext}^{n+1}_R(R/I, R), \ldots, \operatorname{Ext}^{m+n}_R(R/I, R), \ldots}^{\neq 0} \\
H_i(K^R(\mathbf{x}, \mathbf{y})) &: \underbrace{H_{m+n}(K^R(\mathbf{x}, \mathbf{y})), \ldots, H_{m+1}(K^R(\mathbf{x}, \mathbf{y}))}_{=0}, \underbrace{H_m(K^R(\mathbf{x}, \mathbf{y}))}_{\neq 0}, \underbrace{H_{m-1}(K^R(\mathbf{x}, \mathbf{y})), \ldots, H_0(K^R(\mathbf{x}, \mathbf{y}))}_{\neq 0}
\end{aligned}
$$

Therefore, we have an example of a sequence in Ext in which $\operatorname{Ext}^i_R \neq 0$ for all $i \geq \delta$.

(b) Consider $R = k[\mathbf{X}, \mathbf{Y}]/\langle X_1^2, \ldots, X_m^2 \rangle$. Set $x_i = \overline{X_i} \in R$ and $y_j = \overline{Y_j} \in R$ and $I = \langle \mathbf{x}, \mathbf{y} \rangle$. Notice that $\mathbf{y}$ is a maximal weakly R-regular sequence in I so $\delta = n$ because $R \cong \frac{k[\mathbf{X}]}{\langle X_1^2, \ldots, X_m^2 \rangle}[\mathbf{Y}]$. Then we have the following.

$$
\begin{aligned}
\operatorname{Ext}^i_R(R/I, R) &: \overbrace{\operatorname{Ext}^0_R(R/I, R), \ldots, \operatorname{Ext}^{n-1}_R(R/I, R)}^{=0}, \overbrace{\operatorname{Ext}^{n}_R(R/I, R)}^{\neq 0}, \overbrace{\operatorname{Ext}^{n+1}_R(R/I, R), \ldots, \operatorname{Ext}^{m+n}_R(R/I, R), \ldots}^{=0} \\
H_i(K^R(\mathbf{x}, \mathbf{y})) &: \underbrace{H_{m+n}(K^R(\mathbf{x}, \mathbf{y})), \ldots, H_{m+1}(K^R(\mathbf{x}, \mathbf{y}))}_{=0}, \underbrace{H_m(K^R(\mathbf{x}, \mathbf{y}))}_{\neq 0}, \underbrace{H_{m-1}(K^R(\mathbf{x}, \mathbf{y})), \ldots, H_0(K^R(\mathbf{x}, \mathbf{y}))}_{\neq 0}
\end{aligned}
$$

Therefore, we also have an example of a sequence in Ext in which $\operatorname{Ext}^i_R = 0$ for all $i > \delta$.

The next result shows one more connection between depth and the topic of this course.

thm191010d THEOREM III.B.3.16 (Auslander-Buchsbaum). *Let $R = k[X_1, \ldots, X_d]$ and $\mathbf{f} = f_1, \ldots, f_n \in R$, where each f_i is a non-constant homogeneous polynomial. Let $I = \langle \mathbf{f} \rangle \subsetneq R$ and $\overline{R} = R/I$ and $x_i = \overline{X_i} \in \overline{R}$ and $m = \langle \mathbf{x} \rangle \subsetneq \overline{R}$ and $\Delta = \operatorname{depth}(m, \overline{R})$.*

(a) *There exists a free resolution $0 \to F_{d-\Delta} \to F_{d-\Delta-1} \to \cdots \to F_0 \to \overline{R} \to 0$.*

(b) *If* $0 \to G_n \to \cdots \to G_0 \to R \to 0$ *is a free resolution over* R, *then* $n \geq d - \Delta$. *Furthermore,* $0 \to \operatorname{Ker} \partial^G_{d-\Delta-1} \to G_{d-\Delta-1} \to \cdots \to G_0 \to \overline{R} \to 0$ *is exact and* $\operatorname{Ker} \partial^G_{d-\Delta-1}$ *is projective. By a result of Serre,* $\operatorname{Ker} \partial^G_{d-\Delta-1}$ *is also free.*

The slogan here is that the "projective dimension" of $\overline{R}$ over R is $\dim(R) - \operatorname{depth}\big(I, \overline{R}\big) = d - \Delta$.

III.B.4. The Taylor Resolution

ection063021j

In this chapter, assume that $R = k[X_1, \ldots, X_d]$. We want to find explicit resolutions of R/I where I is a monomial ideal.

rec191010e RECALL III.B.4.1. A <u>monomial</u> in R is $\mathbf{X}^{\mathbf{e}} = X_1^{e_1} \cdots X_d^{e_d} \in R$, where $\mathbf{e} = (e_1, \ldots, e_d) \in \mathbb{N}^d$. It is noteworthy that our definition of a monomial requires the coefficient to be 1.

def191010f DEFINITION III.B.4.2. A <u>monomial ideal</u> in R is an ideal generated by monomials. We will use the notation $[\![R]\!]$ to represent the set of all monomials in R and $[\![I]\!] = I \cap [\![R]\!]$ to represent the set of all monomials in I.

ex191010g EXAMPLE III.B.4.3. (a) The ideal $\mathfrak{X} = m = \langle \mathbf{X} \rangle = \langle X_1, \ldots, X_d \rangle \lneq R$ is a monomial ideal.
(b) The ideal $\langle XY, XZ, YZ \rangle \lneq R = k[X, Y, Z]$ is a monomial ideal.

def191010h DEFINITION III.B.4.4 (Taylor resolution). Let $\mathbf{f} = f_1, \ldots, f_n \in [\![R]\!]$. Then the <u>Taylor resolution</u> of $\mathbf{f}$ is

$$T = T^R(\mathbf{f}) = \; (0 \longrightarrow R \xrightarrow{\partial_n^T} R^n \xrightarrow{\partial_{n-1}^T} \cdots \xrightarrow{\partial_{i+1}^T} R^{\binom{n}{i}} \xrightarrow{\partial_i^T} \cdots \xrightarrow{\partial_2^T} R^n \xrightarrow{\partial_1^T} R \longrightarrow 0)$$

where the basis is the same as the exterior basis for the Koszul complex

$$\{e_{j_1, \ldots, j_i} \mid 1 \leq j_1 < \cdots < j_i \leq n\} \subset R^{\binom{n}{i}} = T_i$$

and

$$\partial_i^T(e_{j_1, \ldots, j_i}) = \sum_{p=1}^{i} (-1)^{p-1} \frac{\operatorname{lcm}(f_{j_1}, \ldots, f_{j_i})}{\operatorname{lcm}(f_{j_1}, \ldots, \widehat{f}_{j_p}, \ldots, f_{j_i})} \, e_{j_1, \ldots, \widehat{j}_p, \ldots, j_i}.$$

prop191010i PROPOSITION III.B.4.5. *The Taylor resolution* $T^R(\mathbf{f})$ *is an* R*-complex which satisfies* $H_0(T^R(\mathbf{f})) \cong R/\langle \mathbf{f} \rangle$.

ex191010j EXAMPLE III.B.4.6. We give the following two examples of Taylor resolutions. The first is familiar.
(a) If $\mathbf{f} = X_1, \ldots, X_n$, then $T^R(\mathbf{f}) = K^R(X_1, \ldots, X_n)$, since

$$\frac{\operatorname{lcm}(X_{j_1}, \ldots, X_{j_i})}{\operatorname{lcm}(X_{j_1}, \ldots, \widehat{X}_{j_p}, \ldots, X_{j_i})} = \frac{X_{j_1} \cdots X_{j_p} \cdots X_{j_i}}{X_{j_1} \cdots \widehat{X}_{j_p} \cdots X_{j_i}} = X_{j_p}.$$

(b) Let $I = \langle XY, XZ, YZ \rangle$. Then since I is generated by three elements, the outline for the Taylor resolution of I is

$$T^R(XY, XZ, YZ): \; (0 \longrightarrow R \xrightarrow{\partial_3^T} R^3 \xrightarrow{\partial_2^T} R^3 \xrightarrow{\partial_1^T} R \longrightarrow 0).$$

Next, we determine ∂_j^T for $j = 1, 2, 3$. For ∂_1^T, it is true in general that $e_i \in R^n$ maps to $f_i \in R$ for all i, since

$$e_i \mapsto (-1)^{1-1} \frac{\operatorname{lcm}(f_i)}{\operatorname{lcm}(\,)} e_\emptyset = 1 \cdot \frac{f_i}{1} \cdot 1 = f_i.$$

Therefore, we have $\partial_1^T = \begin{pmatrix} XY & XZ & YZ \end{pmatrix}$. For ∂_2^T, we see the following:

$$e_{12} \mapsto \frac{\operatorname{lcm}(f_1, f_2)}{\operatorname{lcm}(f_2)} e_2 - \frac{\operatorname{lcm}(f_1, f_2)}{\operatorname{lcm}(f_1)} e_1 = \frac{XYZ}{XZ} e_2 - \frac{XYZ}{XY} e_1 = Y e_2 - Z e_1 = \begin{pmatrix} -Z \\ Y \\ 0. \end{pmatrix}$$

$$e_{13} \mapsto \frac{\operatorname{lcm}(f_1, f_3)}{\operatorname{lcm}(f_3)} e_3 - \frac{\operatorname{lcm}(f_1, f_3)}{\operatorname{lcm}(f_1)} e_1 = \frac{XYZ}{YZ} e_3 - \frac{XYZ}{XY} e_1 = X e_3 - Z e_1 = \begin{pmatrix} -Z \\ 0 \\ X. \end{pmatrix}$$

$$e_{23} \mapsto \frac{\operatorname{lcm}(f_2, f_3)}{\operatorname{lcm}(f_3)} e_3 - \frac{\operatorname{lcm}(f_2, f_3)}{\operatorname{lcm}(f_2)} e_2 = \frac{XYZ}{YZ} e_3 - \frac{XYZ}{XZ} e_2 = X e_3 - Y e_2 = \begin{pmatrix} 0 \\ -Y \\ X. \end{pmatrix}$$

Therefore, we have

$$\partial_2^T = \begin{pmatrix} -Z & -Z & 0 \\ Y & 0 & -Y \\ 0 & X & X \end{pmatrix}.$$

For ∂_3^T, we see that

$$\begin{aligned} e_{123} \mapsto & \frac{\operatorname{lcm}(f_1,f_2,f_3)}{\operatorname{lcm}(f_2,f_3)} e_{23} - \frac{\operatorname{lcm}(f_1,f_2,f_3)}{\operatorname{lcm}(f_1,f_3)} e_{13} + \frac{\operatorname{lcm}(f_1,f_2,f_3)}{\operatorname{lcm}(f_1,f_2)} e_{12} \\ &= \frac{XYZ}{XYZ} e_{23} - \frac{XYZ}{XYZ} e_{13} + \frac{XYZ}{XYZ} e_{12} = e_{23} - e_{13} + e_{12} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \end{aligned}$$

Therefore, we have $\partial_3^T = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^T$.

rmk191017a

REMARK III.B.4.7. We will see later that $T^R(\mathbf{f})$ is always a resolution of $R/\langle \mathbf{f} \rangle$ (under monomial assumptions). However, Examples III.B.3.11 and III.B.4.6 show that $T^R(\mathbf{f})$ might not be minimal.

PROOF OF PROPOSITION III.B.4.5. We first check that $\partial_{i-1}^T(\partial_i^T(e_{j_1,\ldots,j_i})) = 0$ for all i. We have

$$\begin{aligned} \partial_{i-1}^T(\partial_i^T(e_{j_1,\ldots,j_i})) &= \sum_{p=1}^{i} (-1)^{p-1} \frac{\operatorname{lcm}(f_{j_1},\ldots,f_{j_i})}{\operatorname{lcm}(f_{j_1},\ldots,\widehat{f}_{j_p},\ldots,f_{j_i})} \partial_{i-1}^T(e_{j_1,\ldots,\widehat{j}_p,\ldots,j_i}) \\ &= \sum_{p=1}^{i} (-1)^{p-1} \frac{\operatorname{lcm}(f_{j_1},\ldots,f_{j_i})}{\operatorname{lcm}(f_{j_1},\ldots,\widehat{f}_{j_p},\ldots,f_{j_i})} \left[\sum_{q=1}^{p-1} (-1)^{q-1} \frac{\operatorname{lcm}(f_{j_1},\ldots,\widehat{f}_{j_p},\ldots,f_{j_i})}{\operatorname{lcm}(f_{j_1},\ldots,\widehat{f}_{j_q},\ldots,\widehat{f}_{j_p},\ldots,f_{j_i})} e_{j_1,\ldots,\widehat{j}_q,\ldots,\widehat{j}_p,\ldots,j_i} \right. \\ &\qquad \left. + \sum_{q=p+1}^{i} (-1)^{q} \frac{\operatorname{lcm}(f_{j_1},\ldots,\widehat{f}_{j_p},\ldots,f_{j_i})}{\operatorname{lcm}(f_{j_1},\ldots,\widehat{f}_{j_p},\ldots,\widehat{f}_{j_q},\ldots,f_{j_i})} e_{j_1,\ldots,\widehat{j}_p,\ldots,\widehat{j}_q,\ldots,j_i} \right] \end{aligned}$$

Then the coefficients of the two inner sums exactly cancel, as in the proof of Theorem III.B.2.14. Therefore, $T^R(\mathbf{f})$ is an R-complex.

Also, we have

$$H_0(T) = \frac{\operatorname{Ker}\ R \longrightarrow 0}{\operatorname{Im}\ R^n \xrightarrow[\partial_1^T]{(f_1,\ldots,f_n)} R} = \frac{R}{\langle \mathbf{f} \rangle}$$

which is the desired result. $\square$

thm191017b

THEOREM III.B.4.8. *The Taylor resolution $T^R(\mathbf{f})$ is a free resolution of $R/\langle \mathbf{f} \rangle$.*

PROOF. We will use induction on n. First consider the base case for $n = 1$, where

$$T^R(\mathbf{f}) = (0 \longrightarrow R \xrightarrow{f_1} R \longrightarrow 0).$$

Since $0 \neq f_1 \in R = k[\mathbf{X}]$, then f_1 is a non-zero-divisor of R. Therefore, $H_i(T) = 0$ for all $i \neq 0$. As in Lemma III.B.1.3, $T^R(\mathbf{f})$ is a free resolution of $R/\langle \mathbf{f} \rangle$.

Now assume that the Taylor resolution on sequences of length $n-1$ resolve appropriately. Set $\mathbf{f}' = f_2, \ldots, f_n$ and $I' = \langle \mathbf{f}' \rangle$. We consider the colon ideal as in Example III.B.1.5c

$$(I' : f_1) = (\langle \mathbf{f}' \rangle : f_1) = \{ g \in R \mid g f_1 \in \langle \mathbf{f}' \rangle \} = \langle g_2, \ldots, g_n \rangle,$$

where $g_i = \mathbf{X}^{(\mathbf{a}_i - \mathbf{a}_1)_+} = X_1^{(a_{i1}-a_{11})_+} \cdots X_d^{(a_{id}-a_{1d})_+}$ and where

$$(a_{ij} - a_{1j})_+ = \begin{cases} 0 & \text{if } a_{ij} - a_{1j} \leq 0 \\ a_{ij} - a_{1j} & \text{if } a_{ij} - a_{1j} \geq 0. \end{cases}$$

Set $\mathbf{g} = g_2, \ldots, g_n$, so $\langle \mathbf{g} \rangle = (I' : f_1)$. By the inductive hypothesis, $T^R(\mathbf{f}')$ is a free resolution of $R/\langle \mathbf{f}' \rangle$ and $T^R(\mathbf{g})$ is a free resolution of $R/\langle \mathbf{g} \rangle$. Our goal from here is to construct a chain map $\Psi^+ : T^R(\mathbf{g})^+ \to T^R(\mathbf{f})^+$ such that Ψ_{-1} is the map $R/\langle \mathbf{g} \rangle = R/(I' : f_1) \xrightarrow{f_1} R/I' = R/\langle \mathbf{f}' \rangle$ and $\operatorname{Cone}(\Psi) \cong T^R(\mathbf{f})$. Then by

Theorem III.B.1.7, $\mathrm{Cone}(\Psi)$ will be a free resolution of $R/\langle \mathbf{f}\rangle$. Our setup for Ψ comes from filling in the question mark in the following diagram:

$$\begin{tikzcd}
T^R(\mathbf{g})^+ = \arrow[d, dashed, "\Psi^+"] & \cdots \arrow[r] & R^n \arrow[r] \arrow[d, dashed, "?"] & R \arrow[r] \arrow[d, dashed, "f_1"] & R/\langle \mathbf{g}\rangle \arrow[r] \arrow[d, hook, "f_1"] & 0 \\
T^R(\mathbf{f}')^+ = & \cdots \arrow[r] & R^n \arrow[r] & R \arrow[r] & R/I' \arrow[r] & 0.
\end{tikzcd}$$

For example, let $\mathbf{f} = X^2, XY, Y^2$. Then $\mathbf{f}' = XY, Y^2$ and $(\langle \mathbf{f}'\rangle : f_1) = (\langle XY, Y^2\rangle : X^2) = \langle Y, Y^2\rangle$, so $\mathbf{g} = Y, Y^2$. Then

$$\begin{tikzcd}
T^R(Y,Y^2)^+ = \arrow[d, dashed, "\Psi^+"] & 0 \arrow[r] & R \arrow[r, "\left(\begin{smallmatrix}-Y\\ 1\end{smallmatrix}\right)"] \arrow[d, dashed, "(**)"] & R^2 \arrow[r, "(Y\ Y^2)"] \arrow[d, dashed, "(*)"] & R \arrow[r, "\pi"] \arrow[d, dashed, "X^2"] & R/\langle \mathbf{g}\rangle \arrow[r] \arrow[d, hook, "X^2"] & 0 \\
T^R(XY,Y^2)^+ = & 0 \arrow[r] & R \arrow[r, "\left(\begin{smallmatrix}-Y\\ X\end{smallmatrix}\right)"'] & R^2 \arrow[r, "(XY\ Y^2)"'] & R \arrow[r, "\tau"] & R/\langle \mathbf{f}'\rangle \arrow[r] & 0.
\end{tikzcd}$$

In order to determine $(*)$ and $(**)$, we chase the respective parts of the above diagram in order to make the diagram commute. For $(*)$, we have

$$\begin{tikzcd}
e_2 \arrow[r, mapsto] \arrow[d, mapsto] & Y \arrow[d, mapsto] & e_3 \arrow[r, mapsto] \arrow[d, mapsto] & Y^2 \arrow[d, mapsto] \\
Xe_2 \arrow[r, mapsto] & X^2Y & X^2e_3 \arrow[r, mapsto] & X^2Y^2 \\
e_2 \arrow[r, mapsto] & XY, & e_3 \arrow[r, mapsto] & Y^2.
\end{tikzcd}$$

Therefore $(*) = \left(\begin{smallmatrix} X & 0 \\ 0 & X^2\end{smallmatrix}\right)$ makes the diagram commute. For $(**)$, we have

$$\begin{tikzcd}
e_{23} \arrow[r, mapsto] \arrow[d, mapsto] & e_3 - Ye_2 \arrow[d, mapsto] \\
Xe_{23} \arrow[r, mapsto] & X^2e_3 - XYe_2 \\
e_{23} \arrow[r, mapsto] & Xe_3 - Ye_2.
\end{tikzcd}$$

Therefore $(**) = X$ makes the diagram commute.

Continuing the proof, we prove two claims.

CLAIM.

$$g_j = \frac{\mathrm{lcm}(f_1, f_j)}{f_1}. \tag{III.B.4.8.1}$$

eqn191022a

PROOF. We prove this claim by comparing the exponents of X_q on each side of the equation. The exponent for X_q on the right hand side is $\max(a_{1q}, a_{jq}) - a_{1q}$, while the exponent for X_q on the left hand side is $(a_{jq} - a_{1q})_+$. There are two cases:

$$\begin{aligned}
&\text{If } a_{jq} \geq a_{1q}, \text{ then } \max(a_{1q}, a_{jq}) - a_{1q} = a_{jq} - a_{1q} = (a_{jq} - a_{1q})_+ \\
&\text{If } a_{jq} \leq a_{1q}, \text{ then } \max(a_{1q}, a_{jq}) - a_{1q} = a_{1q} - a_{1q} = 0 = (a_{jq} - a_{1q})_+
\end{aligned}$$

This proves the first claim. ✓

CLAIM.

$$f_1 \cdot \mathrm{lcm}(g_{j2}, \ldots, g_{ji}) = \mathrm{lcm}(f_1, f_{j2}, \ldots, f_{ji}). \tag{III.B.4.8.2}$$

eqn191022b

PROOF. Using Claim III.B.4.8.1, we have

$$\begin{aligned} f_1 \cdot \operatorname{lcm}(g_{j2}, \ldots, g_{ji}) &= f_1 \cdot \operatorname{lcm}\left(\frac{\operatorname{lcm}(f_1, f_{j_2})}{f_1}, \ldots, \frac{\operatorname{lcm}(f_1, f_{j_i})}{f_1}\right) \\ &= \frac{f_1}{f_1} \cdot \operatorname{lcm}\left(\operatorname{lcm}(f_1, f_{j_2}), \ldots, \operatorname{lcm}(f_1, f_{j_i})\right) \\ &= \operatorname{lcm}(f_1, f_{j2}, \ldots, f_{ji}). \end{aligned}$$

This proves the second claim. ✓

Now we define

$$\Psi_i(e_{j_1,\ldots,j_i}) = \frac{f_1 \cdot \operatorname{lcm}(g_{j_1}, \ldots, g_{j_i})}{\operatorname{lcm}(f_{j_1}, \ldots, f_{j_i})}\, e_{j_1,\ldots,j_i}.$$

By Claim III.B.4.8.2, the coefficients of the above definition are

$$\frac{f_1 \cdot \operatorname{lcm}(g_{j_1}, \ldots, g_{j_i})}{\operatorname{lcm}(f_{j_1}, \ldots, f_{j_i})} = \frac{\operatorname{lcm}(f_1, f_{j_1}, \ldots, f_{j_i})}{\operatorname{lcm}(f_{j_1}, \ldots, f_{j_i})},$$

which means they are elements of R. We want to show that Ψ is a chain map and that $\operatorname{Cone}(\Psi) \cong T^R(\mathbf{f})$. In order to check that Ψ is a chain map, consider the following diagram:

$$\begin{tikzcd} R^{\binom{n}{i}} \arrow[r, "\partial_i^{T^R(\mathbf{g})}"] \arrow[d, "\Psi_i"] & R^{\binom{n}{i-1}} \arrow[d, "\Psi_{i-1}"] \\ R^{\binom{n}{i}} \arrow[r, "\partial_i^{T^R(\mathbf{f}')}"] & R^{\binom{n}{i-1}}. \end{tikzcd}$$

A diagram chase of the above diagram follows:

$$\begin{tikzcd} e_{j_1,\ldots,j_i} \arrow[r, mapsto] \arrow[dd, mapsto] & \displaystyle\sum_{p=1}^{i}(-1)^{p-1}\frac{\operatorname{lcm}(g_{j_1},\ldots,g_{j_i})}{\operatorname{lcm}(g_{j_1},\ldots,\widehat{g_{j_p}},\ldots,g_{j_i})}e_{j_1,\ldots,\widehat{j_p},\ldots,j_i} \arrow[d, mapsto] \\ & \displaystyle\sum_{p=1}^{i}(-1)^{p-1}\frac{\operatorname{lcm}(g_{j_1},\ldots,g_{j_i})}{\operatorname{lcm}(g_{j_1},\ldots,\widehat{g_{j_p}},\ldots,g_{j_i})}\cdot\frac{f_1\cdot\operatorname{lcm}(g_{j_1},\ldots,\widehat{g_{j_p}},\ldots,g_{j_i})}{\operatorname{lcm}(f_{j_1},\ldots,\widehat{f_{j_p}},\ldots,f_{j_i})}e_{j_1,\ldots,\widehat{j_p},\ldots,j_i} \\ \frac{f_1\cdot\operatorname{lcm}(g_{j_1},\ldots,g_{j_i})}{\operatorname{lcm}(f_{j_1},\ldots,f_{j_i})}e_{j_1,\ldots,j_i} \arrow[r, mapsto] & \frac{f_1\cdot\operatorname{lcm}(g_{j_1},\ldots,g_{j_i})}{\operatorname{lcm}(f_{j_1},\ldots,f_{j_i})}\displaystyle\sum_{p=1}^{i}(-1)^{p-1}\frac{\operatorname{lcm}(f_{j_1},\ldots,f_{j_i})}{\operatorname{lcm}(f_{j_1},\ldots,\widehat{f_{j_p}},\ldots,f_{j_i})}e_{j_1,\ldots,\widehat{j_p},\ldots,j_i}. \end{tikzcd}$$

Notice that once we cancel factors in both sums in the bottom right corner, we see that they are equal. Therefore, Ψ is a chain map.

In order to check that $\operatorname{Cone}(\Psi) \cong T^R(\mathbf{f})$, consider the following diagram:

$$\begin{tikzcd} \operatorname{Cone}(\Psi)_i = \arrow[d, "\Phi_i"] & \begin{matrix} T^R(\mathbf{g})_{i-1} \\ \oplus \\ T^R(\mathbf{f}')_i \end{matrix} \arrow[r, "\begin{pmatrix} -\partial_{i-1}^{T^R(\mathbf{g})} & 0 \\ \Psi_{i-1} & \partial_i^{T^R(\mathbf{f}')} \end{pmatrix}"] & \begin{matrix} T^R(\mathbf{g})_{i-2} \\ \oplus \\ T^R(\mathbf{f}') \end{matrix} & = \operatorname{Cone}(\Psi)_{i-1} \arrow[d, "\Phi_{i-1}"] \\ T^R(\mathbf{f})_i = & R^{\binom{n}{i}} \arrow[r, "\partial_i^{T^R(\mathbf{f})}"] & R^{\binom{n}{i-1}} & = T^R(\mathbf{f})_{i-1} \end{tikzcd}$$

where Φ_i is defined on basis vectors as

$$\begin{aligned} \Phi_i \begin{pmatrix} 0 \\ e_{j_1,\ldots,j_i} \end{pmatrix} &= e_{j_1,\ldots,j_i}, \text{ and} \\ \Phi_i \begin{pmatrix} e_{j_2,\ldots,j_i} \\ 0 \end{pmatrix} &= e_{1,j_2,\ldots,j_i}. \end{aligned}$$

We now show that Φ is a chain map by checking commutativity of the above diagram on the basis vectors for $\operatorname{Cone}(\Psi)$:

$$\begin{tikzcd}
\begin{pmatrix} 0 \\ e_{j_1,\dots,j_i} \end{pmatrix} \arrow[r, mapsto] \arrow[d, mapsto] & \begin{pmatrix} 0 \\ \partial_i^{T^R(\mathbf{f}')}(e_{j_1,\dots,j_i}) \end{pmatrix} = \displaystyle\sum_{p=1}^{i} (-1)^{p-1} \frac{\operatorname{lcm}(f_{j_1},\dots,f_{j_i})}{\operatorname{lcm}(f_{j_1},\dots,\widehat{f_{j_p}},\dots,f_{j_i})} \begin{pmatrix} 0 \\ e_{j_1,\dots,\widehat{j_p},\dots,j_i} \end{pmatrix} \arrow[d, mapsto] \\
e_{j_1,\dots,j_i} \arrow[r, mapsto] & \displaystyle\sum_{p=1}^{i} (-1)^{p-1} \frac{\operatorname{lcm}(f_{j_1},\dots,f_{j_i})}{\operatorname{lcm}(f_{j_1},\dots,\widehat{f_{j_p}},\dots,f_{j_i})} e_{j_1,\dots,\widehat{j_p},\dots,j_i},
\end{tikzcd}$$

and

$$\begin{tikzcd}
\begin{pmatrix} e_{j_2,\dots,j_i} \\ 0 \end{pmatrix} \arrow[r, mapsto] \arrow[d, mapsto] & \begin{pmatrix} \partial_{i-1}^{T^R(\mathbf{g})}(e_{j_2,\dots,j_i}) \\ \Psi_{i-1}(e_{j_2,\dots,j_i}) \end{pmatrix} = \begin{array}{l} \left[-\displaystyle\sum_{p=2}^{i} (-1)^{p} \frac{\operatorname{lcm}(g_{j_2},\dots,g_{j_i})}{\operatorname{lcm}(g_{j_2},\dots,\widehat{g_{j_p}},\dots,g_{j_i})} \begin{pmatrix} e_{j_2,\dots,\widehat{j_p},\dots,j_i} \\ 0 \end{pmatrix} \right] \\ \qquad + \left[\dfrac{f_1 \cdot \operatorname{lcm}(g_{j_2},\dots,g_{j_i})}{\operatorname{lcm}(f_{j_2},\dots,f_{j_i})} \begin{pmatrix} 0 \\ e_{j_2,\dots,j_i} \end{pmatrix} \right] \end{array} \arrow[d, mapsto] \\
& \begin{array}{l} \left[\displaystyle\sum_{p=2}^{i} (-1)^{p-1} \frac{\operatorname{lcm}(g_{j_2},\dots,g_{j_i})}{\operatorname{lcm}(g_{j_2},\dots,\widehat{g_{j_p}},\dots,g_{j_i})} e_{1,j_2,\dots,\widehat{j_p},\dots,j_i} \right] \\ \qquad + \left[\dfrac{f_1 \cdot \operatorname{lcm}(g_{j_2},\dots,g_{j_i})}{\operatorname{lcm}(f_{j_2},\dots,f_{j_i})} e_{j_2,\dots,j_i} \right] \end{array} \\
e_{1,j_2,\dots,j_i} \arrow[r, mapsto] & \begin{array}{l} \left[\dfrac{\operatorname{lcm}(f_1,f_{j_2},\dots,f_{j_i})}{\operatorname{lcm}(f_{j_2},\dots,f_{j_i})} e_{j_2,\dots,j_i} \right] \\ + \left[\displaystyle\sum_{p=2}^{i} (-1)^{p-1} \frac{\operatorname{lcm}(f_1,f_{j_2},\dots,f_{j_i})}{\operatorname{lcm}(f_1,f_{j_2},\dots,\widehat{f_{j_p}},\dots,f_{j_i})} e_{1,j_2,\dots,\widehat{j_p},\dots,j_i} \right]. \end{array}
\end{tikzcd}$$

To show that Φ is a chain map, it suffices to show that the two lines in the bottom right corner are equal. By Claim III.B.4.8.2, the coefficients for the $e_{j_2,\dots,j_i}$ basis element are equal. For the terms inside the summation, we just need to multiply by $\frac{f_1}{f_1}$ and use Claim III.B.4.8.2 to see that they are equal:

$$\frac{f_1}{f_1} \cdot \frac{\operatorname{lcm}(g_{j_2},\dots,g_{j_i})}{\operatorname{lcm}(g_{j_2},\dots,\widehat{g_{j_p}},\dots,g_{j_i})} = \frac{\operatorname{lcm}(f_1,f_{j_2},\dots,f_{j_i})}{\operatorname{lcm}(f_1,f_{j_2},\dots,\widehat{f_{j_p}},\dots,f_{j_i})}.$$

This shows that the diagram commutes, so Φ is chain map. Furthermore, it is straightforward to show that Φ induces a bijection between bases, so Φ is an isomorphism. Therefore, $\operatorname{Cone}(\Phi) \cong T^R(\mathbf{f})$, so Theorem III.B.1.7 tells us that $T^R(\mathbf{f})$ is a free resolution of $R/\langle \mathbf{f} \rangle$. □

III.B.5. A Colloquial Presentation of Two Resolutions

ection063021k
trck191022a

Parlor Trick. Let $R = k[X,Y,Z]$ and $I = \langle XY, XZ, YZ \rangle$, and consider the resolution

$$0 \longrightarrow R^2 \xrightarrow[\partial_2]{\begin{pmatrix} -Z & -Z \\ Y & 0 \\ 0 & X \end{pmatrix}} R^3 \xrightarrow{(XY\ XZ\ YZ)} R \longrightarrow 0.$$

Look at the three 2×2 minors (i.e., sub-determinants) of ∂_2:

$$\text{By deleting the first row: } \begin{vmatrix} Y & 0 \\ 0 & X \end{vmatrix} = XY,$$

$$\text{By deleting the second row: } \begin{vmatrix} -Z & -Z \\ 0 & X \end{vmatrix} = -XZ,$$

$$\text{By deleting the third row: } \begin{vmatrix} -Z & -Z \\ Y & 0 \end{vmatrix} = YZ.$$

Notice here that these three minors are the generators of I. This is a special case of the Hilbert-Burch resolution; see Theorem III.B.5.10 below.

To further motivate our use of minors in this chapter, consider a resolution

$$0 \longrightarrow R^{\beta_m} \xrightarrow{\partial_m} \cdots \longrightarrow R^{\beta_0} \longrightarrow 0.$$

In particular, recall that ∂_m is one-to-one, so the columns of the matrix A representing ∂_m are linearly independent over R. If R is a field, then ∂_m is one-to-one if and only if some size-β_m minor of A is non-zero. Our goal in this chapter is to find similar conditions for the case where R is not a field.

defn191024a

Definition III.B.5.1. Let m, n be positive integers and

$$M_{m\times n}(R) = \{m \times n \text{ matrices } (a_{ij}) \mid \text{ all } a_{ij} \in R\} \cong R^{mn}.$$

For all positive integers $r \leq \min(m, n)$, a size-r minor of a matrix $A \in M_{m\times n}(R)$ is the determinant of an $r \times r$ matrix obtained by deleting some number of rows and columns of A. We also call this an $r \times r$ subdeterminant of A. If this deletion leaves rows $\mathbf{i} = i_1, \dots, i_r$ and columns $\mathbf{j} = j_1, \dots, j_r$, then the corresponding size-r minor is of the form

$$\begin{vmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_r} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_r,j_1} & a_{i_r,j_2} & \cdots & a_{i_r,j_r} \end{vmatrix}$$

and is denoted $[i_1, \dots, i_r \mid j_1, \dots, j_r]_A = [\mathbf{i} \mid \mathbf{j}]_A$. Also, define $I_r(A)$ to be the ideal of R generated by all size-r minors of A.

ex191024b

Example III.B.5.2. Let $R = k[X, Y, Z]$ and let

$$A = \begin{pmatrix} -Z & -Z \\ Y & 0 \\ 0 & X \end{pmatrix}.$$

From our computations in the Parlor Trick at the beginning of this chapter, $I_2(A) = \langle XY, XZ, YZ\rangle$. Furthermore, $I_1(A) = \langle X, Y, Z\rangle$, since this is the ideal generated by all 1×1 minors of A. We will later be able to use Proposition III.B.5.5 to show that each ideal $I_r(A)$ is independent of the choice of basis.

lem191024c

Lemma III.B.5.3. *Let m, r be positive integers such that $r \leq m$ and $A \in M_{r\times m}(R)$ and $B \in M_{m\times r}(R)$. Then $|AB| \in I_r(A) \cap I_r(B)$.*

Proof. Notice that the rows of AB are linear combinations of the rows of B. Since $B \in M_{m\times r}(R)$, each row of B has r entries and since $AB \in M_{r\times r}(R)$, AB has r rows. Then

$$|AB| = \begin{vmatrix} \text{linear combination of rows of } B \\ \text{linear combination of rows of } B \\ \vdots \\ \text{linear combination of rows of } B \end{vmatrix} = \text{big linear combination of } \begin{vmatrix} \text{row of } B \\ \text{row of } B \\ \vdots \\ \text{row of } B \end{vmatrix} \in I_r(B).$$

Similarly, notice that the columns of AB are linear combinations of the columns of A. By a similar argument, $|AB| \in I_r(A)$. □

There is an alternate proof of Lemma III.B.5.3 using the Cauchy-Binet Formula.

lem191024d

Lemma III.B.5.4. *Let m, n, p, r be positive integers such that $r \leq \min(m, n, p)$ and $A \in M_{n\times p}(R)$ and $B \in M_{p\times m}(R)$. Then $I_r(AB) \subseteq I_r(A) \cap I_r(B)$.*

Proof. It suffices to show that each size-r minor of AB is in both $I_r(A)$ and $I_r(B)$. We can find that $[\mathbf{i} \mid \mathbf{j}]_{AB} = |A_{\mathbf{i}}B_{\mathbf{j}}|$ by expanding the corresponding matrices, where $A_{\mathbf{i}} \in M_{r\times p}(R)$ consists of rows $i_1, \dots, i_r$ of A and $B_{\mathbf{j}} \in M_{p\times r}$ consists of columns $j_1, \dots, j_r$ of B. Then by Lemma III.B.5.3,

$$|A_{\mathbf{i}}B_{\mathbf{j}}| \in I_r(A_{\mathbf{i}}) \cap I_r(B_{\mathbf{j}}).$$

Since all minors of $A_{\mathbf{i}}$ are also minors of A and all minors of $B_{\mathbf{j}}$ are also minors of B, we have

$$[\mathbf{i} \mid \mathbf{j}]_{AB} = |A_{\mathbf{i}}B_{\mathbf{j}}| \in I_r(A_{\mathbf{i}}) \cap I_r(B_{\mathbf{j}}) \subseteq I_r(A) \cap I_r(B).$$

□

The above two lemmas allow for the following proposition. The slogan is if A and B differ only by a change of basis, then $I_r(A)$ and $I_r(B)$ are equal.

prop191024e

Proposition III.B.5.5. *Let m, n, r be positive integers such that $r \leq \min(m, n)$ and $A \in M_{m\times n}(R)$ and $U \in M_{m\times m}(R)^\times$ and $V \in M_{n\times n}(R)^\times$. Set $B = UAV$. Then $I_r(A) = I_r(B)$. Note that B is defined so that the following diagram commutes:*

$$\begin{tikzcd} R^n \arrow[r, "A"] & R^m \arrow[d, "\cong"', "U"] \\ R^n \arrow[u, "V", "\cong"'] \arrow[r, "B"'] & R^m. \end{tikzcd}$$

Proof. Since U and V are invertible, we write $A = U^{-1}BV^{-1}$. Then we use Lemma III.B.5.4 to show both inclusions:

$$\begin{aligned} I_r(B) &= I_r(UAV) \subseteq I_r(U) \cap I_r(A) \cap I_r(V) \subseteq I_r(A), \text{ and} \\ I_r(A) &= I_r(U^{-1}BV^{-1}) \subseteq I_r(U^{-1}) \cap I_r(B) \cap I_r(V^{-1}) \subseteq I_r(B). \end{aligned}$$

Therefore, $I_r(A) = I_r(B)$. □

defn191024f

Definition III.B.5.6. Let r, m, n be positive integers such that $r \leq \min(m, n)$, let $f \in \mathrm{Hom}_R(R^n, R^m)$, and let $A \in M_{m\times n}(R)$ represent f with respect to some bases. Then $I_r(f) = I_r(A)$. By Proposition III.B.5.5, this definition is independent of the choice of bases. Furthermore, if $s > \min(m, n)$, then $I_s(f) = 0$ and if $s \leq 0$, then $I_s(f) = R$.

prop191024g

Proposition III.B.5.7. *Let r, m, n be positive integers such that $r \leq \min(m, n)$ and let $f \in \mathrm{Hom}_R(R^n, R^m)$. Then*

$$I_0(f) \supseteq I_1(f) \supseteq I_2(f) \supseteq \cdots \supseteq I_r(f) \supseteq \cdots \supseteq \underbrace{I_{\min(m,n)+1}(f)}_{=0}.$$

Proof. Expanding any size-r minor along a single row or column allows us to write it as a linear combination of size-$(r-1)$ minors. Therefore $I_r(f) \subseteq I_{r-1}(f)$ for all r. □

The proof of the following result is outside of the scope of this class. We will use it to verify the two resolutions of interest in Theorems III.B.5.10 and III.B.5.30 below.

thm191024h

Theorem III.B.5.8 (Buchsbaum-Eisenbud). *Assume that R is noetherian and consider an R-complex*

$$F = \left(0 \longrightarrow R^{\beta_m} \xrightarrow{\partial_m^F} R^{\beta_{m-1}} \xrightarrow{\partial_{m-1}^F} \cdots \xrightarrow{\partial_1^F} R^{\beta_0} \longrightarrow 0 \right).$$

For all $i = 1, \ldots, m$, set

$$r_i = \sum_{j=i}^{m} (-1)^{j-i}\beta_j = \beta_i - \beta_{i+1} + \cdots + (-1)^{m-i}\beta_m.$$

Then F is a resolution of $H_0(F)$ if and only if $\operatorname{depth}(I_{r_i}(\partial_i^F), R) \geq i$ for all $i = 1, \ldots, m$.

ex191024i

Example III.B.5.9. Let $R = k[X, Y, Z]$ and $I = \langle XY, XZ, YZ\rangle$ and consider the resolution

$$F = \left(0 \longrightarrow R^2 \xrightarrow{\left(\begin{smallmatrix} -Z & -Z \\ Y & 0 \\ 0 & X \end{smallmatrix}\right)} R^3 \xrightarrow{(XY\ XZ\ YZ)} R \longrightarrow 0 \right).$$

Notice here that $\beta_2 = 2$, $\beta_1 = 3$, and $\beta_0 = 1$. We check $\operatorname{depth}(I_{r_i}(\partial_i^F), R) \geq i$ for $i = 1, 2$:

$i = 2:$ $r_2 = \beta_2 = 2$, so $I_{r_2}(\partial_2^F) = I_2\left(\begin{smallmatrix} -Z & -Z \\ Y & 0 \\ 0 & X \end{smallmatrix}\right) = \langle XY, XZ, YZ\rangle = I$. Then

$$\operatorname{depth}(I_{r_2}(\partial_2^F), R) = \operatorname{depth}(I, R) = 2 \geq 2.$$

$i = 1:$ $r_1 = \beta_1 - \beta_2 = 3 - 2 = 1$, so $I_{r_1}(\partial_1^F) = I_1\begin{pmatrix} XY & XZ & YZ \end{pmatrix} = \langle XY, XZ, YZ\rangle = I$. Then

$$\operatorname{depth}(I_{r_1}(\partial_1^F), R) = \operatorname{depth}(I, R) = 2 \geq 1.$$

Therefore by Theorem III.B.5.8, F is a resolution of R/I.

We next give a result by Hilbert and Burch, an unnumbered example, and the proof of the result.

thm191024j

THEOREM III.B.5.10 (Hilbert-Burch). *Assume that* $R = k[X_1, \ldots, X_d]$.

thm191024j.a

(a) *Let* $f \in \mathrm{Hom}_R(R^n, R^{n+1})$ *for* $n \geq 1$, *and let* $B \in M_{(n+1)\times n}(R)$ *represent* f *with respect to the standard bases. For* $i = 1, \ldots, n+1$, *set* $f_i = |B_i|$, *where* B_i *is obtained from* B *by deleting the* i^{th} *row. In other words, we have*

$$f_i = [1, \ldots, \widehat{i}, \ldots, n+1 \mid 1, \ldots, n]_B.$$

Assume $\mathrm{depth}(I_n(f), R) \geq 2$. *Then*

$$0 \longrightarrow R^n \xrightarrow{f} R^{n+1} \xrightarrow{(\,f_1 \quad -f_2 \quad \cdots \quad (-1)^n f_{n+1}\,)} R \longrightarrow 0$$

is a free resolution of $R/I_n(f)$. *Also, for any non-zero-divisor* $a \in R$, *we get a resolution*

$$0 \longrightarrow R^n \xrightarrow{f} R^{n+1} \xrightarrow[h]{(\,af_1 \quad -af_2 \quad \cdots \quad (-1)^n af_{n+1}\,)} R \longrightarrow 0.$$

of $R/aI_n(f)$.

thm191024j.b

(b) *Conversely, if* $I \leq R$ *with* $I \neq 0$ *such that there exists a free resolution*

$$0 \longrightarrow R^{\beta_2} \xrightarrow{\partial_2^F} R^{\beta_1} \xrightarrow{\partial_1^F} R \longrightarrow 0,$$

of R/I, *then* $\beta_1 = \beta_2 + 1$ *and there exists a non-zero-divisor* $a \in R$ *such that* $I = aI_{\beta_2}(\partial_2^F)$.

ex191029a

EXAMPLE. We saw in the III.B.5 opening this chapter that the complex

$$0 \longrightarrow R^2 \xrightarrow[\partial_2]{\begin{pmatrix} -Z & -Z \\ Y & 0 \\ 0 & X \end{pmatrix}} R^3 \xrightarrow[\partial_1]{(\,XY \quad XZ \quad YZ\,)} R \longrightarrow 0$$

resolves the R-module R/I where $I = (XY\ XZ\ YZ)$. We also saw that the 2×2 minors of ∂_2 are generators of I. By the theorem we also have that

$$0 \longrightarrow R^2 \xrightarrow[\partial_2]{\begin{pmatrix} -Z & -Z \\ Y & 0 \\ 0 & X \end{pmatrix}} R^3 \xrightarrow[\partial_1]{(\,X^2Y^2Z \quad X^2YZ^2 \quad XY^2Z^2\,)} R \longrightarrow 0$$

is a resolution for $R/(X^2Y^2Z,\ X^2YZ^2,\ XY^2Z^2)$.

PROOF. (a) It suffices to prove the last statement of a. We want to apply Buchsbaum-Eisenbud to the sequence

$$0 \longrightarrow R^n \xrightarrow{f} R^{n+1} \xrightarrow{h} R \longrightarrow 0, \qquad \text{(III.B.5.10.1)}$$ eq191030a

which we claim is an R-complex. If $A = (a_{ij})$ is the matrix representing f, then we have

$$\begin{aligned} hf &= \begin{pmatrix} af_1 & -af_2 & \cdots & (-1)^n af_{n+1} \end{pmatrix} (a_{ij}) \\ &= \begin{pmatrix} a\,(f_1a_{11} - f_2a_{21} + \cdots + (-1)^n f_{n+1}a_{n+1,1}) \\ a\,(f_1a_{12} - f_2a_{22} + \cdots + (-1)^n f_{n+1}a_{n+1,2}) \\ \vdots \\ a\,(f_1a_{1,n+1} - f_2a_{2,n+1} + \cdots + (-1)^n f_{n+1}a_{n+1,n+1}) \end{pmatrix}^T \end{aligned}$$

which is a row vector of zeros, since, for instance, the first entry is the product of a and the determinant

$$\begin{vmatrix} a_{11} & a_{11} & a_{12} & \cdots & a_{1,n+1} \\ a_{21} & a_{21} & a_{22} & \cdots & a_{2,n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n+1,1} & a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n+1} \end{vmatrix},$$

which contains a repeated column. Hence (III.B.5.10.1) is an R-complex. In the context of the result by Buchsbaum-Eisenbud, if $i = 2$ then $r_2 = \beta_2 = n$ and

$$\mathrm{depth}(I_{r_2}(f), R) = \mathrm{depth}(I_n(f), R) \geq 2$$

where the inequality holds by assumption. If $i = 1$, then $r_1 = \beta_1 - \beta_2 = (n+1) - n = 1$ and

$$I_{r_1}(h) = I_1\begin{pmatrix} af_1 & \cdots & af_{n+1} \end{pmatrix} = \langle af_1, \ldots, af_{n+1} \rangle = a \cdot \langle f_1, \ldots, f_{n+1} \rangle = a \cdot I_n(f).$$

Hence it now suffices to show that $I_{r_1}(h) = a \cdot I_n(f)$ contains a non-zero-divisor on R. Since $\operatorname{depth}(I_n(f), R) \geq 2$, there exists a non-zero-divisor $b \in I_n(f)$. Since a is a non-zero divisor, so is the product $ab \in a \cdot I_n(f)$.

(b) Assume R/I has a resolution

$$0 \longrightarrow R^{\beta_2} \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R \longrightarrow R/I \longrightarrow 0 \qquad \text{(III.B.5.10.2)}$$ eqn191029b

and that $I \neq 0$. By Theorem III.A.3.8, we know $\beta_1 = \beta_2 + 1$ (here we use our assumption that $R = k[X_1, \ldots, X_d]$). Then since $r_2 = \beta_2$ and $r_1 = \beta_1 - \beta_2 = 1$, by Buchsbaum-Eisenbud we have

$$\operatorname{depth}(I_{\beta_2}(\partial_2), R) \geq 2$$

and

$$\operatorname{depth}(I_1(\partial_1), R) \geq 1.$$

Note that since the determinant of a one-by-one matrix is equal to the lone entry, the ideal $I_1(\partial_1)$ is generated by the entries in the matrix representing ∂_1 and hence $I_1(\partial_1) = \operatorname{Im} \partial_1 = I$. Consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & R^{\beta_2} \arrow[r, "\partial_2"] & R^{\beta_1} \arrow[r] & \operatorname{Im} \partial_1 \arrow[r] & 0 \\
0 \arrow[r] & R^{\beta_2} \arrow[r, "\partial_2"'] \arrow[u, "\mathrm{id}"] & R^{\beta_1} \arrow[r] \arrow[u, "\mathrm{id}"] & I_{\beta_2}(\partial_2) \arrow[r] \arrow[u, dashed, "\exists ! \phi"'] & 0
\end{tikzcd}$$

The top row is exact since (III.B.5.10.2) is exact, the bottom row is exact by Hilbert-Burch a, the first square commutes by construction, and the existence of ϕ follows from the exactness of the rows. Then by the Snake Lemma we have that ϕ is an isomorphism, i.e., $I_{\beta_2}(\partial_2) \cong I$.

We need to show that there is a non-zero-divisor $a \in R$ such that $I = a \cdot I_{\beta_2}(\partial_2)$. Since $\operatorname{depth}(I_{\beta_2}(\partial_2), R) \geq 2$, we know there is a weakly R-regular sequence $g, h \in I_{\beta_2}(\partial_2)$. Since ϕ is a homomorphism we have

$$g \cdot \phi(h) = \phi(gh) = h \cdot \phi(g)$$

and we know $g, h \neq 0$ since the sequence is weakly R-regular. We claim $\phi(g) \in \langle g \rangle$. In $\overline{R} = R/\langle g \rangle$ we have

$$h \cdot \overline{\phi(g)} = \overline{g \cdot \phi(h)} = 0$$

and since h is a non-zero-divisor on $\overline{R}$ by assumption, we also have

$$\overline{\phi(g)} = 0 \in \overline{R},$$

so $\phi(g) \in \langle g \rangle$. Thus since for all $\zeta \in I_{\beta_2}(\partial_2)$ we have

$$g \cdot \phi(\zeta) = \zeta \cdot \phi(g)$$

we also have

$$\phi(\zeta) = \zeta \cdot \frac{\phi(g)}{g}$$

where $\phi(g)/g \in R$ by the claim. Set $a = \phi(g)/g$ note we have shown that $\phi(\zeta) = \zeta \cdot a$ for all $\zeta \in I_{\beta_2}(\partial_2)$, i.e.,

$$I = \operatorname{Im} \phi = a \cdot I_{\beta_2}(\partial_2).$$

□

Buchsbaum-Eisenbud.

ex191029b EXAMPLE III.B.5.11. Set $R = k[X, Y, Z]$ and consider the matrix

$$A = \begin{pmatrix} 0 & X & Y \\ -X & 0 & Z \\ -Y & -Z & 0 \end{pmatrix}.$$

Then we construct generators of an ideal I by taking the square roots of the 2×2 minors of A obtained by deleting the i^{th} row and i^{th} column, $i = 1, 2, 3$, i.e., $I = \langle Z, Y, X \rangle$. Observe that

$$\begin{pmatrix} Z & -Y & X \end{pmatrix} \cdot \begin{pmatrix} 0 & X & Y \\ -X & 0 & Z \\ -Y & -Z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

and we thus have the following R-complex.

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} Z \\ -Y \\ X \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} 0 & X & Y \\ -X & 0 & Z \\ -Y & -Z & 0 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} Z & -Y & X \end{pmatrix}} R \longrightarrow 0$$

It is the Koszul complex $K^R(Z, -Y, X)$! We show how to build resolutions from matrices like A in Theorem III.B.5.30 below.

defn191031a

DEFINITION III.B.5.12. A matrix $A \in M_{n\times n}(R)$ is alternating if $A^T = -A$ and $a_{ii} = 0$ for all $i = 1, \ldots, n$. We denote the set of $n \times n$ alternating matrices in $M_{n\times n}(R)$ by

$$\mathrm{Alt}_n(R) = \{\text{alternating } A \in M_{n\times n}(R)\}.$$

Notice that if 2 is a unit in R (e.g. if $R \supseteq \mathbb{Q}$), then $A^T = -A$ implies that $a_{ii} = 0$ for all $i = 1, \ldots, n$ since $a_{ii} = -a_{ii}$ implies that $2a_{ii} = 0$.

ex191031b

EXAMPLE III.B.5.13. Let $R = k[X, Y, Z]$. The following two matrices are both alternating:

$$A = \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 & X & Y \\ -X & 0 & Z \\ -Y & -Z & 0 \end{pmatrix}.$$

Furthermore, notice that $|A| = X^2$ and

$$|B| = -X \begin{vmatrix} -X & Z \\ -Y & 0 \end{vmatrix} + Y \begin{vmatrix} -X & 0 \\ -Y & -Z \end{vmatrix} = -XYZ + XYZ = 0.$$

thm191031c

THEOREM III.B.5.14 (Cayley). *Let $A \in \mathrm{Alt}_n(R)$. If n is even, then there exists $f \in R$ such that $|A| = f^2$. If n is odd, then $|A| = 0 = 0^2$.*

Before we can prove the above theorem, we need a few more tools.

prop191031d

PROPOSITION III.B.5.15. *If n is odd and $A \in \mathrm{Alt}_n(R)$, then $|A| = 0$.*

PROOF. We split the proof into two cases. For the first case, suppose that 2 is a unit in R. Then we use that n is odd to get

$$|A| = \left|A^T\right| = |-A| = (-1)^n |A| = -|A|.$$

Therefore, $2|A| = 0$. Since 2 is a unit, this implies $|A| = 0$.

For the second case, we do not assume that 2 is a unit in R. Let $A = (a_{ij})$ and set

$$S = \mathbb{Z}[X_{ij} \mid i = 2, \ldots, n, \ j = i+1, \ldots, n],$$

so S is a polynomial ring in $\binom{n-1}{2}$ variables. Then $S \subseteq \mathbb{Q}(X_{ij}) = \mathrm{Frac}(S)$, where $\mathrm{Frac}(S)$ is the field of fractions of S. Define

$$X = \begin{pmatrix} 0 & X_{12} & X_{13} & \cdots & X_{1n} \\ -X_{12} & 0 & X_{23} & \cdots & X_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -X_{1n} & -X_{2n} & -X_{3n} & \cdots & 0 \end{pmatrix} \in \mathrm{Alt}_n(S) \subseteq \mathrm{Alt}_n(\mathrm{Frac}(S)).$$

Then X falls into the first case, so $|X| = 0$. Define a ring homomorphism $\phi : S \to R$ by $\phi(X_{ij}) = a_{ij}$. Then $|A| = \phi(|X|) = \phi(0) = 0$. $\square$

lem191031e

LEMMA III.B.5.16. *If n is even, then*

$$\begin{vmatrix} 0 & b_{12} & 0 & 0 & \cdots & 0 \\ -b_{12} & 0 & b_{23} & 0 & \cdots & 0 \\ 0 & -b_{23} & 0 & b_{34} & \cdots & 0 \\ 0 & 0 & -b_{34} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{vmatrix} = b_{12}^2 b_{34}^2 \cdots b_{n-1,n}^2 = \prod_{\substack{i=1 \\ i \text{ odd}}}^{n-1} b_{i,i+1}^2.$$

PROOF. We prove this by induction. For the base case, consider $n = 2$. Then

$$\begin{vmatrix} 0 & b_{12} \\ -b_{12} & 0 \end{vmatrix} = b_{12}^2.$$

Now suppose that the result holds for $(n-2)\times(n-2)$ matrices of the given form. Then expand along the first column, then along the first row to obtain the first equality in the next display.

$$\begin{vmatrix} 0 & b_{12} & 0 & 0 & \cdots & 0 \\ -b_{12} & 0 & b_{23} & 0 & \cdots & 0 \\ 0 & -b_{23} & 0 & b_{34} & \cdots & 0 \\ 0 & 0 & -b_{34} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{vmatrix} = b_{12}^2 \begin{vmatrix} 0 & b_{34} & \cdots & 0 \\ -b_{34} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix} = b_{12}^2 b_{34}^2 \cdots b_{n-1,n}^2$$

The second equality follows from the inductive hypothesis. $\square$

lem191031f

LEMMA III.B.5.17. *Let $\mathbb{Q} \subseteq K$ be a field extension and let $A \in \mathrm{Alt}_n(K)^\times$. Then there exists $C \in M_{n\times n}(K)^\times$ such that $B = C^T AC \in \mathrm{Alt}_n(K)^\times$ has the form*

$$B = \begin{pmatrix} 0 & b_{12} & 0 & 0 & \cdots & 0 \\ -b_{12} & 0 & b_{23} & 0 & \cdots & 0 \\ 0 & -b_{23} & 0 & b_{34} & \cdots & 0 \\ 0 & 0 & -b_{34} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

PROOF. If n is odd, then $|A| = 0$ by Proposition III.B.5.15, which contradicts that A is invertible. Therefore n must be even. Define $f : K^n \times K^n \to K$ by $f(v, w) = v^T Aw \in M_{1\times 1}(K) \cong K$. Then f is bilinear and

$$f(w, v) = w^T Av = (w^T Av)^T = v^T A^T w = -v^T Aw = -f(v, w),$$

so f is skew-symmetric. Therefore we have $f(v, v) = 0$ because 2 is a unit in $\mathbb{Q} \subseteq K$.

clm191031f.1

CLAIM (1). *There exists $v, w \in K^n$ such that $f(v, w) \neq 0$.*

PROOF. We show that there exist $1 \le i, j \le n$ such that $f(e_i, e_j) \neq 0$. Notice that

$$f(e_i, e_j) = e_i^T Ae_j = a_i^T \operatorname{Col}(A, j) = a_{ij}.$$

Since A is invertible, we must have $a_{ij} \neq 0$ for some $1 \le i, j \le n$, so $f(e_i, e_j) \neq 0$ for those i and j. ✓

Now replace v with $\frac{1}{f(v,w)} v$ to assume that $f(v, w) = 1$. If there exists $\alpha \in K$ such that $v = \alpha w$, then

$$1 = f(v, w) = f(\alpha w, w) = \alpha f(w, w) = 0,$$

which is clearly contradictory. This implies that v and w are linearly independent. Set $V_1 = \mathrm{Span}_K(v, w) \subseteq K^n$, so a basis for V_1 is $\{v, w\}$. Set

$$V_2 = \text{“}V_1^{\perp_f}\text{”} = \{y \in K^n \mid f(y, z) = 0\ \forall\ z \in V_1\} \subseteq K^n.$$

clm191031f.2

CLAIM (2). *$K^n = V_1 \oplus V_2$.*

PROOF. We need to prove that $V_1 \cap V_2 = 0$ and for all $t \in K^n$, $t = z + y$ for some $z \in V_1$ and $y \in V_2$.

(1) We first show that the intersection is trivial. Let $u \in V_1 \cap V_2$, so $u = av + bw$ for $a, b \in K$. Then

$$0 = f(u, v) = f(av + bw, v) = a \underbrace{f(v, v)}_{=0} + b \underbrace{f(w, v)}_{=-1} = -b,$$

so $b = 0$. By a similar argument, $a = 0$. Therefore, $u = av + bw = 0$.

(2) Let $t \in K^n$. We want to find $z \in V_1$ and $y \in V_2$ such that $t = z + y$. Set $z = f(t,w)v - f(t,v)w \in \operatorname{Span}(v,w) = V_1$. Then $t = z + (t-z)$, so it suffices to show that $t - z \in V_2$. To do this, it suffices to show that $f(t-z,v) = 0$ and $f(t-z,w) = 0$:

$$\begin{aligned} f(t-z,v) &= f(t - (f(t,w)v - f(t,v)w), v) \\ &= f(t,v) - f(t,w)\underbrace{f(v,v)}_{=0} + f(t,v)\underbrace{f(w,v)}_{=-1} \\ &= f(t,v) - f(t,v) = 0. \end{aligned}$$

By a similar argument, $f(t-z,w) = 0$. ✓

Let $v_3, \ldots, v_n$ be a basis of V_2, so $v, w, v_3, \ldots, v_n$ is a basis of K^n. Define $f_2 = f|_{V_2 \times V_2} : V_2 \times V_2 \to K$, and let $B_1 = (b_{ij}) \in M_{(n-2)\times(n-2)}(K)$ be the matrix representing f_2 with respect to the basis $v_3, \ldots, v_n$. Then

$$b_{ij} = f(v_{i+2}, v_{j+2}) = -f(v_{j+2}, v_{i+2}) = -b_{ji},$$

so $B_1 \in \operatorname{Alt}_n(K)$. Set $P = (v|w|v_3|\cdots|v_n)$.

clm191031f.3 CLAIM (3).

$$P^T A P = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & B_1 \end{array}\right)$$

PROOF. Consider each entry for the above matrix as follows:

$$\begin{aligned} (P^T A P)_{ij} &= e_i^T P^T A P e_j \\ &= (Pe_i)^T A (Pe_j) \\ &= \operatorname{Col}(P,i)^T A \operatorname{Col}(P,j). \end{aligned}$$

If $i, j \geq 3$, then

$$(P^T A P)_{ij} = v_i^T A v_j = f(v_i, v_j) = b_{i-2,j-2}.$$

The other cases for i and j are computed similarly. ✓

In particular, since P and A are invertible, then $P^T A P$ is also invertible, so B_1 must be invertible since it is a submatrix of $P^T A P$. We can repeat this process as many times as needed to find $Q \in M_{n\times n}(K)^\times$ such that $C = QP$ and $B = C^T A C$ has the desired form. □

lem191031g LEMMA III.B.5.18. *Assume that R is a unique factorization domain and let $g, h \in R$ be such that $h \neq 0$ and $(g/h)^2 \in R$. Then $g/h \in R$.*

PROOF. Since R is a unique factorization domain, we can assume that g/h is in lowest terms. In other words, g and h have no common prime factors. Set $f = g^2/h^2 \in R$, so $h^2 f = g^2$. If h is not a unit, then it has a prime factor p, so

$$p \mid h \quad \Rightarrow \quad p \mid g^2 \quad \Rightarrow \quad p \mid g.$$

This is a contradiction since g and h have no common prime factors. Therefore, h is a unit, so $g/h \in R$. □

Now we are finally ready to prove the result by Cayley.

PROOF OF THEOREM III.B.5.14. If n is odd, then the conclusion follows from Proposition III.B.5.15, so we assume n is even. We proceed by way of cases. As a special case we consider

$$R_0 = \mathbb{Z}[X_{ij} \mid i = 1, \ldots, n-1;\ j = i+1, \ldots, n]$$

and

$$X = \begin{pmatrix} 0 & X_{12} & \cdots & X_{1n} \\ -X_{12} & 0 & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -X_{1n} & -X_{2n} & \cdots & 0 \end{pmatrix}.$$

By Lemma III.B.5.17 there exists a matrix $C \in M_{n\times n}(K)^\times$ such that $C^T X C = B$ is tridiagonal and alternating, where $K = \operatorname{frac}(R_0)$. Then by Lemma III.B.5.16 we have

$$(b_{12} b_{34} \cdots b_{n-1,n})^2 = |B| = |C^T||X||C| = |C|^2|X|$$

and since C is invertible we have

$$|X| = \frac{1}{|C|^2}(b_{12}b_{34}\cdots b_{n-1,n})^2 \in K.$$

Since $|X| \in R_0$, by Lemma III.B.5.18 we have

$$f = \frac{b_{12}b_{34}\cdots b_{n-1,n}}{|C|} \in R_0,$$

which proves this case.

In the general case, we let $A \in \mathrm{Alt}_n(R)$ and consider the ring homomorphism $\phi\colon R_0 \to R$ given by $\phi(X_{ij}) = a_{ij}$. There exists an element $F \in R_0$ such that $|X| = F^2$ and we observe that

$$|A| = \phi(|X|) = \phi(F^2) = \phi(F)^2.$$

Taking $f = \phi(F)$, this proves the general case. □

note191105a NOTE III.B.5.19. Let $A \in \mathrm{Alt}_n(R)$ and $f \in R$ such that $f^2 = |A|$. Then f is not unique, not even up to a sign, in general. For instance, if $x \in R$ such that $x^2 = 0$, then for all $\alpha \in R$ we have

$$\begin{vmatrix} 0 & x \\ -x & 0 \end{vmatrix} = x^2 = 0 = (\alpha x)^2.$$

This motivates the following question: how does one choose f well?

prop191105b PROPOSITION III.B.5.20. *Let R be an integral domain and let $f, g \in R$ such that $f^2 = g^2$. Then $f = \pm g$.*

PROOF. This follows from the equality $0 = g^2 - f^2 = (g - f)(g + f)$. □

prop191105c PROPOSITION III.B.5.21. *Let D be an integral domain and set $R_0 = D[X_1, \ldots, X_d]$.*

rop191105c.a (a) *If $f \in R_0$ is such that $0 \neq f^2$ is homogeneous of degree n, then f is homogeneous, n is even, and $\deg(f) = n/2$.*

rop191105c.b (b) *If $f, g \in R_0$ are such that $0 \neq fg$ is homogeneous of degree n, then f and g are each homogeneous and $\deg(f) + \deg(g) = n$.*

PROOF. (b) We write $f = f_i + \cdots + f_j$ and $g = g_p + \cdots + g_q$ such that $i \leq j$, $p \leq q$, and where f_ℓ and g_m are homogeneous of degree ℓ and m, respectively, and $f_i, f_j, g_p, g_q \neq 0$. Then since we are in a domain we know $f_i g_p$ and $f_j g_q$ are each nonzero. Note in the product fg these are the terms of lowest and highest possible degree, respectively. Since fg is homogeneous this implies $i = j$ and $p = q$, implying f and g are each homogeneous and $\deg(f) + \deg(g) = \deg(fg)$.

(a) This follows directly from part (b). □

note191105d NOTE III.B.5.22. Assume $q \in \mathbb{N}$ and $n = 2q$, and consider the ring R_0 and the matrix $X \in \mathrm{Alt}_n(R)$ as in the proof of Cayley's Theorem. Let $f \in R_0$ be such that $f^2 = |X|$. Then

$$f^2 = |X| = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) x_{1,\sigma(1)} x_{2,\sigma(2)} \cdots x_{n,\sigma(n)} \qquad \text{(III.B.5.22.1)}$$ eqn191105a

where

$$x_{ij} = \begin{cases} X_{ij} & \text{if } i < j \\ 0 & \text{if } i = j \\ -X_{ij} & \text{if } i > j. \end{cases}$$

Consider $\sigma_0 = (1\ 2)(3\ 4)\cdots(n{-}1\ \ n) \in S_n$ with $\mathrm{sgn}(\sigma_0) = (-1)^q$. Then the term of the sum in (III.B.5.22.1) associated with σ_0 is

$$\begin{aligned} (-1)^q X_{12}(-X_{12}) X_{34}(-X_{34}) \cdots X_{n-1,n}(-X_{n-1,n}) &= (-1)^{2q}(X_{12}X_{34}\cdots X_{n-1,n})^2 \\ &= (X_{12}X_{34}\cdots X_{n-1,n})^2. \end{aligned}$$

One can check that σ_0 is the unique element of S_n such that its associated term uses $(X_{12}X_{34}\cdots X_{n-1,n})^2$. Therefore f must contain $\pm X_{12}X_{34}\cdots X_{n-1,n}$ and we multiply f by -1 if necessary to assume f contains $X_{12}X_{34}\cdots X_{n-1,n}$.

depf191105f

DEPFINITION III.B.5.23. Using the notation of Note III.B.5.22, the pfaffian of X is $\mathrm{pf}\, X = f$ where the coefficient of $X_{12}X_{34}\cdots X_{n-1,n}$ is 1. Note $\mathrm{pf}(X)^2 = |X|$. Let $A \in \mathrm{Alt}_n(R)$ and define the ring homomorphism $\phi\colon R_0 \to R$ by $\phi(X_{ij}) = a_{ij}$. The pfaffian of A is $\mathrm{pf}(A) = \phi(\mathrm{pf}(X))$.

prop191105e

PROPOSITION III.B.5.24. *Observe that in the notation of Definition III.B.5.23 we have*

$$\mathrm{pf}(A)^2 = \phi(\mathrm{pf}(X))^2 = \phi(\mathrm{pf}(X)^2) = \phi(|X|) = |A|.$$

The next fact we state without proof.

fact191105g

FACT III.B.5.25. *Let $q \in \mathbb{N}$ and $n = 2q$, and consider R_0 and X as in Note III.B.5.22. Then*

$$\mathrm{pf}(X) = \frac{1}{2^q \cdot q!} \cdot \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{q} X_{\sigma(2i-1),\sigma(2i)}.$$

Set

$$\Pi_n = \{(i_1, j_1, i_2, j_2, \ldots, i_q, j_q) \in \mathbb{N}^n \mid \{i_1, j_1, \ldots, i_q, j_q\} = [n];\ i_1 < i_2 < \cdots < i_q;\ i_m < j_m,\ \forall m = 1, \ldots, q\}.$$

Then we have

$$\begin{aligned} \mathrm{pf}(X) &\overset{(1)}{=} \sum_{(i_1,\ldots,j_q)\in\Pi_n} \mathrm{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ i_1 & j_1 & i_2 & j_2 & \cdots & i_q & j_q \end{pmatrix} X_{i_1,j_1} X_{i_2,j_2} \cdots X_{i_q,j_q} \\ &\overset{(2)}{=} \sum_{j=1, j\neq i}^{n} (-1)^{i+j+1+\theta(j-i)} \cdot a_{ij}\, \mathrm{pf}(A_{ij}) \end{aligned}$$

where

$$\theta(j-i) = \begin{cases} 0 & \text{if } j - i > 0 \\ 1 & \text{if } j - i < 0 \end{cases}$$

and $A_{ij} \in \mathrm{Alt}_{n-2}(R)$ is obtained from A by deleting the i^{th} row and column as well as the j^{th} row and column.

ex191105h

EXAMPLE III.B.5.26. Consider the ring $\mathbb{Z}[a, b, c, x, y, z]$ and the alternating matrix

$$X = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & x & y \\ -b & -x & 0 & z \\ -c & -y & -z & 0 \end{pmatrix}.$$

Expanding $|X|$ along the first column we have

$$|X| = a \cdot \begin{vmatrix} a & b & c \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix} - b \cdot \begin{vmatrix} a & b & c \\ 0 & x & y \\ -y & -z & 0 \end{vmatrix} + c \cdot \begin{vmatrix} a & b & c \\ 0 & x & y \\ -x & 0 & z \end{vmatrix}.$$

Expanding each of these determinants along the top row we find

$$\begin{aligned} |X| &= a^2z^2 - 2abyz + 2acxz - 2bcxy + b^2y^2 + c^2x^2 \\ &= (az - by + cx)^2. \end{aligned}$$

Since $X_{12} = a$ and $X_{34} = z$ we want the coefficient of az to be $+1$, as we have above. Hence $\mathrm{pf}(X) = az - by + cx$.

Now we demonstrate equalities (1) and (2) from Fact III.B.5.25 for X. First, note that

$$\Pi_4 = \{(1, 2, 3, 4),\ (1, 3, 2, 4),\ (1, 4, 2, 3)\}$$

and we compute the signs of the corresponding elements of S_4.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1) \qquad\qquad \mathrm{sgn} = 1$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = (2\ 3) \qquad\qquad \mathrm{sgn} = -1$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = (2\ 4\ 3) \qquad\qquad \mathrm{sgn} = 1$$

Therefore equality (1) gives

$$\operatorname{pf}(X) = X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23} = az - by + cx$$

which agrees with our initial computation.

Second, we confirm (2) for this example. We choose $i = 3$ Then (2) gives

$$\begin{aligned}\operatorname{pf}(X) &= \sum_{j=1, j\neq 3}^{4} (-1)^{3+j+1+\theta(j-3)} a_{3j} \operatorname{pf}(A_{3j}) \\ &= (-1)^{3+1+1+1} a_{31} \operatorname{pf}\left(\begin{smallmatrix} 0 & y \\ -y & 0 \end{smallmatrix}\right) + (-1)^{3+2+1+1} a_{32} \operatorname{pf}\left(\begin{smallmatrix} 0 & c \\ -c & 0 \end{smallmatrix}\right) + (-1)^{3+4+1} a_{34} \operatorname{pf}\left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right) \\ &= -by + xc + za,\end{aligned}$$

which also agrees with our initial computation.

defn191107a

DEFINITION III.B.5.27. Let $A \in \operatorname{Alt}_n(R)$. For all $i = 1, \ldots, n$, let A_i denote the $(n-1) \times (n-1)$ alternating matrix found by deleting the i^{th} row and i^{th} column of A. Set

$$\operatorname{Pf}_{n-1}(A) = \langle \operatorname{pf}(A_1), \ldots, \operatorname{pf}(A_n) \rangle .$$

Then define $P \in M_{1\times n}(R)$ by

$$P = P(A) = \begin{pmatrix} \operatorname{pf}(A_1) & -\operatorname{pf}(A_2) & \cdots & (-1)^{n-1} \operatorname{pf}(A_n) \end{pmatrix}$$

and

$$F = F(A) = \left(0 \longrightarrow R \xrightarrow{P^T} R^n \xrightarrow{A} R^n \xrightarrow{P} R \longrightarrow 0 \right).$$

ex191107b

EXAMPLE III.B.5.28. (a) If n is even, then $\operatorname{Pf}_{n-1}(A) = 0$ since by Proposition III.B.5.15, we have $\operatorname{pf}(A_i) = 0$ for all $i = 1, \ldots, n$.

(b) Let $n = 3$. Then $F(A)$ is the Koszul complex found in Example III.B.5.11.

prop191107c

PROPOSITION III.B.5.29. *If $A \in \operatorname{Alt}_n(R)$, then $F(A)$ is an R-complex with $H_0(F(A)) = R/\operatorname{Pf}_{n-1}(A)$.*

PROOF. Let $i \in [n]$ and set

$$\widetilde{A} = \begin{bmatrix} 0 & \operatorname{Row}(A,i) \\ \operatorname{Col}(A,i) & A \end{bmatrix} = \left(\begin{array}{c|ccccc} 0 & a_{i1} & \cdots & a_{ii} & \cdots & a_{in} \\ \hline a_{1i} = -a_{i1} & a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{ii} = -a_{ii} & a_{i1} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{ni} = -a_{in} & a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{array}\right) \in \operatorname{Alt}_{n+1}(R).$$

Since A is alternating, we have $a_{ii} = 0$. Therefore $\widetilde{A}$ has a repeated column, so $|A| = 0$; since $\widetilde{A}$ is alternating, it follows that $\operatorname{pf}(\widetilde{A}) = 0$. Using Fact III.B.5.25, we have

$$0 = \operatorname{pf}(\widetilde{A}) = \sum_{j=2}^{n+1} (-1)^j \widetilde{a}_{ij} \operatorname{pf}(\widetilde{A}_{1j}) = \sum_{j=1}^{n} (-1)^{j-1} a_{ij} \operatorname{pf}(A_j) = \operatorname{Row}(A,i) \cdot P^T.$$

This is true for any i, so $AP^T = 0$. Also

$$0 = (AP^T)^T = PA^T = -PA,$$

so $PA = 0$ as well. Therefore, $F(A)$ is an R-complex. Furthermore, notice that

$$H_0(F(A)) = R/\operatorname{Im} P = R/\operatorname{Pf}_{n-1}(A).$$

□

thm191107d

THEOREM III.B.5.30 (Buchsbaum-Eisenbud). *Assume R is local and noetherian with maximal ideal $\mathfrak{m}$.*

thm191107d.a

(a) *Let $A \in \operatorname{Alt}_n(R)$ such that $I = \operatorname{Pf}_{n-1}(A)$ satisfies* $\operatorname{depth}(I,R) \geq 3$ *and $a_{ij} \in \mathfrak{m}$. Then $F(A)$ is a free resolution of R/I and n is odd.*

thm191107d.b

(b) *Conversely, if $I \lneq R$ satisfies* $\operatorname{depth}(I,R) \geq 3$ *and R/I has a free resolution*

$$0 \longrightarrow R \longrightarrow R^n \longrightarrow R^n \longrightarrow R \longrightarrow 0,$$

then there exists $A \in \operatorname{Alt}_n(R)$ such that $I = \operatorname{Pf}_{n-1}(A)$ and $F(A)$ is a free resolution of R/I. In particular, n is odd.

note191107e NOTE III.B.5.31. For $A \in \mathrm{Alt}_n(R)$, verifying that $\mathrm{depth}(\mathrm{Pf}_{n-1}(A), R) \geq 3$ can be hard.

PROOF OF BUCHSBAUM-EISENBUD(a). We use the Buchsbaum Eisenbud acyclicity criterion III.B.5.8, beginning with the $i = 3$ and $i = 1$ cases.

$i = 3$: In this case, we have $r_3 = 1$. Then

$$I_{r_3}(\partial_3^F) = I_{r_3}(P^T) = I_1(P^T) = \langle P_{ij}\rangle = \langle \mathrm{pf}(A_1), \dots, \mathrm{pf}(A_n)\rangle = I.$$

By assumption, I satisfies $\mathrm{depth}(I, R) \geq 3$, so $\mathrm{depth}(I_{r_3}(\partial_3^F), R) \geq 3$ as well.

$i = 1$: In this case, we have $r_1 = n - (n - 1) = 1$, so $I_{r_1}(\partial_1^F) = I_1(P) = I$ via the same argument as for the $i = 3$ case. Therefore by assumption, $\mathrm{depth}(I_{r_1}(\partial_1^F), R) \geq 3 \geq 1$.

$i = 2$: In this case, we have $r_2 = n - 1$, so $I_{r_2}(\partial_2^F) = I_{n-1}(A)$. This ideal is related to $\mathrm{Pf}_{n-1}(A)$, but they are not equal. In order to complete this case, we need to show that

$$\mathrm{depth}(\mathrm{Pf}_{n-1}(A), R) = \mathrm{depth}(I_{n-1}(A), R).$$

For this, we need to build up a few more results.

lem191107f LEMMA III.B.5.32. *Set $R = \mathbb{Z}[X_{ij} \mid i, j = 1, \dots, n]$ and $X = (X_{ij})$. Then $|X|$ is prime in R.*

PROOF. Notice that $|X|$ is a homogeneous polynoimal of degree n. A result of Gauss tells us that R is a unique factorization domain, so it suffices to show that $|X|$ is irreducible in R. By Proposition III.B.5.21, if $|X|$ factors in R, then it factors as $|X| = fg$ where f and g are both homogeneous polynomials. Furthermore, since $\mathbb{Z}$ is an integral domain, $\deg_{X_{ij}}$ is additive on products. In particular, X_{11} appears in $|X|$, so X_{11} must appear in f or in g, and moreover

$$1 = \deg_{X_{11}}(|X|) = \deg_{X_{11}}(f) + \deg_{X_{11}}(g).$$

By symmetry, we can assume without loss of generality that $\deg_{X_{11}}(f) = 1$ and $\deg_{X_{11}}(g) = 0$.

CLAIM. $\deg_{X_{1j}}(g) = 0$, *i.e., X_{1j} does not appear in g for any $j = 1, \dots, n$.*

PROOF. By way of contradiction, suppose that $\deg_{X_{1j}}(g) > 0$. Then $\deg_{X_{1j}}(f) = 0$ and $\deg_{X_{1j}}(g) = 1$. Then we can rewrite $|X|$ as

$$|X| = fg = (f_0 + f_1X_{11})(g_0 + g_1X_{1j}),$$

where X_{11} does not appear in f_i and X_{1j} does not appear in g_i for $i = 0, 1$. Multiplying out the above product gives us

$$|X| = f_0g_0 + f_0g_1X_{1j} + f_1g_0X_{11} + f_1g_1X_{11}X_{1j}.$$

Notice that $f_1 \neq 0$ and $g_1 \neq 0$, so the final term in the above equation is non-zero. Since X_{11} does not appear in g, then X_{11} does not appear in g_i for $i = 0, 1$ and since X_{1j} does not appear in f, then X_{1j} does not appear in f_i for $i = 0, 1$. Therefore we have

$$|X| = \underbrace{f_0g_0}_{\text{no } X_{11}X_{1j}} + \underbrace{\underbrace{f_1g_0}_{\text{no } X_{1j}} X_{11}}_{\text{no } X_{11}X_{1j}} + \underbrace{\underbrace{f_0g_1}_{\text{no } X_{11}} X_{1j}}_{\text{no } X_{11}X_{1j}} + \underbrace{f_1g_1X_{11}X_{1j}}_{X_{11}X_{1j} \text{ appears}}.$$

Therefore there is no cancellation, so $|X|$ has a term with $X_{11}X_{1j}$, which contradicts the fact that $|X|$ only contains terms of the form $X_{1*}X_{2*}\cdots X_{n*}$. Therefore, X_{1j} does not appear in g for any $j = 1, \dots, n$. ✓

Through a similar argument, we can show g has no X_{ij} for any $i, j = 1, \dots, n$. So g is a constant polynomial. Furthermore, the terms of $|X|$ each have a coefficient of ± 1, so $g = \pm 1$. Therefore $|X|$ is irreducible. □

lem191107g LEMMA III.B.5.33. *Let $A \in M_{n\times n}(R)$ and*

$$\mathrm{Adj}(A)_{ji} = (-1)^{i+j}[1, \dots, \widehat{i}, \dots, n \mid 1, \dots, \widehat{j}, \dots, n]_A,$$

where we recall that $[1, \dots, \widehat{i}, \dots, n \mid 1, \dots, \widehat{j}, \dots, n]_A$ denotes the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of A. It follows that $A \cdot \mathrm{Adj}(A) = |A|\, I_n = \mathrm{Adj}(A) \cdot A$. Assume that $|A| = 0$. Then for all i, j, p, q,

$$\mathrm{Adj}(A)_{ij}\,\mathrm{Adj}(A)_{pq} = \mathrm{Adj}(A)_{iq}\,\mathrm{Adj}(A)_{pj}.$$

In particular,

$$\mathrm{Adj}(A)_{ij}\,\mathrm{Adj}(A)_{ji} = \mathrm{Adj}(A)_{ii}\,\mathrm{Adj}(A)_{jj}.$$

Proof. The conclusion is trivial if $\operatorname{Adj}(A) = 0$, so assume without loss of generality that $\operatorname{Adj}(A) \neq 0$. First we prove a special case. Assume R is an integral domain and let $K = \operatorname{Frac}(R)$. Since $\operatorname{Adj}(A) \neq 0$ there exists some non-zero size-$(n-1)$ minor of A. The columns used for that minor must be linearly independent over K and therefore $\dim_K \operatorname{Col}_K(A) \geq n-1$. Since the determinant of A is zero, the columns of A are linearly dependent, i.e., $\dim_K \operatorname{Col}_K(A) \leq n-1$. Hence $\dim_K \operatorname{Col}_K(A) = n-1$ and therefore by rank-nullity any two vectors in $\operatorname{Null}_K(A)$ are linearly dependent over K. Now since $A \cdot \operatorname{Adj}(A) = |A| \cdot I_n = 0$, we know every column of $\operatorname{Adj}(A)$ is in $\operatorname{Null}_K(A)$, i.e., every two columns of $\operatorname{Adj}(A)$ are linearly dependent over K. Thus every size-2 minor of $\operatorname{Adj}(A)$ using rows i, p and columns j, q gives the desired result.

Now we prove the general case. Set $R_1 = \mathbb{Z}[X_{ij} \mid i, j = 1, \ldots, n]$ and set $X = (X_{ij}) \in M_{n\times n}(R_1)$. By Lemma III.B.5.32 we know $|X| \in R_1$ is prime and therefore $R_2 = R_1/\langle |X| \rangle$ is an integral domain. Set $x_{ij} = \overline{X_{ij}} \in R_2$ and $x = (x_{ij}) \in M_{n\times n}(R_2)$, and note $|x| = \overline{|X|} = 0$ in R_2. Now, by the special case we have

$$\operatorname{Adj}(x)_{ij} \operatorname{Adj}(x)_{pq} = \operatorname{Adj}(x)_{iq} \operatorname{Adj}(x)_{pj}.$$

Recall the ring homomorphism $\phi\colon R_1 \to R$ given by $\phi(X_{ij}) = a_{ij}$ and note that $\phi(|X|) = |A| = 0$. Therefore there exists a unique, well-defined ring homomorphism $\overline{\phi}$ making the following diagram commute

$$\begin{tikzcd}
R_1 \arrow[r, "\phi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & R \\
R_2 \arrow[ur, dashed, "\exists!\overline{\phi}"'] &
\end{tikzcd}$$

where π is the natural surjection. Furthermore, we have $\overline{\phi}(x_{ij}) = \phi(X_{ij}) = a_{ij}$ for all i, j and

$$\operatorname{Adj}(A)_{ij} = \phi(\operatorname{Adj}(X)_{ij}) = \overline{\phi}(\operatorname{Adj}(x)_{ij}).$$

Thus we conclude as follows.

$$\begin{aligned}
\operatorname{Adj}(A)_{ij} \operatorname{Adj}(A)_{pq} &= \overline{\phi}(\operatorname{Adj}(x)_{ij})\overline{\phi}(\operatorname{Adj}(x)_{pq}) \\
&= \overline{\phi}(\operatorname{Adj}(x)_{ij} \operatorname{Adj}(x)_{pq}) \\
&= \overline{\phi}(\operatorname{Adj}(x)_{iq} \operatorname{Adj}(x)_{pj}) \\
&= \overline{\phi}(\operatorname{Adj}(x)_{iq})\overline{\phi}(\operatorname{Adj}(x)_{pj}) \\
&= \operatorname{Adj}(A)_{iq} \operatorname{Adj}(A)_{pj}
\end{aligned}$$

□

lem191112a

Lemma III.B.5.34. *Let $A \in \operatorname{Alt}_n(R)$ with $n = 2q+1$ for some $q \in \mathbb{N}$. Assume $\operatorname{Pf}_{n-1}(A) \neq 0$. Then*

$$\operatorname{rad}(\operatorname{Pf}_{n-1}(A)) = \operatorname{rad}(I_{n-1}(A)).$$

Proof. For the forward containment, it suffices to show that $\operatorname{pf}(A_i)^2 \in I_{n-1}(A)$ for each $i = 1, \ldots, n$. Since $\operatorname{pf}(A_i)^2 = |A_i|$, which is a size-$(n-1)$ minor of A, this is immediate.

For the reverse containment, we need to show that for every $i, j \in [n]$ we have $(\operatorname{Adj}(A)_{ij})^2 \in \operatorname{Pf}_{n-1}(A)$. Since n is odd, we know $|A| = 0$. Then by Lemma III.B.5.32 we have

$$\begin{aligned}
(\operatorname{Adj}(A)_{ij})^2 &= -\operatorname{Adj}(A)_{ij} \operatorname{Adj}(A)_{ji} \\
&= -\operatorname{Adj}(A)_{ii} \operatorname{Adj}(A)_{jj} \\
&= -|A_i||A_j| \\
&= -\operatorname{pf}(A_i)^2 \operatorname{pf}(A_j)^2 \in \operatorname{Pf}_{n-1}(A)
\end{aligned}$$

where the first equality holds since A is alternating. □

prop191112b

Proposition III.B.5.35. *Let $I, J \leq R$ be ideals such that $\operatorname{rad}(I) = \operatorname{rad}(J)$. Then $\operatorname{depth}(I, R) = \operatorname{depth}(J, R)$.*

Proof. If $I = R$, then $\operatorname{rad}(J) = \operatorname{rad}(I) = R$, so $J = R$. Therefore we assume without loss of generality that I and J are proper ideals. Let $\mathbf{f} = f_1, \ldots, f_n \in I$ be a weakly R-regular sequence.

clm191112b.1

Claim (1). *If $r_1, \ldots, r_n \in R$ such that $\sum_{i=1}^n f_i r_i = 0$, then $r_i \in \langle \mathbf{f} \rangle$ for all $i \in [n]$.*

PROOF. We induct on n. Suppose $n=1$. Then $f_1r_1=0$ implies $r_1=0\in\langle\mathbf{f}\rangle$, since f_1 is a non-zero-divisor. We therefore proceed with the inductive step. Then by assumption we have $f_nr_n=-\sum_{i=1}^{n-1}f_ir_i$, implying $f_n\overline{r_n}=0$ in the ring $R/\langle f_1,\ldots,f_{n-1}\rangle$. Since f_n is a non-zero-divisor on $R/\langle f_1,\ldots,f_{n-1}\rangle$, this implies $r_n\in\langle f_1,\ldots,f_{n-1}\rangle\subseteq\langle\mathbf{f}\rangle$. So let $s_1,\ldots,s_{n-1}\in R$ be such that $r_n=\sum_{i=1}^{n-1}f_is_i$. Substituting we have

$$0=\sum_{i=1}^{n}f_ir_i=\sum_{i=1}^{n-1}f_ir_i+f_n\cdot\sum_{i=1}^{n-1}f_is_i=\sum_{i=1}^{n-1}f_i(r_i+f_ns_i).$$

Then the inductive hypothesis implies

$$r_i+f_ns_i\in\langle f_1,\ldots,f_{n-1}\rangle\subseteq\langle\mathbf{f}\rangle$$

for each $i=1,\ldots,n-1$. Then since each $f_ns_i\in\langle\mathbf{f}\rangle$, it follows that $r_i\in\langle\mathbf{f}\rangle$ for $i=1,\ldots,n-1$. ✓

clm191112b.2 CLAIM (2). *Let $m\in\mathbb{Z}_{m\geq 1}$. Then $\mathbf{f}^{(m)}:=f_1^m,f_2,\ldots,f_n$ is a weakly R-regular sequence.*

PROOF. We induct on m. The base case $m=1$ holds by assumption, so we proceed with the inductive step. Assume $m\geq 2$ and $\mathbf{f}^{(m-1)}$ is weakly R-regular. Since f_1 is a non-zero-divisor on R by assumption, it follows that f_1^m is a non-zero-divisor on R as well. For $i\geq 2$ set $\overline{R}=R/\langle f_1^m,f_2,\ldots,f_{i-1}\rangle$ and we need to show that f_i is a non-zero-divisor on $\overline{R}$. Let $\overline{r}\in\overline{R}$ be such that $f_i\overline{r}=0\in\overline{R}$. Then $r\in R$ satisfies $f_ir\in\langle f_1^m,f_2,\ldots,f_{i-1}\rangle$ and we let $t_1,\ldots,t_{i-1}\in R$ such that

$$f_ir=t_1f_1^m+\sum_{j=2}^{i-1}t_jf_j=(t_1f_1)f_1^{m-1}+\sum_{j=2}^{i-1}t_jf_j. \qquad \text{(III.B.5.35.1)}$$ eqn191112b.1

Since $\mathbf{f}^{(m-1)}$ is weakly R-regular we know $r\in\left\langle f_1^{m-1},f_2,\ldots,f_{i-1}\right\rangle$ and thus there exist $u_1,\ldots,u_{i-1}\in R$ such that

$$r=u_1f_1^{m-1}+\sum_{j=2}^{i-1}u_jf_j.$$

Rearranging (III.B.5.35.1) and substituting we obtain the following.

$$\begin{aligned}0&=f_ir-t_1f_1^m-\sum_{j=2}^{i-1}t_jf_j\\&=u_1f_if_1^{m-1}+\sum_{j=2}^{i-1}f_iu_jf_j-t_1f_1^m-\sum_{j=2}^{i-1}t_jf_j\\&=f_1^{m-1}(u_1f_i-t_1f_1)+\sum_{j=2}^{i-1}f_j(f_iu_j-t_j)\end{aligned}$$

Since $f_1^{m-1},f_2,\ldots,f_{i-1}$ is weakly R-regular, by Claim (1) we have

$$u_1f_i-f_1t_1,f_iu_j-t_j\in\left\langle f_1^{m-1},f_2,\ldots,f_{i-1}\right\rangle$$

for all $j=2,\ldots,i-1$. Therefore $u_1f_i\in\langle f_1,f_2,\ldots,f_{i-1}\rangle$ and since $f_1,\ldots,f_{i-1}$ is weakly R-regular, we know $u_1\in\langle f_1,\ldots,f_{i-1}\rangle$. Hence $u_1=\sum_{j=1}^{i-1}v_jf_j$ for some $v_1,\ldots,v_{i-1}\in R$ and we have

$$\begin{aligned}r&=u_1f_1^{m-1}+\sum_{j=2}^{i-1}u_jf_j\\&=f_1^{m-1}\sum_{j=1}^{i-1}v_jf_j+\sum_{j=2}^{i-1}u_jf_j\\&=v_1f_1^m+\sum_{j=2}^{i-1}f_j(f_1^{m-1}v_j+u_j)\in\langle f_1^m,f_2,\ldots,f_{i-1}\rangle.\end{aligned}$$

Therefore $\overline{r}=0\in\overline{R}$ and thus f_i is a non-zero-divisor on $\overline{R}$. ✓

clm191112b.3 CLAIM (3). *Let $m_1,\ldots,m_n\in\mathbb{Z}_{\geq 1}$. Then $f_1^{m_1},f_2^{m_2},\ldots,f_n^{m_n}$ is weakly R-regular.*

Proof. We again induct on n and note the base case $n = 1$ is done by Claim (2). For the inductive step, note that $f_1^{m_1}$ is a non-zero-divisor on R and $f_2, \ldots, f_n$ is weakly regular on $R/\langle f_1^{m_1}\rangle$ by Claim (2). Therefore by the inductive hypothesis $f_2^{m_2}, \ldots, f_n^{m_n}$ is weakly regular on $R/\langle f_1^{m_1}\rangle$. ✓

Claim (4). *Finally, we claim* $\operatorname{depth}(I, R) \leq \operatorname{depth}(J, R)$*, which will complete the proof by symmetry.*

Proof. Let $n = \operatorname{depth}(I, R)$ and let $\mathbf{f} = f_1, \ldots, f_n \in I$ be weakly R-regular. Since $\operatorname{rad}(I) = \operatorname{rad}(J)$, there exist $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 1}$ such that $f_i^{m_i} \in J$ for $i = 1, \ldots, n$. Then by Claim (3) we know $f_1^{m_1}, \ldots, f_n^{m_n} \in J$ is weakly R-regular. By the definition of depth we have

$$\operatorname{depth}(J, R) \geq n = \operatorname{depth}(I, R). \qquad ✓$$

□

Proof of Buchsbaum-Eisenbud (Continued). Recall that the only remaining case is for $i = 2$, and we needed to show that $\operatorname{depth}(I_{n-1}(A), R) \overset{?}{\geq} 2$. By Lemma III.B.5.34, we have that

$$\operatorname{rad}(\operatorname{Pf}_{n-1}(A)) = \operatorname{rad}(I_{n-1}(A))$$

and by Proposition III.B.5.35 this implies

$$\operatorname{depth}(\operatorname{Pf}_{n-1}(A), R) = \operatorname{depth}(I_{n-1}(A), R).$$

Therefore,

$$\operatorname{depth}(I_{n-1}(A), R) = \operatorname{depth}(\operatorname{Pf}_{n-1}(A), R) \geq 3 \geq 2.$$

□

Exercises

Let $x_1, \ldots, x_n \in R$, and let σ be an element of the symmetric group S_n. The goal of the following three exercises is to prove that there is an isomorphism of Koszul complexes

$$K^R(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \cong K^R(x_1, \ldots, x_n). \qquad (\dagger)$$ eq190916a

Note that Exercise III.B.5.36 can be done with no knowledge of Koszul complexes.

e190910a Exercise III.B.5.36. Let the following commutative diagram of chain maps be given.

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\alpha"'] & Y \arrow[d, "\gamma"] \\ A' \arrow[r, "\phi'"] & Y' \end{tikzcd}$$

e190910a.a (a) Prove that α and γ induce a well-defined chain map $\Lambda\colon \operatorname{Cone}(\phi) \to \operatorname{Cone}(\phi')$.
e190910a.b (b) Prove that if α and γ are isomorphisms, then so is Λ.

e190910b Exercise III.B.5.37. Let $x, y \in R$.

(a) Prove that there is an isomorphism between Koszul complexes $K^R(x, y) \cong K^R(y, x)$.
(b) More generally, let A be an R-complex, and set $K^R(x; A) = \operatorname{Cone}(A \xrightarrow{x} A)$ and $K^R(x, y; A) = \operatorname{Cone}(K^R(y; A) \xrightarrow{x} K^R(y; A))$. Define $K^R(y, x; A)$ similarly. Prove that $K^R(x, y; A) \cong K^R(y, x; A)$.

e190910c Exercise III.B.5.38. (a) Prove that if σ is an adjacent transposition $\sigma = (i \;\; i+1)$, then there is an isomorphism $(\dagger)$.
(b) Prove that if $\sigma \in S_n$ is arbitrary, then there is an isomorphism $(\dagger)$.

For the following two exercises, let R be a commutative ring with identity, and let $\mathbf{x} = x_1, \ldots, x_n \in R$. The goal of the following two exercises is to show that $\langle \mathbf{x}\rangle \operatorname{H}_i(K^R(\mathbf{x})) = 0$ for all i. In particular, this proves that if $\langle \mathbf{x}\rangle = R$, then $K^R(\mathbf{x})$ is exact.

exr210722d

Exercise III.B.5.39. Let $\phi\colon A \to C$ be a chain map. We say that ϕ is <u>null-homotopic</u> if there is a sequence of R-module homomorphisms $s = \{s_i\colon A_i \to C_{i+1} \mid i \in \mathbb{Z}\}$ such that $\phi_i = s_{i-1}\partial_i^A + \partial_{i+1}^C s_i$ for all i. Diagrammatically, s and ϕ look like this

$$\begin{tikzcd}
\cdots \arrow[r] & A_{i+1} \arrow[r] \arrow[d] & A_i \arrow[r, "\partial_i^A"] \arrow[d, "\phi_i"'] \arrow[dl, "s_i"'] & A_{i-1} \arrow[r] \arrow[d] \arrow[dl, "s_{i-1}"] & \cdots \\
\cdots \arrow[r] & C_{i+1} \arrow[r, "\partial_{i+1}^C"'] & C_i \arrow[r] & C_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

though the triangles in this diagram do not commute in general.
Prove that if ϕ is null-homotopic, then $\mathrm{H}_i(\phi) = 0$ as a map $\mathrm{H}_i(A) \to \mathrm{H}_i(C)$ for all $i \in \mathbb{Z}$.

exr210722e

Exercise III.B.5.40. (a) Prove that the homothety $K^R(\mathbf{x}) \xrightarrow{x_1} K^R(\mathbf{x})$ is null-homotopic, and conclude that $x_1 \mathrm{H}_i(K^R(\mathbf{x})) = 0$ for all i.
(b) Prove that $x_j \mathrm{H}_i(K^R(\mathbf{x})) = 0$ for all i, j, and conclude that $\langle \mathbf{x} \rangle \mathrm{H}_i(K^R(\mathbf{x})) = 0$ for all i.
(c) Prove that the following conditions are equivalent.
 (i) $\langle \mathbf{x} \rangle = R$
 (ii) $\mathrm{H}_i(K^R(\mathbf{x})) = 0$ for all i
 (iii) $\mathrm{H}_0(K^R(\mathbf{x})) = 0$

For the following two exercises, let k be a field, and set $R = k[W,X,Y,Z]$. Consider the ideals $I = \langle WX, XY, YZ \rangle$ and $J = \langle WZ, WX, XY, YZ \rangle$.

exr210722f

Exercise III.B.5.41. (a) Compute the following Taylor resolutions, describing each differential as a matrix: $T = T^R(WX, XY, YZ)$ and $U = T^R(WZ, WX, XY, YZ)$.
(b) Verify directly (without invoking Theorem III.B.4.8) that T is a resolution of R/I and that U is a resolution of R/J.

exr210722g

Exercise III.B.5.42. Consider the natural surjection $\pi\colon R/I \to R/J$ induced by the inclusion $I \subseteq J$.
(a) Explicitly construct a chain map $\Phi^+\colon T^+ \to U^+$ such that $\Phi_{-1} = \pi$, describing the maps Φ_i for $i \geq 0$ as matrices, and verifying that Φ^+ is a chain map.
(b) Explicitly compute $\mathrm{Cone}(\Phi)$, describing each differential as a matrix.
(c) Is $\mathrm{Cone}(\Phi)$ a resolution? Justify your answer.

exr210722h

Exercise III.B.5.43. Let k be a field. Set $R = k[X,Y]$ and consider the following matrices.

$$A = \begin{pmatrix} X & 0 \\ Y & X \\ 0 & Y \end{pmatrix} \qquad B = \begin{pmatrix} X & 0 & 0 \\ Y & X & 0 \\ 0 & Y & X \\ 0 & 0 & Y \end{pmatrix}$$

exr210722h.a (a) Compute $I = I_2(A)$ and $J = I_3(B)$ and verify that $\mathrm{depth}(I,R), \mathrm{depth}(J,R) \geq 2$.
exr210722h.b (b) Compute the Hilbert-Burch resolutions associated to A and B. Denote them by F and G respectively.
exr210722h.c (c) Verify directly (without invoking Theorem III.B.5.10) that F is a resolution of R/I and that G is a resolution of R/J.
exr210722h.d (d) For $n \geq 4$, describe a free resolution of $R/\langle X,Y\rangle^n$.

exr210722i

Exercise III.B.5.44. Let k be a field. Set $S = k[V,W,X,Y,Z]$ and consider the following alternating matrix.

$$C = \begin{pmatrix} 0 & 0 & V & W & 0 \\ 0 & 0 & 0 & X & Y \\ -V & 0 & 0 & 0 & Z \\ -W & -X & 0 & 0 & 0 \\ 0 & -Y & -Z & 0 & 0 \end{pmatrix}$$

(a) Compute $K = \mathrm{Pf}_4(C)$ and verify that $\mathrm{depth}(K,S) \geq 3$.
itemb (b) Compute the Buchsbaum-Eisenbud resolution associated to C. Denote it by L.
(c) Verify directly (without invoking Theorem III.B.5.30) that L is a resolution of S/K.

(d) Compute the Taylor resolution T associated to S/K, and explicitly construct a surjective chain map $\Phi\colon T \to L$ such that $\mathrm{H}_0(\Phi)$ is an isomorphism. Explain why $\mathrm{Ker}(\Phi)$ is an exact sequence.

CHAPTER III.C

Differential Graded Algebra Resolutions

apter290721c

We have covered several different types of resolutions so far, and some of them are a lot nicer than others! However, each of them has a downside as well.

(1) Taylor resolutions for monomial ideals have a closed formula! ☺ However, these resolutions are most likely not minimal, so they are not efficient. ☹
(2) A minimal resolution for an ideal always exist! ☺ However, they can be difficult to compute. ☹
(3) DG algebra resolutions have an extra ring structure, so they convey more information than other types of resolutions! ☺ However, they are not usually minimal. ☹

In this part, we will first discuss some definitions, properties, and examples relevant to DG algebra resolutions, then talk about several applications of DG algebra resolutions.

III.C.1. Definitions, Properties, and Examples

ection0630211

Throughout this chapter, assume that R is a commutative ring with identity. Recall that if X is an R-complex and $0 \neq x \in X_i$, then the homological degree of x is $|x| = i$.

defn191114a DEFINITION III.C.1.1. A commutative <u>differential graded R-algebra</u> (DG R-algebra) is an R-complex

$$A = \left(\cdots \xrightarrow{\partial_2^A} A_1 \xrightarrow{\partial_1^A} A_0 \longrightarrow 0 \right)$$

equipped with a binary operation $\mu_{ij} : A_i \times A_j \to A_{i+j}$ (we will write $\mu_{ij}(a,b) = ab$) satisfying the following properties.

- μ_{ij} is R-bilinear. Therefore, μ_{ij} is also distributive. In particular, $0 \cdot b = 0 = b \cdot 0$ for all $b \in A$.
- μ_{ij} is unital, i.e., there exists $1 \in A_0$ such that $1 \cdot a = a = a \cdot 1$ for all $a \in A_i$.
- μ_{ij} is associative.
- μ_{ij} is graded commutative, i.e., for all $a, b \in A \setminus \{0\}$ one has $ba = (-1)^{|a|\cdot|b|}ab$ and $a^2 = 0$ whenever $|a|$ is odd. The second condition is automatic if 2 is a unit in R.
- μ_{ij} satisfies the *Leibniz rule*, i.e., for all $a, b \in A \setminus \{0\}$ one has $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$.

The convention for determining signs is that if we switch the order of two factors, multiply that term by $(-1)^{\text{product of degrees}}$.

note191114b NOTE III.C.1.2. Let A be a complex of free R-modules, with $A_i = R^{\beta_i}$ for all $i \geq 0$ and $A_i = 0$ for all $i < 0$. Let B_i be a basis of A_i over R.

(a) Any function $f_{ij} : B_i \times B_j \to A_{i+j}$ extends uniquely to an R-bilinear function $\mu_{ij} : A_i \times A_j \to A_{i+j}$ so that $f_{ij} = \mu_{ij}\big|_{B_i \times B_j}$ as in Exercise III.A.2.10. Therefore to define μ_{ij} it suffices to specify it on the basis vectors.
(b) The operation μ_{ij} is unital in general if and only if it is unital on the basis vectors, and similarly for associativity, graded commutativity, and the Leibniz rule. Exercises III.A.2.11 and III.A.2.12 show this for the unital, associative, and graded commutative properties.
(c) There are a few ways to make the Leibniz rule easier to verify:

CLAIM. *The Leibniz rule is automatic for products of the form $1 \cdot b$ and $b \cdot 1$.*

PROOF. Since $1 \cdot b = b$, then $\partial(1 \cdot b) = \partial(b)$. On the other hand,

$$\underbrace{\partial(1)}_{=0} \cdot b + \underbrace{(-1)^{|1|}}_{=1} 1 \cdot \partial(b) = \partial(b)$$

because $|1| = 0$ and because the mapping $\partial : A_0 \to 0$ satisfies $\partial(1) = 0$. ✓

Claim. *The Leibniz rule is automatic for a^2 when $|a|$ is odd.*

Proof. Since $a^2 = 0$, then $\partial(a^2) = \partial(0) = 0$. On the other hand,

$$\begin{aligned}\partial(a)\cdot a + \underbrace{(-1)^{|a|}}_{=-1} a \cdot \partial(a) &= \partial(a)\cdot a - a\partial(a)\\ &= \partial(a)\cdot a - \underbrace{(-1)^{|a||\partial(a)|}}_{=1}\partial(a)\cdot a\\ &= \partial(a)\cdot a - \partial(a)\cdot a = 0\end{aligned}$$

✓

Claim. *The Leibniz rule holds for ab if and only if the Leibniz rule holds for ba.*

Proof. We show that if Leibniz rule hold for ab, then Leibniz rule holds for ba. The other implication is by symmetry. We will use that $|\partial(a)| = |a| - 1$. We have

$$\begin{aligned}\partial(ba) &= (-1)^{|a||b|}\partial(ab)\\ &= (-1)^{|a||b|}[\partial(a)b + (-1)^{|a|}a\partial(b)]\\ &= (-1)^{|a||b|}\partial(a)b + (-1)^{|a||b|+|a|}a\partial(b)\\ &= (-1)^{|a||b|+|\partial(a)||b|}b\partial(a) + (-1)^{|a||b|+|a|+|a||\partial(b)|}\partial(b)a\\ &= \underbrace{(-1)^{|a||b|+|a||b|-|b|}}_{=(-1)^{|b|}}b\partial(a) + \underbrace{(-1)^{|a||b|+|a|+|a||b|-|a|}}_{=1}\partial(b)a\\ &= \partial(b)a + (-1)^{|b|}b\partial(a).\end{aligned}$$

✓

ex191114c

Example III.C.1.3. Let $x, y \in R$ and set $K = K^R(x, y)$ with exterior basis

$$K = \left(0 \longrightarrow \underset{e_{12}}{R} \xrightarrow{\binom{-y}{x}} \underset{\substack{e_1\\ e_2}}{R^2} \xrightarrow{(x\ y)} \underset{1=e_\emptyset}{R} \longrightarrow 0 \right).$$

The rules for multiplication on K are as follows:

- $1 \cdot e_* = e_* = e_* \cdot 1$ for $* \in \{1, 2, 12, \emptyset\}$.
- $e_i^2 = 0$ for $i = 1, 2$.
- $e_1e_2 = e_{12} = -e_2e_1$.
- $e_ie_{12} = 0 = e_{12}e_i$ and $e_{12}^2 = 0$ for degree reasons.

With these definitions, Note III.C.1.2 implies that the unital, R-bilinear, and graded commutative properties are automatically satisfied. To show associatitivity, we need to show $a(bc) \overset{?}{=} (ab)c$ for basis vectors a, b, c. We split this into two cases below.

(a) If $b = 1$, then $a(1 \cdot c) = ac = (a \cdot 1) \cdot c$. If $a = 1$ or $c = 1$, then we are done similarly.
(b) If $|a|, |b|, |c| > 1$, then $|a(bc)| \geq 3$, so $a(bc) = 0 = (ab)c$.

We also need to check the Leibniz rule for the basis vectors. By our rules for multiplication above and by symmetry, we need only check the following.

$$\partial(e_1e_2) = \partial(e_{12}) = xe_2 - ye_1 = \partial(e_1)e_2 + (-1)^{|e_1|}e_1\partial(e_2)\ ✓$$
$$\partial(e_2e_{12}) = 0 = ye_{12} - xe_2^2 - ye_{12} = ye_{12} - xe_2^2 + ye_2e_1 = ye_{12} - e_2(xe_2 - ye_1) = \partial(e_2)e_{12} - e_2\partial(e_{12})\ ✓$$
$$\partial(e_{12}^2) = \partial(0) = 0 = (xe_2 - ye_1)e_{12} + e_{12}(xe_2 - ye_1) = \partial(e_{12})e_{12} + e_{12}\partial(e_{12})\ ✓$$

Our next result expands on Example III.C.1.3 by showing that all resolutions of R/I with length at most 2 have DG algebra structures. Exercise III.C.3.18 deals with resolutions of length 3.

thm191119a

Theorem III.C.1.4. *Let $I \leq R$ be an ideal such that R/I has a resolution of the form*

$$F = \left(0 \longrightarrow R^n \longrightarrow R^m \longrightarrow R \longrightarrow 0 \right).$$

Then F has the structure of a DG algebra.

Proof. Let $1 \in R$ and $e_1, \ldots, e_m \in R^m$ and $f_1, \ldots, f_n \in R^n$ be bases of R, R^m, and R^n, respectively. Define multiplication on the basis vectors in the following way.

$$\begin{aligned} &1 \cdot a = a = a \cdot 1 \\ &e_i^2 = 0, \quad \forall i \\ &e_i f_j = 0 = f_j e_i, \quad \forall i, j \\ &f_i f_j = 0, \quad \forall i, j \end{aligned}$$

It remains to define $e_i e_j$ for $i \neq j$. The Leibniz rule dictates that

$$\partial(e_i e_j) = \partial(e_i) e_j - e_i \partial(e_j) = x_i e_j - x_j e_i,$$

where $x_i = \partial(e_i) \in R$. Now we observe

$$\partial(x_i e_j - x_j e_i) = x_i x_j - x_j x_i = 0$$

and therefore $x_i e_j - x_j e_i \in \operatorname{Im} \partial_2^F$. Thus there exists some $\gamma_{ij} \in F_2 = R^n$ such that $\partial(\gamma_{ij}) = x_i e_j - x_j e_i$. Since ∂_2^F is injective, this γ_{ij} is unique and we define $e_i e_j = \gamma_{ij}$, i.e., $e_i e_j$ is the unique element of F_2 such that $\partial(e_i e_j) = x_i e_j - x_j e_i$.

As in Example III.C.1.3, associativity follows for degree reasons. To show graded commutivity we need to show that $e_j e_i = -e_i e_j$ for all i, j, for which it suffices to show that $\partial(e_j e_i) = -\partial(e_i e_j)$. By definition we have

$$\partial(e_j e_i) = x_j e_i - x_i e_j = -(x_i e_j - x_j e_i) = -\partial(e_i e_j),$$

as desired. For the most part, the Leibniz rule is satisfied by definition. For instance, we have $f_i f_j \in F_4 = 0$ and $\partial(f_i f_j) \in F_3 = 0$, so the Leibniz rule is satisfied. What about $e_i f_j$? For degree reasons we have $\partial(e_i f_j) = 0$ and we therefore need to show $0 = \partial(e_i) f_j - e_i \partial(f_j)$. Again noting that ∂_2^F is injective, it suffices to show that $\partial\left(\partial(e_i) f_j - e_i \partial(f_j)\right) = 0$. The Leibniz rule in degree 1 is satisfied, so we have

$$\partial\left(\partial(e_i) f_j - e_i \partial(f_j)\right) = \left[\partial(\partial(e_i)) f_j + \partial(e_i)\partial(f_j)\right] - \left[\partial(e_i)\partial(f_j) - e_i \partial(\partial(f_j))\right] = 0,$$

as desired. □

Next, we demonstrate some general properties of DG algebras.

prop191119b

Proposition III.C.1.5. *Let A be a DG algebra. Then A_0 is a commutative ring with identity under the operations from A.*

Proof. Since A_0 is an R-module, it is an additive abelian group. Since A is a DG algebra, the multiplication $A_0 \times A_0 \to A_0$ is well-defined. It is also associative, unital, and distributive by assumption. Finally, we see that for any $a, b \in A_0$ we have

$$ba = (-1)^{|a||b|} ab = (-1)^{0 \cdot 0} ab = ab. \qquad \square$$

The following result shows that the homology modules of a DG algebra each have more than just a module structure. We will use this in our applications to show that certain collections of homology modules form graded commutative rings.

thm191119c

Theorem III.C.1.6. *Assume A is a DG algebra. Then*

$$H(A) := \left(\cdots \xrightarrow{0} H_1(A) \xrightarrow{0} H_0(A) \longrightarrow 0 \right)$$

is also a DG algebra. Therefore $\displaystyle\bigoplus_{i=0}^{\infty} H_i(A)$ is a graded commutative ring with identity.

Proof Sketch. Recall that $Z_i(A) = \operatorname{Ker} \partial_i^A$. One first shows that

$$Z(A) = \left(\cdots \xrightarrow{0} Z_1(A) \xrightarrow{0} Z_0(A) \xrightarrow{0} 0 \right)$$

is a DG sub-algebra of A, i.e., that $Z(A)$ is a subcomplex of A such that the multiplication on A induces a multiplication on $Z(A)$ making $Z(A)$ into a DG algebra. One does this using a DG subalgebra test, which is analogous to the familiar subring and subgroup tests: we need to show $Z(A)$ is a subcomplex that is closed under multiplication with $1 \in Z_0(A)$. It is a subcomplex since $\partial_i^A|_{Z_i(A)} = 0$ for all i by definition of $Z_i(A)$. We also know $1 \in A_0 = Z_0(A)$ and if we let $z, w \in Z(A)$, then it follows that $zw \in Z(A)$ since

$$\partial(zw) = \underbrace{\partial(z)}_{=0} w \pm z \underbrace{\partial(w)}_{=0} = 0.$$

Next, recalling that $B_i(A) = \operatorname{Im} \partial_{i+1}^A$, one sets

$$B(A) = \left(\cdots \xrightarrow{0} B_1(A) \xrightarrow{0} B_0(A) \longrightarrow 0 \right)$$

and proves this is a DG ideal of $Z(A)$, i.e., that $B(A)$ is a subcomplex of $Z(A)$ that absorbs multiplication by elements of $Z(A)$. The subcomplex condition is straightforward since $B_i(A) \subseteq Z_i(A)$. Let $b \in B(A)$ and $z \in Z(A)$ and in order to show that $zb, bz \in B(A)$, it suffices to show that $bz \in B(A)$ (because of graded commutivity). Let $a \in A$ such that $b = \partial(a)$ and observe that

$$\partial(az) = \partial(a)z + \underbrace{(-1)^{|a|} a\partial(z)}_{=0} = bz,$$

so $bz \in B(A)$.

For the third and final step, one shows that $H(A) = Z(A)/B(A)$ is a DG algebra with differential and multiplication induced from $Z(A)$. Certainly the 0-differential on $H(A)$ makes it into an R-complex. Most of the work is done showing that the multiplication

$$H_i(A) \times H_j(A) \longrightarrow H_{i+j}(A)$$

$$(\overline{a}, \overline{b}) \longmapsto \overline{ab}$$

is well-defined. If $\overline{a} = \overline{a'}$ and $\overline{b} = \overline{b'}$, then $a - a', b - b' \in B(A)$. Therefore $a - a' = \partial(c)$ and $b - b' = \partial(d)$ for some $c, d \in Z(A)$. Thus $a = a' + \partial(c)$ and $b = b' + \partial(d)$, and since $B(A)$ is a DG ideal we have

$$ab = a'b' + \underbrace{a'\partial(d) + \partial(c)b' + \partial(c)\partial(d)}_{\in B(A)},$$

so $\overline{ab} = \overline{a'b'}$. To show that this makes $H(A)$ a DG algebra, one checks that all other DG algebra axioms are inherited from the corresponding axioms on $Z(A)$. ☐

ex191121a

EXAMPLE III.C.1.7. Let $R = k[X,Y]/\langle XY \rangle$ and $x = \overline{X}$ and $y = \overline{Y}$ and

$$K = K^R(x,y) = \left(0 \longrightarrow R \xrightarrow{\binom{-y}{x}} R^2 \xrightarrow{(x\ y)} R \longrightarrow 0 \right).$$

Recall that $H_0(K) = R/\langle x, y \rangle \cong k$, $H_1(K) \cong k$, and $H_2(K) = 0$. Then

$$H = H(K) \cong \left(0 \longrightarrow 0 \longrightarrow \underset{\varepsilon}{k} \xrightarrow{0} \underset{1}{k} \longrightarrow 0 \right)$$

and

$$\begin{aligned} \bigoplus_{i=0}^{\infty} H_i(K) &= k \cdot 1 \oplus k \cdot \varepsilon && \text{(since } \varepsilon^2 = 0) \\ &\cong k[Z]/\langle Z^2 \rangle && \text{(where } \overline{Z} \sim \varepsilon). \end{aligned}$$

Exterior DG Algebra Structure on the Koszul Complex. Our next goal is to extend Example III.C.1.3 again by showing that every Koszul complex has a DG algebra structure.

defn191121b

DEFINITION III.C.1.8. Let $[n] = \{1, \ldots, n\}$ for a positive integer n and $\mathbf{x} = x_1, \ldots, x_n \in R$ and $K = K^R(\mathbf{x})$. Let e_Λ and e_Γ be basis vectors of K, where $\Lambda, \Gamma \subseteq [n]$. Define

$$e_\Lambda e_\Gamma = \begin{cases} 0 & \text{if } \Lambda \cap \Gamma \neq \emptyset \\ \operatorname{sgn}(\Lambda, \Gamma) e_{\Lambda \cup \Gamma} & \text{if } \Lambda \cap \Gamma = \emptyset, \end{cases}$$

where $\operatorname{sgn}(\Lambda, \Gamma)$ is the sign of the permutation used to put $\Lambda \cup \Gamma$ into strictly increasing order. To understand the two cases in the display, notice that if $\Lambda = \{\lambda_1 < \cdots < \lambda_j\}$ and $\Gamma = \{\gamma_1 < \cdots < \gamma_k\}$, then $|e_\Lambda| = j$ and

$|e_\Gamma| = k$, so we must have $|e_\Lambda e_\Gamma| = j + k$, that is, in order for $e_\Lambda e_\Gamma$ to possibly be a non-zero multiple of $e_{\Lambda\cup\Gamma}$, we must have $\Lambda \cap \Gamma \neq \emptyset$.

ex191121c

EXAMPLE III.C.1.9. Consider the following three products of basis vectors:

- Let $\Lambda = \{1 < 3\}$ and $\Gamma = \{2 < 3\}$. Then $e_{13}e_{23} = 0$ since $\Lambda \cap \Gamma \neq \emptyset$.
- Let $\Lambda = \{1 < 3\}$ and $\Gamma = \{2 < 4\}$, and consider $e_{13}e_{24}$. Then
$$\Lambda \cup \Gamma = \{1 < 3, 2 < 4\} = \begin{pmatrix}2 & 3\end{pmatrix}\{1 < 2 < 3 < 4\},$$
so the permutation used to put $\Lambda\cup\Gamma$ in order is odd (i.e., $\operatorname{sgn}(\Lambda, \Gamma) = -1$). Therefore $e_{13}e_{24} = -e_{1234}$.
- Consider $e_{24}e_{13}$. Then as in the preceding bullet, we find that $e_{24}e_{13} = -e_{1234}$. Let's check graded commutativity for this product:
$$e_{24}e_{13} \overset{?}{=} (-1)^{|e_{13}||e_{24}|}e_{13}e_{24} = (-1)^{2\cdot 2}e_{13}e_{24} = e_{13}e_{24} = -e_{1234} = e_{24}e_{13}.\checkmark$$

Also, we check the Leibniz rule for the first product above. On the one hand, $\partial(e_{13}e_{23}) = \partial(0) = 0$. On the other hand, we have

$$\begin{aligned}\partial(e_{13})e_{23} + e_{13}\partial(e_{23}) &= (x_1e_3 - x_3e_1)e_{23} + e_{13}(x_2e_3 - x_3e_2)\\ &= x_1\underbrace{e_3e_{23}}_{=0} - x_3\underbrace{e_1e_{23}}_{=e_{123}} + x_2\underbrace{e_{13}e_3}_{=0} - x_3\underbrace{e_{13}e_2}_{=-e_{123}}\\ &= -x_3e_{123} + x_3e_{123} = 0.\checkmark\end{aligned}$$

Next, we work to make our treatment of $\operatorname{sgn}(\Lambda, \Gamma)$ rigorous.

defn191121d

DEFINITION III.C.1.10. Let $\mathbf{i} = (i_1, \ldots, i_n)$ where all of the i_k are distinct positive integers. Define
$$S(\mathbf{i}) = \{(p, q) \in [n] \times [n] \mid p < q \text{ and } i_p > i_q\}.$$
In words, $S(\mathbf{i})$ counts the entries of $\mathbf{i}$ not in strictly ascending order. Also define
$$\sigma(\mathbf{i}) = (-1)^{|S(\mathbf{i})|}.$$

ex191121e

EXAMPLE III.C.1.11. (a) Let $\mathbf{i} = (1, 3, 2, 4)$. The second and third positions are out of order, so $S(\mathbf{i}) = \{(2, 3)\}$ and $\sigma(\mathbf{i}) = (-1)^1 = -1$.
(b) Let $\mathbf{j} = (2, 4, 1, 3)$. Then there are three pairs that are out of order, so
$$S(\mathbf{j}) = \{(1, 3), (2, 3), (2, 4)\} \text{ and } \sigma(\mathbf{j}) = (-1)^3 = -1.$$

prop191121f

PROPOSITION III.C.1.12. *Let $\mathbf{i}$ be as in Definition III.C.1.10 and let $\tau \in S_n$. Define $\tau\cdot\mathbf{i} = (i_{\tau(1)}, \ldots, i_{\tau(n)})$. Write τ as a product of adjacent transpositions $\tau = \tau_1 \cdots \tau_\ell$. Then*
$$\sigma(\tau \cdot \mathbf{i}) = (-1)^\ell\sigma(\mathbf{i}).$$

PROOF. We prove this by induction on ℓ.
Base case: Let $\ell = 1$. Then τ is an adjacent transposition, so can be written as $\tau = \begin{pmatrix}x & x+1\end{pmatrix}$. Then
$$\begin{aligned}S(\tau \cdot \mathbf{i}) &= S(i_1, \ldots, i_{x-1}, i_{x+1}, i_x, i_{x+2}, \ldots, i_n)\\ &= \begin{cases} S(\mathbf{i}) \cup \{(x, x+1)\} & \text{if } i_x < i_{x+1}\\ S(\mathbf{i}) \setminus \{(x, x+1)\} & \text{if } i_{x+1} < i_x.\end{cases}\end{aligned}$$
Then
$$|S(\tau \cdot \mathbf{i})| = \begin{cases} |S(\mathbf{i})| + 1 & \text{if } i_x < i_{x+1}\\ |S(\mathbf{i})| - 1 & \text{if } i_{x+1} < i_x.\end{cases}$$
Therefore
$$\sigma(\tau \cdot \mathbf{i}) = (-1)^{|S(\mathbf{i})|\pm 1} = -(-1)^{|s(\mathbf{i})|} = -\sigma(\mathbf{i}).$$
We omit the inductive case here since it is routine. $\square$

cor191121g

COROLLARY III.C.1.13. *Let $\Lambda = \{\lambda_1 < \cdots < \lambda_j\}$ and $\Gamma = \{\gamma_1 < \cdots < \gamma_k\}$. Assume $\Lambda \cap \Gamma = \emptyset$. Then*
$$\operatorname{sgn}(\Lambda, \Gamma) = \sigma(\lambda_1, \ldots, \lambda_j, \gamma_1, \ldots, \gamma_k).$$

PROOF. Set $\mathbf{i} = (i_1, \ldots, i_n)$. Let $\tau = \tau_1, \ldots, \tau_\ell$ be as in Proposition III.C.1.12 so that $\tau \cdot \mathbf{i}$ is in strictly ascending order. Then $S(\tau \cdot \mathbf{i}) = \emptyset$, so $\sigma(\tau \cdot \mathbf{i}) = (-1)^0 = 1$. Therefore,

$$\sigma(\mathbf{i}) = (-1)^\ell \sigma(\tau \cdot \mathbf{i}) = (-1)^\ell = \operatorname{sgn}(\Lambda, \Gamma).$$

□

Now we are in position to verify that every Koszul complex is a DG algebra. We accomplish this in a sequence of theorems.

thm191121h

THEOREM III.C.1.14. *The exterior product on K is associative.*

PROOF. It suffices to show that $(e_\Lambda e_\Gamma)e_\Omega = e_\Lambda(e_\Gamma e_\Omega)$ for all $\Lambda, \Gamma, \Omega \subseteq [n]$. If $\Lambda \cap \Gamma \neq \emptyset$, then both sides are zero, and similarly if $\Lambda \cap \Omega \neq \emptyset$ or $\Gamma \cap \Omega \neq \emptyset$. Thus we assume that Λ, Γ, and Ω are pairwise disjoint. Then

$$(e_\Lambda e_\Gamma)e_\Omega = \operatorname{sgn}(\Lambda, \Gamma) \cdot \operatorname{sgn}(\Lambda \cup \Gamma, \Omega) e_{\Lambda \cup \Gamma \cup \Omega}$$
$$e_\Lambda(e_\Gamma e_\Omega) = \operatorname{sgn}(\Gamma, \Omega) \cdot \operatorname{sgn}(\Lambda, \Gamma \cup \Omega) e_{\Lambda \cup \Gamma \cup \Omega}.$$

Notice that the coefficients of the above items are both equal to

$$\sigma(\lambda_1, \ldots, \lambda_j, \gamma_1, \ldots, \gamma_k, \omega_1, \ldots, \omega_\ell),$$

so the two products are equal. □

thm191121i

THEOREM III.C.1.15. *The exterior product on K is graded commutative.*

PROOF. If $\Lambda \neq 0$, then $e_\Lambda^2 = 0$. If $\Lambda \cap \Gamma \neq \emptyset$, then $e_\Lambda e_\Gamma = 0 = (-1)^{|\Lambda||\Gamma|} e_\Gamma e_\Lambda$. So assume $\Lambda \cap \Gamma = \emptyset$. Then we claim

$$\operatorname{sgn}(\Gamma, \Lambda) = (-1)^{|\Gamma||\Lambda|} \operatorname{sgn}(\Lambda, \Gamma).$$

To show this, we want to show that the right hand side describes the sign of a permutation that puts $\Gamma \cup \Lambda$ in strictly ascending order. This happens in two steps: first, $(-1)^{|\Lambda||\Gamma|}$ moves all $\gamma \in \Gamma$ to the right of Λ; second, $\operatorname{sgn}(\Lambda, \Gamma)$ puts $\Lambda \cup \Gamma$ into strictly ascending order. Therefore,

$$e_\Gamma e_\Lambda = \operatorname{sgn}(\Gamma, \Lambda) e_{\Gamma \cup \Lambda} = (-1)^{|\Gamma||\Lambda|} \operatorname{sgn}(\Lambda, \Gamma) e_{\Lambda \cup \Gamma} = (-1)^{|\Gamma||\Lambda|} e_\Lambda e_\Gamma.$$

□

rmk191121j

REMARK. *Even though squares of basis vectors are equal to zero, squares of linear combinations of basis vectors are not necessarily equal to zero. Consider the following two examples assuming $2 \neq 0$ in R:*

(a) $(e_1 + e_2)^2 = e_1^2 + e_1 e_2 + e_2 e_1 + e_2^2 = e_1 e_2 - e_1 e_2 = 0$.
(b) $(e_{12} + e_{34})^2 = e_{12}^2 + e_{12} e_{34} + e_{34} e_{12} + e_{34}^2 = e_{12} e_{34} + e_{12} e_{34} = 2 e_{12} e_{34} \neq 0$.

thm191126a

THEOREM III.C.1.16. *The Koszul complex $K = K^R(\mathbf{x})$ is a DG algebra with exterior multiplication.*

PROOF. We have already shown that every property other than the Leibniz rule is satisfied, so we want to show that

$$\partial(e_\Lambda e_\Gamma) \overset{?}{=} \partial(e_\Lambda) e_\Gamma + (-1)^{|\Lambda|} e_\Lambda \partial(e_\Gamma).$$

Define $\phi(m, \Lambda) = |\{\lambda \in \Lambda \mid \lambda < m\}|$, so

$$\partial(e_\Lambda) = \sum_{\lambda \in \Lambda} (-1)^{\phi(\lambda, \Lambda)} x_\lambda e_{\Lambda \setminus \{\lambda\}}.$$

Then the right hand side of the Leibniz rule looks like

$$\sum_{\lambda \in \Lambda} (-1)^{\phi(\lambda, \Lambda)} x_\lambda e_{\Lambda \setminus \{\lambda\}} e_\Gamma + (-1)^{|\Lambda|} \sum_{\gamma \in \Gamma} (-1)^{\phi(\gamma, \Gamma)} x_\gamma e_\Lambda e_{\Gamma \setminus \{\gamma\}}.$$

We have three cases to consider:

Case 1: Suppose $|\Lambda \cap \Gamma| \geq 2$. Then $|(\Lambda \setminus \{\lambda\}) \cap \Gamma| \geq 1$, so $e_{\Lambda \setminus \{\lambda\}} e_\Gamma = 0$ for all $\lambda \in \Lambda$. Similarly, for all $\gamma \in \Gamma$, we have $e_\Lambda e_{\Gamma \setminus \{\gamma\}} = 0$. Therefore, the entire right hand side of the Leibniz rule is equal to 0. Also, the left hand side is $\partial(e_\Lambda e_\Gamma) = \partial(0) = 0$, so the Leibniz rule is satisfied.

Case 2: Suppose $|\Lambda \cap \Gamma| = 1$, say $\lambda_{p_0} = \gamma_{q_0} \in \Lambda \cap \Gamma$. The left hand side of the Leibniz rule is 0, so we want to show the right hand side is also 0. For $\lambda \neq \lambda_{p_0}$, we have $\lambda_{p_0} \in (\Lambda \setminus \{\lambda\}) \cap \Gamma$, so $e_{\Lambda \setminus \{\lambda\}} e_\Gamma = 0$. Similarly, for $\gamma \neq \gamma_{q_0}$, we have $e_\Lambda e_{\Gamma \setminus \{\gamma\}} = 0$. So the right hand side reduces to

$$(-1)^{\phi(\lambda_{p_0}, \Lambda)} x_{\lambda_{p_0}} e_{\Lambda \setminus \{\lambda_{p_0}\}} e_\Gamma + (-1)^{|\Lambda| + \phi(\gamma_{q_0}, \Gamma)} x_{\gamma_{q_0}} e_\Lambda e_{\Gamma \setminus \{\gamma_{q_0}\}}.$$

Notice that $(\Lambda \setminus \{\lambda_{p_0}\}) \cup \Gamma = \Lambda \cup \Gamma = \Lambda \cup (\Gamma \setminus \{\gamma_{q_0}\})$ because $\lambda_{p_0} \in \Gamma$ and $\gamma_{q_0} \in \Lambda$. Then the right hand side simplifies to

$$(-1)^{\phi(\lambda_{p_0},\Lambda)} \operatorname{sgn}(\Lambda \setminus \{\lambda_{p_0}\}, \Gamma) x_{\lambda_{p_0}} e_{\Lambda\cup\Gamma} + (-1)^{|\Lambda|+\phi(\gamma_{q_0},\Gamma)} \operatorname{sgn}(\Lambda, \Gamma \setminus \{\gamma_{q_0}\}) x_{\gamma_{q_0}} e_{\Lambda\cup\Gamma}.$$

We want to show that the two terms in the display above have opposite signs. Consider listing the elements in $(\Lambda \setminus \{\lambda_{p_0}\}) \cup \Gamma$:

$$(\lambda_1, \ldots, \lambda_{p_0-1}, \lambda_{p_0+1}, \ldots, \lambda_j, \gamma_1, \ldots, \gamma_{q_0-1}, \gamma_{q_0}, \ldots, \gamma_k).$$

Notice that $(q_0 - 1) + (j - p_0)$ adjacent transpositions are needed to move γ_{q_0} to between λ_{p_0-1} and λ_{p_0+1}, and $\phi(\lambda_{p_0}, \Lambda) = p_0 - 1$. Then the sign of the first term is

$$(-1)^{\phi(\lambda_{p_0},\Lambda)} \operatorname{sgn}(\Lambda \setminus \{\lambda_{p_0}\}, \Gamma) = (-1)^{(p_0-1)+(q_0-1)+(j-p_0)} \operatorname{sgn}(\Lambda, \Gamma \setminus \{\gamma_{q_0}\}) = (-1)^{q_0+j} \operatorname{sgn}(\Lambda, \Gamma \setminus \{\gamma_{q_0}\}).$$

Also, notice that $|\Lambda| = j$ and $\phi(\gamma_{q_0}, \Gamma) = q_0 - 1$, so the sign of the second term is

$$(-1)^{|\Lambda|+\phi(\gamma_{q_0},\Gamma)} \operatorname{sgn}(\Lambda, \Gamma \setminus \{\gamma_{q_0}\}) = (-1)^{j+q_0-1} \operatorname{sgn}(\Lambda, \Gamma \setminus \{\gamma_{q_0}\}) = -(-1)^{j+q_0} \operatorname{sgn}(\Lambda, \Gamma \setminus \{\gamma_{q_0}\}).$$

Therefore the signs of the two terms are opposites, so the two terms cancel.

Case 3: Suppose $\Lambda \cap \Gamma = \emptyset$. Then $(\Lambda \setminus \{\lambda\}) \cap \Gamma = \emptyset = \Lambda \cap (\Gamma \setminus \{\gamma\})$ for all $\lambda \in \Lambda$ and $\gamma \in \Gamma$. The right hand side of the Leibniz rule is

$$\begin{aligned}
&\sum_{\lambda\in\Lambda} (-1)^{\phi(\lambda,\Lambda)} x_\lambda e_{\Lambda\setminus\{\lambda\}} e_\Gamma + (-1)^{|\Lambda|} \sum_{\gamma\in\Gamma} (-1)^{\phi(\gamma,\Gamma)} x_\gamma e_\Lambda e_{\Gamma\setminus\{\gamma\}} \\
&\qquad = \sum_{\lambda\in\Lambda} (-1)^{\phi(\lambda,\Lambda)} \operatorname{sgn}(\Lambda \setminus \{\lambda\}, \Gamma) x_\lambda e_{(\Lambda\setminus\{\lambda\})\cup\Gamma} + \sum_{\gamma\in\Gamma} (-1)^{|\Lambda|+\phi(\gamma,\Gamma)} \operatorname{sgn}(\Lambda, \Gamma \setminus \{\gamma\}) x_\gamma e_{\Lambda\cup(\Gamma\setminus\{\gamma\})}.
\end{aligned}$$

The left hand side of the Leibniz rule is

$$\begin{aligned}
\partial(e_\Lambda e_\Gamma) &= \operatorname{sgn}(\Lambda, \Gamma) \partial(e_{\Lambda\cup\Gamma}) \\
&= \operatorname{sgn}(\Lambda, \Gamma) \sum_{\zeta\in\Lambda\cup\Gamma} (-1)^{\phi(\zeta,\Lambda\cup\Gamma)} x_\zeta e_{(\Lambda\cup\Gamma)\setminus\{\zeta\}} \\
&= \operatorname{sgn}(\Lambda, \Gamma) \left[\sum_{\lambda\in\Lambda} (-1)^{\phi(\lambda,\Lambda\cup\Gamma)} x_\lambda e_{(\Lambda\setminus\{\lambda\})\cup\Gamma} + \sum_{\gamma\in\Gamma} (-1)^{\phi(\gamma,\Lambda\cup\Gamma)} x_\gamma e_{\Lambda\cup(\Gamma\setminus\{\gamma\})} \right].
\end{aligned}$$

Now we compare the signs of both terms in the left hand side and right hand side of the Leibniz rule:

$$\underbrace{(-1)^{\phi(\lambda,\Lambda)}}_{\substack{\text{move } \lambda \text{ to the} \\ \text{beginning of } \Lambda}} \underbrace{\operatorname{sgn}(\Lambda \setminus \{\lambda\}, \Gamma)}_{\substack{\text{order the} \\ \text{leftovers}}} \stackrel{?}{=} \underbrace{\operatorname{sgn}(\Lambda, \Gamma)}_{\substack{\text{order all} \\ \text{elements}}} \underbrace{(-1)^{\phi(\lambda,\Lambda\cup\Gamma)}}_{\substack{\text{move } \lambda \text{ to the} \\ \text{beginning of } \Lambda\cup\Gamma}}, \checkmark$$

$$\underbrace{(-1)^{\phi(\gamma,\Gamma)}}_{\substack{\text{move } \gamma \text{ to the} \\ \text{beginning of } \Gamma}} \underbrace{(-1)^{|\Lambda|}}_{\substack{\text{move } \gamma \text{ past} \\ \text{all elements of } \Lambda}} \underbrace{\operatorname{sgn}(\Lambda, \Gamma \setminus \{\gamma\})}_{\substack{\text{order the} \\ \text{leftovers}}} \stackrel{?}{=} \underbrace{\operatorname{sgn}(\Lambda, \Gamma)}_{\substack{\text{order all} \\ \text{elements}}} \underbrace{(-1)^{\phi(\gamma,\Lambda\cup\Gamma)}}_{\substack{\text{move } \gamma \text{ to the} \\ \text{beginning of } \Lambda\cup\Gamma}}. \checkmark$$

Therefore the Leibniz rule is satisfied. □

Next, we use the results about the Koszul complex to verify that every Taylor resolution is a DG algebra.

defn191126b

DEFINITION III.C.1.17. Let $R = k[X_1, \ldots, X_d]$ and $f_1, \ldots, f_n \in [\![R]\!]$. For all $\Lambda = \{\lambda_1 < \cdots < \lambda_j\} \subseteq [n]$, set

$$f_\Lambda = \operatorname{lcm}(\{f_\lambda \mid \lambda \in \Lambda\}) = \operatorname{lcm}(f_{\lambda_1}, \ldots, f_{\lambda_j}).$$

For $\Lambda, \Gamma \subseteq [n]$, consider $e_\Lambda, e_\Gamma \in T = T^R(\mathbf{f})$. Define

$$e_\Lambda e_\Gamma = \begin{cases} 0 & \text{if } \Lambda \cap \Gamma \neq \emptyset \\ \operatorname{sgn}(\Lambda, \Gamma) \frac{f_\Lambda f_\Gamma}{f_{\Lambda\cup\Gamma}} e_{\Lambda\cup\Gamma} & \text{if } \Lambda \cap \Gamma = \emptyset. \end{cases}$$

ex191126c

EXAMPLE III.C.1.18. Consider the Taylor resolution $T = T^R(XY, XZ, YZ)$:

$$0 \longrightarrow \underset{e_{123}}{R} \xrightarrow{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \underset{\substack{e_{12} \\ e_{13} \\ e_{23}}}{R^3} \xrightarrow{\begin{pmatrix} -Z & -Z & 0 \\ Y & 0 & -Y \\ 0 & X & X \end{pmatrix}} \underset{\substack{e_1 \\ e_2 \\ e_3}}{R^3} \xrightarrow{(XY\ XZ\ YZ)} \underset{1=e_\emptyset}{R} \longrightarrow 0.$$

Then:

$$\begin{aligned}
e_2e_1 &= -\frac{f_2f_1}{f_{12}}e_{12} = -\frac{XZ\cdot XY}{XYZ}e_{12} = -Xe_{12},\\
e_3e_{13} &= 0,\\
e_3e_{12} &= +\frac{f_3f_{12}}{f_{123}}e_{123} = \frac{YZ\cdot XYZ}{XYZ} = YZe_{123}.
\end{aligned}$$

thm191126d

THEOREM III.C.1.19. *Multiplication on T is associative.*

PROOF. It suffices to show that $(e_\Lambda e_\Gamma)e_\Omega = e_\Lambda(e_\Gamma e_\Omega)$ where $\Lambda,\Gamma,\Omega\subseteq[n]$ are pairwise disjoint. The left hand side simplifies to

$$\begin{aligned}
(e_\Lambda e_\Gamma)e_\Omega &= \left(\operatorname{sgn}(\Lambda,\Gamma)\frac{f_\Lambda f_\Gamma}{f_{\Lambda\cup\Gamma}}e_{\Lambda\cup\Gamma}\right)e_\Omega\\
&= \operatorname{sgn}(\Lambda,\Gamma)\operatorname{sgn}(\Lambda\cup\Gamma,\Omega)\frac{f_\Lambda f_\Gamma}{f_{\Lambda\cup\Gamma}}\frac{f_{\Lambda\cup\Gamma}f_\Omega}{f_{(\Lambda\cup\Gamma)\cup\Omega}}e_{(\Lambda\cup\Gamma)\cup\Omega}\\
&= \operatorname{sgn}(\Lambda,\Gamma)\operatorname{sgn}(\Lambda\cup\Gamma,\Omega)\frac{f_\Lambda f_\Gamma f_\Omega}{f_{\Lambda\cup\Gamma\cup\Omega}}e_{\Lambda\cup\Gamma\cup\Omega}.
\end{aligned}$$

The right hand side simplifies to

$$\begin{aligned}
e_\Lambda(e_\Gamma e_\Omega) &= e_\Lambda\left(\operatorname{sgn}(\Gamma,\Omega)\frac{f_\Gamma f_\Omega}{f_{\Gamma\cup\Omega}}e_{\Gamma\cup\Omega}\right)\\
&= \operatorname{sgn}(\Gamma,\Omega)\operatorname{sgn}(\Lambda,\Gamma\cup\Omega)\frac{f_\Lambda f_{\Gamma\cup\Omega}}{f_{\Lambda\cup(\Gamma\cup\Omega)}}\frac{f_\Gamma f_\Omega}{f_{\Gamma\cup\Omega}}e_{\Lambda\cup(\Gamma\cup\Omega)}\\
&= \operatorname{sgn}(\Gamma,\Omega)\operatorname{sgn}(\Lambda,\Gamma\cup\Omega)\frac{f_\Lambda f_\Gamma f_\Omega}{f_{\Lambda\cup\Gamma\cup\Omega}}e_{\Lambda\cup\Gamma\cup\Omega}.
\end{aligned}$$

Notice that the monomial coefficients agree and the signs agree using the same proof as for the Koszul complex in Theorem III.C.1.14. $\square$

thm191126e

THEOREM III.C.1.20. *Multiplication on T is graded commutatitive.*

PROOF. First, $e_\Lambda^2 = 0$ for all $\Lambda\neq\emptyset$. We want to show that the following equation holds for all $\Lambda,\Gamma\subseteq[n]$:

$$e_\Lambda e_\Gamma \stackrel{?}{=} (-1)^{|\Lambda||\Gamma|}e_\Gamma e_\Lambda.$$

This is automatic if $\Lambda\cap\Gamma\neq\emptyset$, so assume $\Lambda\cap\Gamma=\emptyset$. The signs agree using the same proof as for the Koszul complex in Theorem III.C.1.15 and the monomial coefficients are

$$\frac{f_\Gamma f_\Lambda}{f_{\Gamma\cup\Lambda}} = \frac{f_\Lambda f_\Gamma}{f_{\Lambda\cup\Gamma}}.$$

$\square$

thm191126f

THEOREM III.C.1.21. *The Taylor resolution $T = T^R(\mathbf{f})$ is a DG algebra.*

PROOF. We want to show that the Leibniz rule is satisfied on basis vectors e_Λ, e_Γ. We have three cases to consider, as in the proof of Theorem III.C.1.16.

Case 1: Suppose $|\Lambda\cap\Gamma|\geq 2$. This case follows the same process as in the proof of Theorem III.C.1.16.

Case 2: Suppose $|\Lambda\cap\Gamma| = 1$, say $\lambda_{p_0} = \gamma_{q_0}\in\Lambda\cap\Gamma$. The left hand side of the Leibniz rule is 0, so we want to show the right hand side is also 0. For $\lambda\neq\lambda_{p_0}$, $\lambda_{p_0}\in(\Lambda\setminus\{\lambda\})\cap\Gamma$, so $e_{\Lambda\setminus\{\lambda\}}e_\Gamma = 0$. Similarly, for $\gamma\neq\gamma_{q_0}$, $e_\Lambda e_{\Gamma\setminus\{\gamma\}} = 0$. So the right hand side of the Leibniz rule reduces to

$$\begin{aligned}
&(-1)^{\phi(\lambda_{p_0},\Lambda)}\frac{f_\Lambda}{f_{\Lambda\setminus\{\lambda_{p_0}\}}}e_{\Lambda\setminus\{\lambda_{p_0}\}}e_\Gamma + (-1)^{|\Lambda|+\phi(\gamma_{q_0},\Gamma)}\frac{f_\Gamma}{f_{\Gamma\setminus\{\gamma_{p_0}\}}}e_\Lambda e_{\Gamma\setminus\{\gamma_{q_0}\}}\\
&\quad= \pm\frac{f_\Lambda}{f_{\Lambda\setminus\{\lambda_{p_0}\}}}\frac{f_{\Lambda\setminus\{\lambda_{p_0}\}}f_\Gamma}{f_{(\Lambda\setminus\{\lambda_{p_0}\})\cup\Gamma}}e_{(\Lambda\setminus\{\lambda_{p_0}\})\cup\Gamma} \mp \frac{f_\Gamma}{f_{\Gamma\setminus\{\gamma_{q_0}\}}}\frac{f_\Lambda f_{\Gamma\setminus\{\gamma_{q_0}\}}}{f_{\Lambda\cup(\Gamma\setminus\{\gamma_{q_0}\})}}e_{\Lambda\cup(\Gamma\setminus\{\gamma_{q_0}\}}\\
&\quad= \pm\frac{f_\Lambda f_\Gamma}{f_{\Lambda\cup\Gamma}}e_{\Lambda\cup\Gamma} \mp \frac{f_\Gamma f_\Lambda}{f_{\Lambda\cup\Gamma}}e_{\Lambda\cup\Gamma} = 0
\end{aligned}$$

where the signs in the terms are opposites as in the proof of Theorem III.C.1.16.

Case 3: Suppose $\Lambda\cap\Gamma=\emptyset$. As in Case 2, this is similar to Case 3 in the proof of Theorem III.C.1.16. $\square$

III.C.2. General Construction of DG Algebra Resolutions

ection063021m

Throughout this chapter, assume R is a noetherian commutative ring with identity and $I \leq R$ is an ideal and $\overline{R} = R/I$.

def191203a

Definition III.C.2.1. A DG algebra resolution of $\overline{R}$ over R is a free resolution A of $\overline{R}$ over R such that A is a DG R-algebra.

ex191202b

Example III.C.2.2. (a) If $\mathbf{f} \in R$ is a weakly R-regular sequence, then the Koszul complex $K^R(\mathbf{f})$ is a DG algebra resolution over R of $R/\langle \mathbf{f} \rangle$.

(b) If $R = k[X_1, \ldots, X_d]$ is a polynomial ring and $\mathbf{f} \in [|R|]$, then the Taylor resolution $T^R(\mathbf{f})$ is a DG algebra resolution of $R/\langle \mathbf{f} \rangle$.

Goals. (1) There exists a DG algebra resolution of $\overline{R}$ over R.

(2) If $R = k[X_1, \ldots, X_d]$, then $\overline{R}$ has a bounded DG algebra resolution over R.

strat210801a

Strategies. (1) Start with $K = K^R(\mathbf{f})$ where $I = \langle \mathbf{f} \rangle$. Then K is a DG algebra and $H_0(K) = R/I$, but this is not generally a resolution, because usually $H_1(K) \neq 0$. Then we will reduce the homology degree-by-degree in a manner similar to the algorithm kernel-surject-kernel-surject-....

rat210801a.b

(2) For the special case when $R = k[X_1, \ldots, X_d]$, Hilbert's Syzygy Theorem implies we can truncate to get a bounded resolution. We then need to show that this truncation is a DG algebra.

disc191203c

Discussion III.C.2.3. For strategy (1), how does one reduce homology? Start with any DG algebra A (some approximation of a resolution of $\overline{R}$). Let $z \in Z_i(A)$ such that $0 \neq \overline{z} \in H_i(A)$. Build a new DG algebra $A[y]$ such that A is a DG subalgebra of $A[y]$ and y is a variable and

$$H_j(A[y]) = \begin{cases} H_j(A) & \forall j < i \\ \dfrac{H_i(A)}{S} & j = i, \end{cases}$$

where in the latter case $S \subseteq H_i(A)$ is a submodule such that $\overline{z} \in S$. So the homology of $A[y]$ in degree less than i is the same as the homology of A, but the homology of $A[y]$ in degree i requires one fewer generator (assuming $\overline{z}$ is a generator of $H_i(A)$). Then start with $A = K^R(\mathbf{f})$ where $I = \langle \mathbf{f} \rangle$. Let $z_{1,1}, \ldots, z_{1,m} \in Z_1(A)$ such that $H_1(A) = \langle \overline{z_{1,1}}, \ldots, \overline{z_{1,m}} \rangle$. Construct $A^{(1)} = A[y_{1,1}, \ldots, y_{1,m}]$ such that $H_0(A^{(1)}) = \overline{R}$ and

$$H_1(A^{(1)}) = H_1(A)/\langle \overline{z_{1,1}}, \ldots, \overline{z_{1,m}} \rangle = 0.$$

We then repeat this process for $H_\ell(A^{(1)})$ for $\ell \geq 2$ in order to construct a DG algebra resolution of $\overline{R}$. Most of the remainder of this chapter is devoted to filling in the details of this argument.

def191203d

Definition III.C.2.4. Let A be a DG algebra over R and let $z \in Z_i(A)$ such that $i \geq 0$ is even. Let y be a symbol and define the degree of y to be $|y| = i+1$ (odd). Define $A[y]$ as follows. The R-modules in the resolution are given by

$$A[y]_n = A_n \oplus A_{n-(i+1)}y = \left\{ \alpha + ay \;\middle|\; \alpha \in A_n \text{ and } a \in A_{n-(i+1)} \right\}$$

(so $A[y]_n$ is an R module) and the differentials in the resolution are given by

$$\partial_n^{A[y]}(\alpha + ay) = \partial_n^A(\alpha) + \partial_{n-(i+1)}^A(a)y + (-1)^{|a|}az,$$

so they are R-linear, satisfy the Leibniz rule for elements of the form ay with $\partial_{i+1}^{A[y]}(y) = z$. We define multiplication as follows. Since $|y|$ is odd (i.e,, $|y| \equiv 1 \pmod 2$), we set $y^2 = 0$ and

$$yb = (-1)^{|y||b|}by = (-1)^{|b|}by.$$

We also set

$$\begin{aligned}(\alpha + ay)(\beta + by) &= \alpha\beta + \alpha by + ay\beta + \underbrace{ayby}_{=0 \because y^2=0} \\ &= \alpha\beta + (\alpha b + (-1)^{|\beta|}a\beta)y.\end{aligned}$$

These definitions merit reality checks. For the differential, since $\alpha + ay \in A[y]_n$, we know $\alpha \in A_n$ and $a \in A_{n-(i+1)}$. Also note

$$\partial_n^A(\alpha) \in A_{n-1} \subseteq A_{n-1} \oplus A_{n-1-(i+1)}y = A[y]_{n-1}$$

and

$$\underbrace{\partial^A_{n-(i+1)}(a)}_{\in A_{n-(i+1)-1}} \cdot \underbrace{y}_{\in A[y]_{i+1}} \stackrel{\cdot}{\in} A[y]_{n-1}.$$

Since $a \in A_{n-(i+1)}$ and $z \in A_i$, we also have $az \in A_{n-1} \subseteq A[y]_{n-1}$, so the differential lands as we would like. Similarly, multiplication defined $A[y]_n \times A[y]_m \to A[y]_{m+n}$ lands well also.

thm191203e

Theorem III.C.2.5. *Using the notation of Definition III.C.2.4 we have the following.*

thm191203e.a (a) *$A[y]$ is a DG algebra.*
thm191203e.b (b) *If A_n is free for all n, then $A[y]_n$ is free for all n.*
thm191203e.c (c) *$A[y]_n = A_n$ and $\partial^{A[y]}_n = \partial^A_n$ for all $n \leq i$ and therefore $H_n(A[y]) = H_n(A)$ for all $n < i$.*
thm191203e.d (d) *$A \subseteq A[y]$ is a DG subalgebra.*
thm191203e.e (e) *$H_i(A[y]) \cong H_i(A)/S$ where $S \subseteq H_i(A)$ is a submodule such that $\overline{z} \in S$.*
thm191203e.f (f) *If $w \in Z_i(A)$, then $w \in Z_i(A[y])$, i.e., $Z_i(A) \subseteq Z_i(A[y])$.*

Proof. (a) This part is tedious, but routine. For instance, $A[y]$ is a complex since using the Leibniz rule in A we see that

$$\begin{aligned}
\partial(\partial(\alpha + ay)) &= \partial\left(\partial(\alpha) + \partial(a)y + (-1)^{|a|}az\right) \\
&= \underbrace{\partial(\partial(\alpha))}_{=0} + \partial(\partial(a)y) + (-1)^{|a|}\partial(az) \\
&= \underbrace{\partial(\partial(a))}_{=0} y + (-1)^{|a|-1}\partial(a)\underbrace{\partial(y)}_{=z} + (-1)^{|a|}[\partial(a)z + (-1)^{|a|}a\underbrace{\partial(z)}_{=0}] \\
&= 0.
\end{aligned}$$

As another for instance, one considers

$$1_{A[y]} = 1_A + 0_A y \in A[y]_0$$

and checks that this is the multiplicative identity.

(b) This holds since the direct sum of free modules is free.

(c) If $n \leq i$, then

$$n - (i+1) = n - i - 1 < n - i \leq 0$$

and therefore

$$A[y]_n = A_n \oplus \underbrace{A_{n-(i+1)}y}_{=0} = A_n \oplus 0 = A_n.$$

Hence we have $\partial(\alpha + 0y) = \partial(\alpha)$ and therefore $\partial^{A[y]}_n = \partial^A_n$ (for these n). Thus we have

$$\begin{tikzcd}
A[y] = & \cdots \arrow[r] & A[y]_{i+1} \arrow[r] & A[y]_i \arrow[r, "\partial^{A[y]}_i", "\partial^A_i"', "\parallel" description] \arrow[d, phantom, "\parallel"] & A[y]_{i-1} \arrow[r, "\partial^{A[y]}_{i-1}", "\partial^A_{i-1}"', "\parallel" description] \arrow[d, phantom, "\parallel"] & \cdots \\
& & & A_i & A_{i-1} &
\end{tikzcd}$$

and therefore $H_n(A[y]) = H_n(A)$ for $n < i$.

(d) $\partial(\alpha + 0y) = \partial(\alpha)$ and $(\alpha + 0y)(\beta + 0y) = \alpha\beta$ etc.

(e) Since the inclusion $A \subseteq A[y]$ is a chain map by part d, there exists a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\subseteq"] & A[y] \arrow[r, "\pi"] & A[y]/A \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & y \arrow[dd, bend left, mapsto] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{i+1} \arrow[r] \arrow[d] & A_{i+1} \oplus A_0 y \arrow[rr] \arrow[d] & \rotatebox{45}{$\ni$} & A_0 y \arrow[r] \arrow[d] & 0 \\
 & & & z & & \\
0 \arrow[r] & A_i \arrow[r] \arrow[d] & A_i \arrow[rr] \arrow[d] & \rotatebox{45}{$\ni$} & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r] \arrow[d] & A_{i-1} \arrow[rr] \arrow[d] & & 0 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & & \vdots &
\end{tikzcd}$$

Then the long exact sequence in homology yields

$$\begin{tikzcd}
\cdots \arrow[r] & H_{i+1}\left(A[y]/A\right) \arrow[r, "\eth"] \arrow[d, phantom, "\parallel"] & H_i(A) \arrow[r] & H_i(A[y]) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0. \\
 & \dfrac{A_0 y}{\operatorname{Im} \partial} \ni \overline{y} & & \dfrac{H_i(A)}{\operatorname{Im} \eth} &
\end{tikzcd}$$

We set $S = \operatorname{Im} \eth$ and a diagram chase shows that $\overline{z} = \eth(\overline{y}) \in S$ by definition of $\eth$.

(f) We simply observe that

$$\partial^{A[y]}(w) = \partial^{A[y]}(w + 0y) = \partial^{A}(w) = 0.$$

$\square$

note191205a NOTE III.C.2.6. Use the notation from Definition III.C.2.4.

note191205a.a (a) First, observe that $\Sigma^i A \xrightarrow{z} A$ is a chain map. Then as an R-complex, we get

$$A[y] \cong \operatorname{Cone}(\Sigma^i A \xrightarrow{z} A).$$

However, $A[y]$ has an extra DG algebra structure that the mapping cone does not convey.

note191205a.b (b) Second, we make two observations about the Leibniz rule. First, use graded commutativity on the Leibniz rule to rewrite the second term as follows:

$$\begin{aligned}
\partial(ab) &= \partial(a)b + (-1)^{|a|} a \partial(b) \\
&= \partial(a)b + (-1)^{|a| + |a||\partial(b)|} \partial(b) a \\
&= \partial(a)b + (-1)^{|a||b|} \partial(b) a
\end{aligned}$$

Then we generalize the Leibniz rule to a product of m terms inductively as follows:

$$\begin{aligned}
\partial(a_1 \cdots a_m) &= \sum_{j=1}^{m} (-1)^{\sum_{t=1}^{j-1} |a_t|} a_1 \cdots a_{j-1} \partial(a_j) a_{j+1} \cdots a_m \\
&= \sum_{j=1}^{m} (-1)^{\left(\sum_{t=1}^{j-1} |a_t|\right)|a_j|} \partial(a_j) a_1 \cdots a_{j-1} a_{j+1} \cdots a_m.
\end{aligned}$$

Now we extend our definition for multiple elements of $Z_i(A)$ where i is fixed and even.

defn191205b

DEFINITION III.C.2.7. Let A be a DG algebra and $z_1, \ldots, z_m \in Z_i(A)$ such that $i \geq 0$ is even. Define

$$A[\mathbf{y}] = A[y_1, \ldots, y_m] = A[y_1, \ldots, y_{m-1}][y_m]$$

and $\partial(y_j) = z_j$ for all $j = 1, \ldots, m$. This is inductively well-defined because $z_m \in Z_i(A) \subseteq Z_i(A[y_1, \ldots, y_{m-1}])$ by Theorem III.C.2.5f.

thm191205c

THEOREM III.C.2.8. *Using the notation of Definition III.C.2.7 we have the following.*

thm191205c.a (a) $A[\mathbf{y}]$ *is a DG algebra.*

thm191205c.b (b) *If* A_n *is free for all* n*, then* $A[\mathbf{y}]_n$ *is free for all* n*.*

thm191205c.c (c) $A[\mathbf{y}]_n = A_n$ *and* $\partial_n^{A[\mathbf{y}]} = \partial_n^A$ *for all* $n \leq i$ *and therefore* $H_n(A[\mathbf{y}]) = H_n(A)$ *for all* $n < i$*.*

thm191205c.d (d) $A \subseteq A[\mathbf{y}]$ *is a DG subalgebra.*

thm191205c.e (e) $H_i(A[\mathbf{y}]) \cong H_i(A)/S$ *where* $S \subseteq H_i(A)$ *is a submodule such that* $\overline{z}_1, \ldots, \overline{z}_m \in S$*. In particular, if* $H_i(A) = \langle \overline{z}_1, \ldots, \overline{z}_m \rangle$*, then* $H_i(A[\mathbf{y}]) = 0$*.*

thm191205c.f (f) *If* $w \in Z_i(A)$*, then* $w \in Z_i(A[\mathbf{y}])$*, i.e.,* $Z_i(A) \subseteq Z_i(A[\mathbf{y}])$*.*

PROOF. Induct on m. $\square$

note191205d

NOTE III.C.2.9. Use the notation from Definition III.C.2.7. The elements in $A[\mathbf{y}]$ are finite sums of terms of the form $ay_{p_1} \cdots y_{p_\ell}$, where $a \in A$ and $1 \leq p_1 < \cdots < p_\ell \leq m$. Also,

$$|ay_{p_1} \cdots y_{p_\ell}| = |a| + |y_{p_1}| + \cdots + |y_{p_\ell}| = |a| + \ell(i+1).$$

and by Note III.C.2.6b the differential applied to such terms yields

$$\begin{aligned}
\partial(ay_{p_1} \cdots y_{p_\ell}) &= \partial(a)y_{p_1} \cdots y_{p_\ell} + (-1)^{|a|} a \partial(y_{p_1} \cdots y_{p_\ell}) \\
&= \partial(a)y_{p_1} \cdots y_{p_\ell} + (-1)^{|a|} a \sum_{j=1}^{\ell} (-1)^{|y_{p_j}| \sum_{t=1}^{j-1} |y_{p_t}|} \partial(y_{p_j}) y_{p_1} \cdots y_{p_{j-1}} y_{p_{j+1}} \cdots y_{p_\ell} \\
&= \partial(a)y_{p_1} \cdots y_{p_\ell} + (-1)^{|a|} a \sum_{j=1}^{\ell} (-1)^{j-1} \partial(y_{p_j}) y_{p_1} \cdots y_{p_{j-1}} y_{p_{j+1}} \cdots y_{p_\ell}
\end{aligned}$$

where the last line comes about because $|y_{p_t}| = i + 1$ is odd, i.e., $|y_{p_t}| \equiv 1 \pmod 2$. Notice that these operations are similar to those on the Koszul complex. Furthermore, multiplication of two such terms is given by

$$(ay_{p_1} \cdots y_{p_\ell})(by_{q_1} \cdots y_{q_k}) = \begin{cases} 0 & \text{if } p_r = q_s \text{ for some } r, s \\ (-1)^{|b|\ell} \sigma(p_1, \ldots, p_\ell, q_1, \ldots, q_k) ab y_{t_1} \cdots y_{t_{\ell+k}} & \text{otherwise,} \end{cases}$$

where $t_1 < \cdots < t_{\ell+k}$ and $\{t_1, \ldots, t_{\ell+k}\} = \{p_1, \ldots, p_\ell, q_1, \ldots, q_k\}$.

Now we move to the case where i is odd.

defn191205e

DEFINITION III.C.2.10. Let A be a DG algebra and $z \in Z_i(A)$ such that $i > 0$ is odd. Let y be a symbol so that $|y| = i + 1$ is even. Define $A[y]$ so that the R-modules are as follows:

$$A[y]_n = A_n \oplus A_{n-(i+1)} y \oplus A_{n-2(i+1)} y^2 \oplus \cdots.$$

This sum is finite because $n - j(i+1) < 0$ for $j \gg 0$, so $A_{n-j(i+1)} = 0$ for all $j \gg 0$. Furthermore, elements in $A[y]_n$ look like $a_0 + a_1 y + a_2 y^2 + \cdots$ for $a_j \in A_{n-j(i+1)}$. Let $\partial(y) = z$, then

$$\partial(y^2) = \partial(y \cdot y) = \partial(y)y + \underbrace{(-1)^{|y|}}_{=1} y \partial(y) = 2\partial(y)y = 2zy.$$

Inductively, we have $\partial(y^j) = jzy^{j-1}$ for $j \geq 0$. Notice that this looks like the power rule for derivatives. Then we can define the differential on $A[y]_n$ as

$$\partial\left(\sum_j a_j y^j\right) = \sum_j \left(\partial(a_j) y^j + (-1)^{|a_j|} a_j \cdot jzy^{j-1}\right).$$

Multiplication is similar to the multiplication in Definition III.C.2.4, but now there are more terms. We define multiplication as

$$\left(\sum_j a_j y^j\right)\left(\sum_k b_k y^k\right) = \sum_{j,k} a_j b_k y^{j+k},$$

where we can swap the order of y^j and b_k because $\left|y^j\right| = j\,|y|$ is even.

thm191205f THEOREM III.C.2.11. *Using the notation of Definition III.C.2.10 we have the following.*

thm191205f.a (a) *$A[y]$ is a DG algebra.*
thm191205f.b (b) *If A_n is free for all n, then $A[y]_n$ is free for all n.*
thm191205f.c (c) *$A[y]_n = A_n$ and $\partial_n^{A[y]} = \partial_n^A$ for all $n \leq i$ and therefore $H_n(A[y]) = H_n(A)$ for all $n < i$.*
thm191205f.d (d) *$A \subseteq A[y]$ is a DG subalgebra.*
thm191205f.e (e) *$H_i(A[y]) \cong H_i(A)/S$ where $S \subseteq H_i(A)$ is a submodule such that $\overline{z} \in S$.*
thm191205f.f (f) *If $w \in Z_i(A)$, then $w \in Z_i(A[y])$, i.e., $Z_i(A) \subseteq Z_i(A[y])$.*

PROOF. Argue as in the proof of Theorem III.C.2.5. □

defn191205g DEFINITION III.C.2.12. Let A be a DG algebra and $z_1, \ldots, z_m \in Z_i(A)$ such that $i > 0$ is odd. Define

$$A[\mathbf{y}] = A[y_1, \ldots, y_m] = A[y_1, \ldots, y_{m-1}][y_m]$$

and $\partial(y_j) = z_j$ for all $j = 1, \ldots, m$. This is inductively well-defined because $z_m \in Z_i(A) \subseteq Z_i(A[y_1, \ldots, y_{m-1}])$ by Theorem III.C.2.11(f).

thm191205h THEOREM III.C.2.13. *Using the notation of Definition III.C.2.12 we have the following.*

thm191205h.a (a) *$A[\mathbf{y}]$ is a DG algebra.*
thm191205h.b (b) *If A_n is free for all n, then $A[\mathbf{y}]_n$ is free for all n.*
thm191205h.c (c) *$A[\mathbf{y}]_n = A_n$ and $\partial_n^{A[\mathbf{y}]} = \partial_n^A$ for all $n \leq i$ and therefore $H_n(A[\mathbf{y}]) = H_n(A)$ for all $n < i$.*
thm191205h.d (d) *$A \subseteq A[\mathbf{y}]$ is a DG subalgebra.*
thm191205h.e (e) *$H_i(A[\mathbf{y}]) \cong H_i(A)/S$ where $S \subseteq H_i(A)$ is a submodule such that $\overline{z}_1, \ldots, \overline{z}_m \in S$. In particular, if $H_i(A) = \langle \overline{z}_1, \ldots, \overline{z}_m \rangle$, then $H_i(A[\mathbf{y}]) = 0$.*
thm191205h.f (f) *If $w \in Z_i(A)$, then $w \in Z_i(A[\mathbf{y}])$, i.e., $Z_i(A) \subseteq Z_i(A[\mathbf{y}])$.*

PROOF. Induct on m. □

We can now prove our first goal of the section.

thm191205i THEOREM III.C.2.14. *There exists a DG algebra resolution of $\overline{R}$ over R.*

PROOF. Recall that $I = \overline{\mathbf{f}}$ and $\overline{R} = R/I$. We construct an ascending chain of DG algebras that approximate the desired resolution. Let

$$A^{(0)} = K^R(\mathbf{f}).$$

Because of our noetherian assumption, there exist $z_{1,1}, \ldots, z_{1,m_1} \in Z_1(A^{(0)})$ so that $H_1(A^{(0)}) = \langle \overline{z}_{1,1}, \ldots, \overline{z}_{1,m_1} \rangle$. Then define

$$A^{(1)} = A^{(0)}[y_{1,1}, \ldots, y_{1,m_1}],$$

where $\partial(y_{1,j}) = z_{1,j}$ for all $j = 1, \ldots, m_1$ and $|y_{1,j}| = 2$. We must have $H_0(A^{(1)}) = \overline{R}$ and $H_1(A^{(1)}) = 0$ by Theorem III.C.2.13(e). Furthermore, the $A_j^{(1)}$ are finitely generated and free for all j. Therefore, there exist $z_{2,1}, \ldots, z_{2,m_2} \in Z_2(A^{(1)})$ so that $H_2(A^{(1)}) = \langle \overline{z}_{2,1}, \ldots, \overline{z}_{2,m_2} \rangle$. Then define

$$A^{(2)} = A^{(1)}[y_{2,1}, \ldots, y_{2,m_2}],$$

where $\partial(y_{2,j}) = z_{2,j}$ for all $j = 1, \ldots, m_2$ and $|y_{2,j}| = 3$. Continuing in this fashion, we find that $A^{(h)}$ for $h \geq 1$ is a free DG algebra that satisfies $H_0(A^{(h)}) = \overline{R}$ and $H_j(A^{(h)}) = 0$ for all $j = 1, \ldots, h$ and $A_j^{(h)} = A_j^{(h-1)}$ for all $j \leq h-1$. Also, we have an ascending chain of DG algebras

$$A^{(0)} \subseteq A^{(1)} \subseteq A^{(2)} \subseteq \cdots \subseteq A^{(h)} \subseteq \cdots.$$

We claim that $A = \bigcup_{h=0}^{\infty} A^{(h)}$ is well-defined and is a DG algebra resolution of $\overline{R}$ over R. For all $j \geq 0$, we have

$$A_j^{(0)} \subseteq \cdots \subseteq A_j^{(j)} = A_j^{(j+1)} = \cdots,$$

so $A_j = A_j^{(j)}$ is the stable value in the ascending chain. To check the differential, consider the following commutative diagram:

$$\begin{tikzcd}
A_j^{(0)} \arrow[d, "\partial_j^{A^{(0)}}"] & \subseteq & \cdots & \subseteq & A_j^{(j)} \arrow[d, "\partial_j^{A^{(j)}}"] & = & A_j^{(j+1)} \arrow[d, "\partial_j^{A^{(j+1)}}"] & = & A_j^{(j+2)} \arrow[d, "\partial_j^{A^{(j+2)}}"] & = & \cdots \\
A_{j-1}^{(0)} & \subseteq & \cdots & = & A_{j-1}^{(j)} & = & A_{j-1}^{(j+1)} & = & A_{j-1}^{(j+2)} & = & \cdots .
\end{tikzcd}$$

Then $\partial_j^A = \partial_j^{A^{(j)}}$ is the differential of the stable value in the ascending chain. Furthermore, A_j is free and finitely generated because $A_j^{(j)}$ is free and finitely generated. To check multiplication, let $a \in A_j$ and $b \in A_k$ and $\ell \geq j, k$. Then $a = A_j = A_j^{(j)} = A_j^{(\ell)}$ and $b = A_k = A_k^{(k)} = A_k^{(\ell)}$, so the multiplication ab makes sense in $A^{(\ell)}$. Furthermore, this is independent of the choice of ℓ because $A^{(1)} \subseteq A^{(2)} \subseteq \cdots$ are subalgebras. The axioms for a DG algebra are inherited from $A^{(h)}$, so A is a DG algebra. Finally,

$$H_j(A) = H_j(A^{(j+1)}) = H_j(A^{(j)}) = 0$$

for all $j \geq 1$, so A is a DG algebra resolution. $\square$

thm191209a

THEOREM III.C.2.15. *Let $R = k[X_1, \ldots, X_d]$ be a polynomial ring over a field, let $I \leq R$ be an ideal, and set $\overline{R} = R/I$. Then there exists a DG algebra resolution $\overline{A}$ of $\overline{R}$ over R such that $\overline{A}_i = 0$ for all $i > d$.*

PROOF. Hilbert's Syzygy Theorem implies $\overline{R}$ has a free resolution F over R such that $F_i = 0$ for all $i > d$. Theorem III.C.2.14 implies there exists a DG algebra resolution A of $\overline{R}$ over R. Then Schanuel's Lemma implies that

$$B_{d-1}^A = Z_{d-1}^A = \operatorname{Ker} \partial_{d-1}^A$$

is projective. A result of Serre implies $\operatorname{Ker} \partial_{d-1}^A$ is free. The sequence

$$\overline{A} = \qquad 0 \longrightarrow B_{d-1}^A \longrightarrow A_{d-1} \longrightarrow \cdots \longrightarrow A_0 \longrightarrow 0$$

is exact in all degrees except in degree 0 and consists of free modules. Therefore it is a free resolution of $\overline{R}$ over R and thus we need only show $\overline{A}$ has a DG algebra structure.

We observe

$$B_{d-1}^A = \operatorname{Im} \partial_d^A \cong \frac{A_d}{\operatorname{Ker} \partial_d^A}$$

and produce the following commutative diagram.

$$\begin{tikzcd}
\mathcal{I} = \arrow[d, hook] & \cdots \arrow[r, "\partial_{d+3}^A"] & A_{d+2} \arrow[r, "\partial_{d+2}^A"] \arrow[d, "\mathrm{id}"'] & A_{d+1} \arrow[r, "\partial_{d+1}^A"] \arrow[d, "\mathrm{id}"'] & \operatorname{Ker} \partial_d^A \arrow[r] \arrow[d, "\subseteq"'] & 0 & \\
A = & \cdots \arrow[r, "\partial_{d+3}^A"'] & A_{d+2} \arrow[r, "\partial_{d+2}^A"'] & A_{d+1} \arrow[r, "\partial_{d+1}^A"'] & A_d \arrow[r, "\partial_d^A"'] & A_{d-1} \arrow[r] & \cdots
\end{tikzcd}$$

$\mathcal{I} \subseteq A$ is a subcomplex such that $\overline{A} \cong A/\mathcal{I}$. As in the proof of Theorem III.C.1.6, it suffices to show that $\mathcal{I}$ is a DG ideal of A, i.e., is a subcomplex that absorbs multiplication by elements of A, i.e., it suffices to show that for all $a \in A$ and for all $x \in \mathcal{I}$ we have $ax \in \mathcal{I}$.

Assume without loss of generality that x is nonzero. Then $|x| \geq d$. If $|ax| > d$, then we are done since in this case $ax \in A_{|ax|} = \mathcal{I}_{|ax|}$. Therefore again without loss of generality assume that $|ax| = d$, i.e., $|x| = d$ and $|a| = 0$. This implies $x \in Z_d^A$ and $a \in R$, which then implies $ax \in Z_d^A = \mathcal{I}_d$. $\square$

Note if $\Delta = \operatorname{depth}(\overline{R}, R)$, then Theorem III.B.3.16 implies $p := \operatorname{pd}_R(\overline{R}) = d - \Delta$. The same proof as for Theorem III.C.2.15 yields a DG algebra resolution A' of $\overline{R}$ such that $A'_i = 0$ for all $i > d - \Delta$.

III.C.3. Applications

The Tor Algebra. Assume R is a noetherian commutative ring with identity and $I, J \leq R$ are ideals. Recall that for R-modules M and N, if we let P be a projective resolution of M over R, then we have

$$\operatorname{Tor}_i^R(M,N) = H_i(P \otimes_R N).$$

If M is finitely generated then P is of the form

$$P = \qquad \cdots \xrightarrow{\partial_3} R^{\beta_2} \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \longrightarrow 0$$

and we tensor to obtain

$$P \otimes_R N = \qquad \cdots \xrightarrow{\partial_3} N^{\beta_2} \xrightarrow{\partial_2} N^{\beta_1} \xrightarrow{\partial_1} N^{\beta_0} \longrightarrow 0.$$

We use the same matrices in P and $P \otimes_R N$ for the differential.

rmk191209c REMARK III.C.3.1. By Theorem II.G.1.7, we have

$$\operatorname{Tor}_i^R(M,N) = H_i(P \otimes_R N) \cong H_i(P \otimes_R Q) \cong H_i(M \otimes_R Q)$$

where Q is a projective resolution of N. (This says that Tor is "balanced".) The differential on $P \otimes_R Q$ is defined as

$$\begin{array}{ccccc} (P \otimes_R Q)_n & = & \bigoplus_{i+j=n}(P_i \otimes_R Q_j) & \supseteq & P_i \otimes_R Q_j \ni p \otimes q \\ \Big\downarrow{\scriptstyle \partial_n^{P\otimes Q}} & & & & \\ (P \otimes_R Q)_{n-1} & = & \bigoplus_{k+1=n-1}(P_k \otimes_R Q_\ell) & \supseteq & (P_{i-1} \otimes_R Q_j) \oplus (P_i \otimes_R Q_{j-1}) \end{array}$$

where

$$\partial(p \otimes q) := \partial(p) \otimes q + (-1)^{|p|} p \otimes \partial(q).$$

One has to check that $P \otimes_R Q$ is an R-complex. We consider augmentations $P \xrightarrow[\simeq]{\tau} M$ and $Q \xrightarrow[\simeq]{\pi} N$. Then we have

$$P \otimes N \xleftarrow[P\otimes\pi]{\simeq} P \otimes_R Q \xrightarrow[\simeq]{\tau\otimes Q} M \otimes_R Q.$$

(We use $\simeq$ to denote that the induced map on homology is an isomorphism, i.e., the map is a quasiisomorphism.)

thm191209d THEOREM III.C.3.2. *Let A be a DG algebra resolution of R/I over R.*

(a) $A' := A \otimes_R (R/J)$ *is a DG algebra.*

(b) $H(A') = \oplus_n H_n(A') = \oplus_n \operatorname{Tor}_n^R(R/I, R/J)$ *is a graded commutative ring, "the Tor algebra".*

note191209e NOTE III.C.3.3. This shows that $\operatorname{Tor}_n^R(R/I, R/J)$ is not a random list of modules. They fit together with a strong structure, and so there are restrictions on what these modules can look like.

ex191209f EXAMPLE III.C.3.4. Let $R = k[X_1, \ldots, X_d]$ be a polynomial ring over a field and set $I = \langle \underline{X} \rangle$. The DG algebra resolution of $R/I \cong k$ is the Koszul complex $A = K^R(\underline{X})$ and thus

$$A' = K^R(\underline{X}) \otimes_R (R/J) \cong K^{R/J}(\underline{x})$$

where $x_i = \overline{X_i} \in R/J$, and the isomorphism is described in the following diagram.

$$\begin{tikzcd} e_\Lambda \otimes \overline{1} \arrow[d, mapsto] \arrow[rrrr, bend left, mapsto] & \in \; R^{\binom{d}{i}} \otimes_R (R/J) \arrow[d, "\partial_i^{K^R(\underline{X})} \otimes (R/J)"'] & & (R/J)^{\binom{d}{i}} \arrow[ll, "\Phi_i"', "\cong"] \arrow[d, "\partial_i^{K^{R/J}(\underline{x})}"] \arrow[dll, phantom, "\circlearrowleft"] \; \ni & \overline{e_\Lambda} \arrow[d, mapsto] \\ \sum_{\lambda\in\Lambda} x_\lambda e_{\Lambda\setminus\{\lambda\}} \otimes \overline{1} \arrow[rrrr, bend right, mapsto] & \in \; R^{\binom{d}{i-1}} \otimes_R (R/J) & & (R/J)^{\binom{d}{i-1}} \arrow[ll, "\cong", "\Phi_{i-1}"'] \; \ni & \sum_{\lambda\in\Lambda} (-1)^{\phi(\lambda,\Lambda)} x_\lambda \overline{e_{\Lambda\setminus\{\lambda\}}} \end{tikzcd}$$

Then Φ is an isomorphism of DG algebras, i.e., it is an isomorphism of R-complexes that respects multiplication and multiplicative identities.

$$\Phi\left(1_{K^{R/J}(\underline{x})}\right) = \Phi\left(\overline{1}\right) = 1 \otimes \overline{1} = 1_{K^R(\underline{X}) \otimes_R (R/J)}$$

$$\Phi(\overline{e_\Lambda e_\Gamma}) = \Phi(\overline{e_\Lambda})\Phi(\overline{e_\Gamma})$$

(Check this.) In particular, it follows in this case that

$$\operatorname{Tor}^R(R/(\underline{X}), R/J) \cong H\left(K^{R/J}(\underline{x})\right)$$

and the graded algebra structure on the Tor algebra is the same as the algebra structure on the homology of $K^{R/J}(\underline{x})$ induced by the exterior algebra structure on $K^{R/J}(\underline{x})$.

Proof of Theorem III.C.3.2. (a) We use an alternate description of A' as in Example III.C.3.4. Set $R' = R/J$, then

$$\begin{tikzcd}
a \otimes \overline{1} \arrow[d, mapsto] \arrow[rrrr, mapsto, bend left] & \in & (A')_i = A_i \otimes_R R' \arrow[r, "\cong"'] \arrow[d, "\partial_i^{A\otimes R'}"'] & A_i/JA_i \arrow[d] & \ni \quad \overline{a} \arrow[d, mapsto] \\
\partial(a) \otimes \overline{1} \arrow[rrrr, mapsto, bend right] & \in & A_{i-1} \otimes_R R' \arrow[r, "\cong"] & A_{i-1}/JA_{i-1} & \ni \quad \overline{\partial(a)}
\end{tikzcd}$$

is an isomorphism of R-complexes. Multiplication on the right side of the diagram is defined by

$$\begin{aligned}
\tfrac{A_i}{JA_i} \times \tfrac{A_j}{JA_j} &\longrightarrow \tfrac{A_{i+j}}{JA_{i+j}} \\
(\overline{a}, \overline{b}) &\longmapsto \overline{ab}.
\end{aligned}$$

To show this is well defined, let $\overline{a} = \overline{a'} \in \frac{A_i}{JA_i}$ and $\overline{b} = \overline{b'} \in \frac{A_j}{JA_j}$. Then $a - a' \in JA_i$ and $b - b' \in JA_j$, so $ab - a'b' \in JA_{i+j}$ and thus $\overline{ab} = \overline{a'b'}$. Therefore $\overline{a}\overline{b} = \overline{ab}$, and it is straightforward to show that the DG axioms for $A' \cong A/JA$ are inherited from A.

(b) Since A' is a DG algebra, then

$$\operatorname{Tor}^R(R/I, R/J) \cong H(A')$$

is a graded commutative ring by Theorem III.C.1.6 $\square$

Avramov's Hammer. For this subsection, assume $R = k[X_1, \ldots, X_d]$ is a polynomial ring and $J \leq R$ is an ideal generated by homogeneous polynomials and $R' = R/J$.

note191209g Note III.C.3.5. We have a general strategy in commutative algebra for proving results.

note191209g.a (1) Prove the result for the case when R' is a finite dimensional vector space over k.

note191209g.b (2) If R' is Cohen-Macaulay, then there is a maximal weakly R-regular sequence $\mathbf{f} \in R'$ which satisfies $\dim_k(R'/\langle \mathbf{f}\rangle) < \infty$. By step 1, the result holds over $R'/\langle \mathbf{f}\rangle$. Furthermore, sometimes the weakly R-regular sequence guarantees that the result then holds over R'.

note191209g.c (3) If R' is not Cohen-Macaulay... ¯_(ツ)_/¯

Avramov's hammer is a tool to help us deal with 3 by producing a finite dimensional DG algebra U which behaves like $R'/\langle \mathbf{f}\rangle$. We prove the result over U, then use general machinery to deduce the result for R'. The downside of Avramov's hammer is that U is a DG algebra, so we need to track more data. The payoff, however, is that we can drop the Cohen-Macaulay assumption.

Another strategy we use is to prove a result for certain finite dimensional rings, then deduce the result for certain Cohen-Macaulay rings. With Avramov's hammer, we can prove a result for certain finite dimensional DG algebras, then deduce the result for certain non-Cohen-Macaulay rings.

prob191209h Problem III.C.3.6. Let M be a finitely generated graded R'-module (i.e., a module having a free resolution with matrices of homogeneous polynomials). If $\operatorname{Tor}_i^{R'}(M, M) = 0$ for all $i \gg 0$, must $\operatorname{pd}_{R'} M$ be finite (i.e., must M have a bounded free resolution over R')?

note191209i Note III.C.3.7. Note that $\operatorname{Tor}_i^{R'}(M, N) = 0$ for all $i \gg 0$ implies neither $\operatorname{pd}_{R'} M < \infty$ nor $\operatorname{pd}_{R'} N < \infty$.

defn191209j Definition III.C.3.8. Let A and B be DG algebras.

(a) A chain map $\Phi : A \to B$ is a <u>morphism of DG algebras</u> if it respects multiplication and multiplicative identities (i.e., $\Phi(1_A) = 1_B$ and $\Phi(aa') = \Phi(a)\Phi(a')$ for all $a, a' \in A$).

(b) A <u>quasiisomorphism of DG algebras</u> is a morphism of DG algebras that induces isomorphisms on homology in all degrees.

ex191209k EXAMPLE III.C.3.9. (a) Let A be a DG algebra resolution of R' over R and let

$$A^+ = \left(\cdots \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\tau} R' \longrightarrow 0 \right).$$

Then τ is a quasiisomorphism of DG algebras as in the following diagram:

$$\begin{tikzcd}
A = \arrow[d, "\tau", "\simeq"'] & \cdots \arrow[r, "\partial_3"] & A_2 \arrow[r, "\partial_2"] \arrow[d] & A_1 \arrow[r, "\partial_1"] \arrow[d] & A_0 \arrow[r] \arrow[d, "\tau"] & 0 \arrow[d] \\
R' = & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & R' \arrow[r] & 0.
\end{tikzcd}$$

ex191209k.b (b) If $\mathbf{f} \in R'$, then the following diagram is a morphism of DG alrebras:

$$\begin{tikzcd}
R' = \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R' \arrow[r] \arrow[d, "1"] & 0 \arrow[d] \\
K^{R'}(\mathbf{f}) = & \cdots \arrow[r] & (R')^{\binom{n}{2}} \arrow[r] & (R')^n \arrow[r] & R' \arrow[r] & 0.
\end{tikzcd}$$

This is not a quasiisomorphism unless $\mathbf{f} = \emptyset$. More generally, if A is a DG R-algebra, then $R' \to A_0$ defined by $\overline{r} \mapsto \overline{r} \cdot 1_A$ induces a morphism of DG algebras $R' \to A$.

ex191209k.c (c) Consider the following alternate description of $\operatorname{Tor}^R(R/I, R/J)$. Let A be a DG algebra resolution of R/I over R and let B be a DG algebra resolution of R/J over R. Then $A \otimes_R B$ is a DG algebra where

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|}(aa') \otimes (bb').$$

Furthermore,

$$\begin{tikzcd}
A \otimes_R B \arrow[d, "A\otimes\pi", "\simeq"'] \\
A \otimes_R (R/J)
\end{tikzcd}$$

is a quasiisomorphism of DG algebras, so $\operatorname{Tor}^R(R/I, R/J) = H(A \otimes_R B)$.

con191209l CONSTRUCTION III.C.3.10 (Avramov's Hammer). Let A be a bounded DG algebra resolution of $R' = R/J$ over $R = k[X_1, \ldots, X_d]$. Then Avramov's hammer is constructed via the following chain:

$$R' \xrightarrow[\substack{\text{DG algebra}\\ \text{morphism by}\\ \text{Example III.C.3.9b}}]{} K^{R'}(\mathbf{x}) \xrightarrow[\substack{\text{by}\\ \text{Example III.C.3.4}}]{\cong} K^R(\mathbf{X}) \otimes_R R' \xleftarrow[\substack{\text{by}\\ \text{Example III.C.3.9c}}]{\simeq} K^R(\mathbf{X}) \otimes_R A \xrightarrow[\substack{\text{by}\\ \text{Example III.C.3.9c}}]{\simeq} k \otimes_R A = U,$$

where

$$A = \qquad 0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_2} \longrightarrow R^{\beta_1} \longrightarrow R \longrightarrow 0,$$

$$U = k \otimes_R A = \qquad 0 \longrightarrow k^{\beta_d} \longrightarrow \cdots \longrightarrow k^{\beta_2} \longrightarrow k^{\beta_1} \longrightarrow k \longrightarrow 0,$$

and $\dim_k(U) = 1 + \beta_1 + \cdots + \beta_d$ is finite.

defn191209m DEFINITION III.C.3.11. Let A be a DG algebra over R. A DG A-module is an R-complex Y equipped with an R-bilinear multiplication $A_i \times Y_j \to Y_{i+j}$ denoted $(a, y) \mapsto ay$ that is unital, associative, and satisfies the Liebniz rule.

ex191209n EXAMPLE III.C.3.12. (a) First, A is a DG A-module. Moreover, $\Sigma^n A$ is a DG A-module.

(b) Let $\Phi : A \to B$ be a morphism of DG algebras. Then:
- B is a DG A-module with $ab = \Phi(a)b$.
- If Y is a DG B-module, then Y is a DG A-module with $ay = \Phi(a)y$. This is a "restriction of scalars".
- There is a notion of tensor product over A such that if Z is a DG A-module, then $B \otimes_A Z$ is a DG B-module with $b(b' \otimes z) = (bb') \otimes z$.

defn191209o DEFINITION III.C.3.13. Let A be a DG algebra and let L and Y be DG A-modules such that $L_i = 0$ for all $i \ll 0$.

(a) A semi-basis for L is a subset $E \subseteq L$ such that every element of L can be written uniquely as a linear combination of elements of E with coefficients from A.
(b) We call L semi-free if it has a semi-basis.
(c) A semi-free resolution of Y is a quasiisomorphism $L \xrightarrow{\simeq} Y$ such that L is semi-free and the quasiisomorphism respects scalar multiplication.

ex191209p EXAMPLE III.C.3.14. Let $K = K^{R'}(\mathbf{x})$ and let $R' \to K$ be a morphism of DG algebras. If M is an R'-module with free resolution F over R', then $K \otimes_R M$ is a DG K-module and $K \otimes_R F$ is a semi-free DG K-module. Furthermore, the augmentation $F \xrightarrow{\simeq} M$ induces a semi-free resolution over K

$$K \otimes_R F \xrightarrow{\simeq} K \otimes_R M.$$

defn191209q DEFINITION III.C.3.15. Let A be a DG algebra and let X and Y be DG A-modules and let $L \xrightarrow{\simeq} X$ be a semi-free resolution. Then

$$\operatorname{Tor}_i^A(X,Y) = H_i(L \otimes_A Y).$$

disc191209r DISCUSSION III.C.3.16. Let M be a finitely generated graded R'-module, and construct a semi-free resolution using Construction III.C.3.10:

$$R' \longrightarrow K^{R'}(\mathbf{x}) \quad \cong \quad K^R(\mathbf{X}) \otimes_R R' \xleftarrow{\simeq} \underbrace{K^R(\mathbf{X}) \otimes_R A}_{=B} \xrightarrow{\simeq} k \otimes_R A = U$$

$$M \rightsquigarrow \underbrace{K \otimes_R M}_{\text{DG } K\text{-module}} \overset{\simeq}{\leftsquigarrow} \underbrace{L}_{\substack{\text{semi-free}\\ \text{resolution}\\ \text{over B}}} \overset{\simeq}{\rightsquigarrow} \underbrace{U \otimes_B L.}_{\text{DG } U\text{-module}}$$

Notice here that $K \otimes_R M$ is a DG B-module by restriction of scalars and that the final complex is a semi-free DG U-module. Then we have

$$\begin{aligned}
\operatorname{Tor}^{R'}_{\gg 0}(M,M) = 0 &\iff \operatorname{Tor}^{K}_{\gg 0}(L,L) = 0 \\
&\iff \operatorname{Tor}^{B}_{\gg 0}(L,L) = 0 \\
&\iff \operatorname{Tor}^{U}_{\gg 0}(U \otimes_B L, U \otimes_B L) = 0 \\
&\overset{?}{\Rightarrow} \operatorname{pd}_U(U \otimes_B L) < \infty \\
&\Rightarrow \operatorname{pd}_B(L) < \infty \\
&\Rightarrow \operatorname{pd}_K(L) < \infty \\
&\Rightarrow \operatorname{pd}_{R'}(M) < \infty,
\end{aligned}$$

where the last implication is where we use the graded assumption on M. The implication labelled with a question has not been proven. The point is that this argument reduces Problem III.C.3.6 to a similar question over a finite dimensional DG algebra where the problem might be easier to solve.

Exercises

Let R be a commutative ring with identity. Let I be an ideal of R, and let F be an R-free resolution of R/I with $F_0 = R$ and $F_i = R^{\beta_i}$ with $\beta_i \in \mathbb{N}$ for all i.

191114a EXERCISE III.C.3.17. Prove that F almost has the structure of a DG R-algebra, specifically, that there is a binary multiplication on F that satisfies all of the axioms for a DG R-algebra except possibly the associativity axiom.
Hint: Let $e_{i,1}, \ldots, e_{i,\beta_i} \in F_i$ be the standard basis. Argue as in the proof of Theorem III.E.4 to define the products $e_{i,m}e_{j,n}$ such that the unital axiom, graded commutative axiom, and Leibniz rule are satisfied.

exr210722j EXERCISE III.C.3.18. Prove that if $F_i = 0$ for all $i \geq 4$, then F has the structure of a DG R-algebra In particular, if $A \in \operatorname{Alt}_n(R)$ with n odd is such that $I = \operatorname{Pf}_{n-1}(A)$ satisfies $\operatorname{depth}(I,R) \geq 3$, then the Buchsbaum-Eisenbud resolution of R/I has the structure of a DG R-algebra.
Hint: Use Exercise III.C.3.17. To verify the associative axiom, use the fact that ∂_3^F is injective.

Part IV

Homological Properties of Rings

part210604b

CHAPTER IV.A

Introduction

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ction063021o

IV.A.1. Niceness

One can ask what makes a ring "nice", but in fact it really depends.

ex200820a EXAMPLE IV.A.1.1. Noetherian rings are nice, but they are too restrictive for some. Maybe you like integral domains, but those also come with some restrictions. Even the subjectively basic assumption that one has a non-zero commutative ring with identity can be too restrictive for some. There are also those that like to work with non-associative groups; we don't talk to those people.

In this class we will focus on niceness conditions that can be characterized by some condition on homology. Similarly, most of the presentation (or much of it) will be colloquial. We begin with the premise that fields are particularly "simple", as justified in the following result.

thm200820b2 THEOREM IV.A.1.2. *Assume R is a non-zero commutative ring with identity. Then the following are equivalent.*

thm200820b.i (i) *R is a field.*
hm200820b.ii (ii) *R has exactly two ideals, the zero ideal and the ring itself.*
m200820b.iii (iii) *$0 \lneq R$ is a maximal ideal.*
hm200820b.iv (iv) *R is simple as an R-module.*
thm200820b.v (v) *Every R-module is free, i.e., has a basis over R.*
hm200820b.vi (vi) *Every finitely generated R-module is isomorphic to R^n for some n.*

Note that conditions (ii) *and* (iii) *say that R only has boring ideals, and that conditions* (v) *and* (vi) *say that R doesn't have many modules.*

For this class we set

$$R^n = \left\{(r_1 \cdots r_n)^T \mid r_i \in R\right\} = \underbrace{R \times \cdots \times R}_{n \text{ copies}} = \underbrace{R \oplus \cdots \oplus R}_{n \text{ copies}}.$$

Many of the rings we encounter in nature are not fields (e.g., coordinate rings of varieties and $k[X_1, \ldots, X_d]$ for $d \geq 1$ from algebraic geometry; Stanley-Reisner rings from combinatorial algebra; rings of integers inside number fields from algebraic number theory), so we seek the sweet spot where the niceness conditions are such that we can draw interesting conclusions, but are not so restrictive that we can still apply these conclusions to other interesting classes of rings.

ex200820c EXAMPLE IV.A.1.3. Integral domains are nice and of course every field is a domain. In fact one has an integral domain if and only if one has a subring of a field. Noetherian rings are also nice and again every field is a noetherian ring. One reason these two classes of rings are of interest is that they are not comparable, i.e., noetherian does not imply domain, nor does domain imply noetherian.

Rings that contain a field (called "equicharacteristic" rings) and rings satisfying the descending chain condition on ideals (called "artinian" rings) are two other interesting classes of rings. Every field is also both equicharacteristic and artinian, and every artinian ring is also noetherian (but the converse of the latter statement fails in general).

$$\begin{tikzcd}
\text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Gorenstein} \arrow[r, Rightarrow] & \text{Cohen-Macaulay} \arrow[r, Rightarrow] & \text{noetherian} \\
\text{domain} & \text{field} \arrow[ul, Rightarrow] \arrow[r, Rightarrow] & \text{artinian} \arrow[u, Rightarrow] &
\end{tikzcd}$$

In this class we will focus on these conditions for two classes of non-zero noetherian rings:

- local rings (which possess a unique maximal ideal) and
- graded rings (which for our purposes will come from graphs, simplicial complexes, and geometry).

We begin the next chapter with the definition of local rings and the definition of graded rings will be forthcoming.

IV.A.2. Local Rings

ection063021p

We will assume for this chapter that R is a non-zero noetherian commutative ring with identity. In particular, R has a maximal ideal. Assume k is a field.

defn200820d

DEFINITION IV.A.2.1. R is local if its maximal ideal is unique. (Note we assume R is noetherian. For this class we will call a non-noetherian ring with a unique maximal ideal "quasilocal".)

notn200820e

NOTATION IV.A.2.2. If R is local with maximal ideal $\mathfrak{m}$, then we say $(R, \mathfrak{m})$ is local. Furthermore, if $k \cong R/\mathfrak{m}$, then we say $(R, \mathfrak{m}, k)$ is local.

ex200820f

EXAMPLE IV.A.2.3.

(a) Every field is a local ring.
(b) Neither $\mathbb{Z}$ nor $k[X_1, \ldots, X_d]$ is local ($d \geq 1$).

Here are some properties of the local condition.

fact200820g

FACT IV.A.2.4.

act200820g.a (a) *If $(R, \mathfrak{m}, k)$ is local and $I \lneq R$, then $(R/I, \mathfrak{m}/I, k)$ is local by the Third Isomorphism Theorem. The converse fails in general: observe that $k[X]$ is not local, but $k[X]/\langle X\rangle \cong k$ is local.*

act200820g.b (b) *Define* $\operatorname{Spec}(R) = \{$*prime ideals* $\mathfrak{p} \lneq R\}$. *For all $\mathfrak{p} \in \operatorname{Spec}(R)$, we have the localization*

$$R_{\mathfrak{p}} = R[(R \setminus \mathfrak{p})^{-1}] = \left\{ \frac{r}{t} \mid r \in R,\, t \in R \setminus \mathfrak{p} \right\}$$

where $\frac{r}{t} = \frac{s}{u}$ if and only if there exists some $v \in R \setminus \mathfrak{p}$ such that $v(ru - st) = 0$. Any such localization $(R_{\mathfrak{p}}, \mathfrak{p}_{\mathfrak{p}})$ is local and $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \cong \operatorname{Frac}(R/\mathfrak{p})$.

act200820g.c (c) *The ring R is local if and only if the set of non-units of R is an ideal of R.*

act200820g.d (d) *(Nakayama's Lemma) If M is a finitely generated R-module and $(R, \mathfrak{m})$ is local, then $M = 0$ if and only if $M/\mathfrak{m}M = 0$.*

Next, we discuss an important local construction, the formal power series ring.

con200825a

CONSTRUCTION IV.A.2.5. Let A be a non-zero noetherian commutative ring with identity. We construct the power series ring in one variable with coefficents in A to be

$$A[\![X]\!] = \left\{ \sum_{i=0}^{\infty} a_i X^i \mid a_i \in A \right\} \left(= \prod_{i=0}^{\infty} A \right)$$

with

- the standard calculus definitions for addition, subtraction, and multiplication, and
- the additive and multiplicative identities defined by

$$0_{A[\![X]\!]} = 0_A \quad \text{and} \quad 1_{A[\![X]\!]} = 1_A.$$

fact200825b

FACT IV.A.2.6. *Let A be a non-zero noetherian commutative ring with identity.*

act200825b.a (a) *The power series ring $A[\![X]\!]$ is a non-zero commutative ring with identity.*

act200825b.b (b) *Furthermore, $A[\![X]\!]$ is noetherian by a version of the Hilbert Basis Theorem using the axiom of choice.*

act200825b.c (c) *The quotient ring $A[\![X]\!]/\langle X\rangle$ is isomorphic to A.*

act200825b.d (d) *As subrings, we have $A \subseteq A[X] \subseteq A[\![X]\!]$.*

act200825b.e (e) *We have that $f = \sum_{i=0}^{\infty} a_i X^i$ is a unit in $A[\![X]\!]$ if and only if a_0 is a unit in A. For example, notice that*

$$(1 - X)\left(\sum_{i=0}^{\infty} X^i\right) = 1,$$

so $1 - X$ is a unit in $A[\![X]\!]$ with multiplicative inverse $\sum_{i=0}^{\infty} X^i$.

In particular, consider the power series ring with coefficients in a field k and let

$$f = \sum_{i=0}^{\infty} a_i X^i \in k\,[\![X]\!]\,.$$

Then f is a unit in $k\,[\![X]\!]$ if and only if a_0 is a unit in k if and only if $a_0 \neq 0$. Also, f is a non-unit in $k\,[\![X]\!]$ if and only if $a_0 = 0$ if and only if

$$f = a_1 X + a_2 X^2 + \cdots = X(a_1 + a_2 X + \cdots) \in \langle X \rangle\,.$$

Then the set of all non-units of $k\,[\![X]\!]$ is just the ideal generated by X, so is an ideal of $k\,[\![X]\!]$. By Fact IV.A.2.4c, we can conclude that $(k\,[\![X]\!]\,, \langle X \rangle\,, k)$ is local.

As with polynomial rings, we next iterate the power series ring construction.

con200825c

CONSTRUCTION IV.A.2.7. We construct the power series ring in two variables with coefficients in A by

$$A\,[\![X, Y]\!] = A\,[\![X]\!]\,[\![Y]\!]$$

with elements written as

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} X^i Y^j.$$

Inductively, we construct the power series ring in d variables with coefficients in A by

$$A\,[\![X_1, \ldots, X_d]\!] = A\,[\![X_1, \ldots, X_{d-1}]\!]\,[\![X_d]\!]$$

with elements written as

$$\sum_{i_1, \ldots, i_d = 0}^{\infty} a_{i_1, \ldots, i_d} X_1^{i_1} \cdots X_d^{i_d}.$$

As a word of warning, the power series ring in infinitely many variables is not as straightforward to construct as with the polynomial ring.

note200825d

NOTE IV.A.2.8. Similar properties to those in Fact IV.A.2.6 hold for $A\,[\![X_1, \ldots, X_d]\!]$. In particular, the power series ring

$$(k\,[\![X_1, \ldots, X_d]\!]\,, \langle X_1, \ldots, X_d \rangle\,, k)$$

is local.

ection063021q

IV.A.3. Graded Rings

Graded rings are a useful setting where it makes sense to talk about homogeneous elements and degrees of elements. They are important in other areas of math as well. As a couple examples, they are used in algebraic geometry to understand projective varieties, and they are used in combinatorics to understand structures like graphs, simplicial complexes, and partially ordered sets via monomial ideals.

We will assume for this chapter that R is a non-zero commutative ring with identity and that k is a field. We will also use the notation that $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$ and that

$$\mathbb{N}^d = \left\{ \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix} \middle| n_i \in \mathbb{N} \right\}.$$

defn200825e

DEFINITION IV.A.3.1. The ring R is $\mathbb{N}$-graded if there exists additive abelian subgroups $R_0, R_1, R_2, \cdots \subseteq R$ such that $R = \bigoplus_{n=0}^{\infty} R_n$ is an additive abelian group with the property that for all $r_i \in R_i$ and $r_j \in R_j$, one has $r_i r_j \in R_{i+j}$, i.e.,multiplication in R respects "the grading" $R_i \times R_j \to R_{i+j}$.

The equality $R = \bigoplus_{n=0}^{\infty} R_n$ means that every $r \in R$ has a unique representation $r = \sum_n^{\text{finite}} r_n$ such that $r_n \in R_n$ for all $n \in \mathbb{N}$. Elements of R_n are homogeneous of degree n (0 does not have a well-defined degree but is homogeneous of all degrees). R_n is sometimes called the homogeneous/graded piece/component of R.

ex200825f

EXAMPLE IV.A.3.2. Let A be a non-zero commutative ring with identity. Then we can show that $R = A[X]$ is $\mathbb{N}$-graded. Let

$$\begin{aligned} R_0 &= A[X]_0 = A \subseteq A[X] \\ R_1 &= A[X]_1 = A \cdot X (\text{not an ideal}) \\ &\vdots \\ R_n &= A[X]_n = A \cdot X^n. \end{aligned}$$

So $A[X] = \bigoplus_{n=0}^{\infty} A[X]_n$ because for every $f \in A[X]$, there exists a unique representation $f = \sum_i^{\text{finite}} a_i X^i$. Also, notice that

$$\begin{array}{ccc} A[X]_i \times A[X]_j & \xrightarrow{?} & A[X]_{i+j} \\ aX^i \cdot bX^j & = & abX^{i+j}. \end{array}$$

Therefore $A[X]$ is $\mathbb{N}$-graded.

Similarly for $R = A[X_1, \dots, X_d]$, every homogeneous polynomial of degree n is a finite sum of terms of the form $aX_1^{i_1} \cdots X_d^{i_d} = a\mathbf{X}^{\mathbf{i}}$ such that $|\mathbf{i}| = \sum_{j=1}^d i_j = n$. Here, the multiplication is given by $\mathbf{X}^{\mathbf{i}} \cdot \mathbf{X}^{\mathbf{j}} = \mathbf{X}^{\mathbf{i}+\mathbf{j}}$.

Next we summarize some foundational properties of graded rings.

fact200825g

FACT IV.A.3.3. *Let R be $\mathbb{N}$-graded.*

fact200825g.a
(a) *We have $1 \in R_0$.*

fact200825g.b
(b) *$R_0 \subseteq R$ is a subring and therefore R is an R_0-module.*

fact200825g.c
(c) *$R_n \subseteq R$ is an R_0-submodule.*

fact200825g.d
(d) *We have $\bigoplus_{n \geq n_0} R_n \leq R$ is an ideal for each fixed $n_0 \in \mathbb{N}$. In particular, we call $R_+ = \bigoplus_{n \geq 1} R_n \leq R$ the "irrelevant ideal" and we have $R/R_+ \cong R_0$ due to the first isomorphism theorem: the ring isomorphism*

$$\begin{array}{ccc} R & \twoheadrightarrow & R_0 \\ \sum_i a_i & \longmapsto & a_0 \end{array}$$

has kernel R_+.

fact200825g.e
(e) *Let $x_1, \dots, x_d \in R_1$ and set $S = R_0[X_1, \dots, X_d]$. The following are equivalent:*
   (i) *For all $r \in R$, there exists $f \in S$ such that $r = f(x_1, \dots, x_d)$.*
   (ii) *For all $n \in \mathbb{N}$ and for all $r \in R_n$, there exists $f \in S_n$ such that $r = f(x_1, \dots, x_d)$.*
   (iii) *The ring homomorphism $\phi : S \to R$ given by $X_i \mapsto x_i$ is surjective.*

PROOF. (a) Since $1 \in R$, then $1 = \sum_i a_i$ for $a_i \in R_i$. Let $b_m \in R_m$. Then

$$\underbrace{b_m}_{\text{homog deg } m} = 1 \cdot b_m = \sum_i a_i b_m = \underbrace{a_0 b_m}_{\text{homog deg } m} + \underbrace{a_1 b_m}_{\text{homog deg } m+1} + \underbrace{\cdots}_{\text{homog deg } \geq m+2}.$$

By uniqueness of representation as a sum of homogeneous elements, we must have $b_m = a_0 b_m$ and $a_1 b_m = 0 = a_2 b_m = \cdots$. Since this is true for all b_m, then for all $i \geq 1$, we have

$$\begin{aligned} a_i \cdot \sum_j b_j &= 0 \\ a_i \cdot \underbrace{\sum_j a_j}_{=1} &= 0 \\ a_i &= 0 \end{aligned}$$

Therefore $1 = a_0 + a_1 + \cdots = a_0 \in R_0$.

(b) R_0 is a subring of R by the subring test because $1 \in R_0$ and $R_0 \times R_0 \to R_{0+0} = R_0$, so R_0 is closed under multiplication.

(c) This is similar to the proof of b because $R_0 \times R_i \to R_{0+i} = R_i$. $\square$

Next, we define an important class of graded rings for geometry and combinatorics.

defn200825h

Definition IV.A.3.4. If the equivalent conditions in Fact IV.A.3.3e hold for some $x_1, \ldots, x_d$ and R_0 is a field, then R is standard graded.

fact200825i

Fact IV.A.3.5. *Let S and ϕ be as in Fact IV.A.3.3*(e).

act200825i.a (a) *If R is standard graded, then R is noetherian (by the Hilbert Basis Theorem) and* $\operatorname{Ker}\phi$ *is generated by finitely many homogeneous polynomials.*

act200825i.b (b) *If $f_1, \ldots, f_n \in S = k[X_1, \ldots, X_d]$ such that each f_i is homogeneous, then $R = S/\langle f_1, \ldots, f_n\rangle$ is standard graded such that $R_i = \left\{\overline{f} \in R \mid f \in S_i\right\}$.*

ex200825j

Example IV.A.3.6. Let $R = k[X, Y, Z]/\left\langle X^2 - YZ\right\rangle$ and notice $X^2 - YZ$ is homogeneous of degree 2. Then R is standard graded by Fact IV.A.3.5(b). Similarly,

$$R = k[X, Y, Z, W]/\left\langle X^2 - YZ, W^3 - XYZ\right\rangle$$

is standard graded, with $X^2 - YZ$ being homogeneous of degree 2 and $W^3 - XYZ$ being homogeneous of degree 3.

The following extension of $\mathbb{N}$-grading is important for combinatorial applications.

defn200827a

Definition IV.A.3.7. R is $\mathbb{N}^d$-graded if there exists additive abelian subgroups $R_{\mathbf{n}} \subseteq R$ such that $R = \oplus_{\mathbf{n}\in\mathbb{N}^d} R_{\mathbf{n}}$ as additive abelian groups and for all $r_{\mathbf{i}} \in R_{\mathbf{i}}$, $r_{\mathbf{j}} \in R_{\mathbf{j}}$ one has $r_{\mathbf{i}}r_{\mathbf{j}} \in R_{\mathbf{i}+\mathbf{j}}$, i.e., multiplication in R respects "the grading" $R_{\mathbf{i}} \times R_{\mathbf{j}} \to R_{\mathbf{i}+\mathbf{j}}$. Note that $\mathbf{n} \in \mathbb{N}^d$, where $\mathbb{N}^d$ contains column vectors of size d with entries in $\mathbb{N}$ (per the opening remarks for this chapter).

The equality $R = \oplus_{\mathbf{n}\in\mathbb{N}^d} R_{\mathbf{n}}$ means every $r \in R$ has a unique representation $r = \sum_{\mathbf{n}}^{\text{finite}} r_{\mathbf{n}}$ such that $r_{\mathbf{n}} \in R_{\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{N}^d$. Elements of $R_{\mathbf{n}}$ are homogeneous of multidegree $\mathbf{n}$. (0 does not have well-defined multidegree, but it is homogeneous of all multidegrees.) $R_{\mathbf{n}}$ is sometimes called a homogeneous/multi-graded piece/component of R.

ex200827b

Example IV.A.3.8. Let A be a non-zero commutative ring with identity. Then the ring $R = A[\mathbf{X}] = \oplus_{\mathbf{n}} A[\mathbf{X}]_{\mathbf{n}}$ is $\mathbb{N}^d$-graded, where $\mathbf{X} = X_1, \ldots, X_d$, because $f = \sum_{\mathbf{i}}^{\text{finite}} a_{\mathbf{i}} X^{\mathbf{i}}$ uniquely, where $a_{\mathbf{i}} \in A$ and $\mathbf{i} \in \mathbb{N}^d$. We also have $R_{\mathbf{0}} = A[\mathbf{X}]_{\mathbf{0}} = A \subseteq A[\mathbf{X}]$ and $R_{\mathbf{n}} = A[\mathbf{X}]_{\mathbf{n}} = A\mathbf{X}^{\mathbf{n}}$ for $\mathbf{n} \in \mathbb{N}^d$ (e.g., if $d = 4$ and $\mathbf{n} = (1, 3, 0, 1)^T$, then $R_{\mathbf{n}} = AX_1X_2^3X_4$). The multiplication in R also respects the grading:

$$\begin{array}{ccc} A[\mathbf{X}]_{\mathbf{i}} \times A[\mathbf{X}]_{\mathbf{j}} & \longrightarrow & A[\mathbf{X}]_{\mathbf{i}+\mathbf{j}} \\ a\mathbf{X}^{\mathbf{i}} \cdot b\mathbf{X}^{\mathbf{j}} & = & ab\mathbf{X}^{\mathbf{i}+\mathbf{j}} \end{array}$$

(e.g., if again $d = 4$, then we have $X_1^3X_2X_4^2 \cdot X_1X_2X_3^6 = X_1^4X_2^2X_3^6X_4^2$).

The following properties are $\mathbb{N}^d$-analogues of items from Fact IV.A.3.3.

fact200827c

Fact IV.A.3.9. *Let R be $\mathbb{N}^d$-graded.*

act200827c.a (a) *We have $1 \in R_{\mathbf{0}}$.*

act200827c.b (b) *$R_{\mathbf{0}} \subseteq R$ is a subring and therefore R is an $R_{\mathbf{0}}$-module.*

act200827c.c (c) *$R_{\mathbf{n}} \subseteq R$ is an $R_{\mathbf{0}}$-submodule.*

act200827c.d (d) *We call $R_+ = \oplus_{\mathbf{n}\neq\mathbf{0}} R_{\mathbf{n}} \leq R$ the "irrelevant ideal" and we have $R/R_+ \cong R_{\mathbf{0}}$ for the same reasoning as in Fact IV.A.3.3.*

ex200827d

Example IV.A.3.10. Let $R = A[\mathbf{X}]$ where A is a non-zero commutative ring with identity.

ex200827d.a (a) $R_+ = \langle \mathbf{X}\rangle$ and $A[\mathbf{X}]/\langle \mathbf{X}\rangle \cong A$.

ex200827d.b (b) If $I = \langle f_1, \ldots, f_n\rangle$ such that $f_i = \mathbf{X}^{\underline{a}_i}$, then R/I is $\mathbb{N}^d$-graded where $\overline{f} \in (R/I)_{\mathbf{n}}$ if $f \in R_{\mathbf{n}}$ and $(R/I)_{\mathbf{n}} = A \cdot \overline{\mathbf{X}^{\mathbf{n}}}$.

note200827c

Note IV.A.3.11. In general, $\mathbb{N}^d$-graded implies $\mathbb{N}$-graded, i.e., if R is $\mathbb{N}^d$-graded and for all $n \in \mathbb{N}$ we set $R_n = \oplus_{|\mathbf{i}|=n} R_{\mathbf{i}}$ where $|\mathbf{i}| = i_1 + \cdots + i_d$, then $R = \oplus_{n=0}^{\infty} R_n$ is $\mathbb{N}$-graded. We call the $\mathbb{N}^d$-grading the "fine grading" and we call the $\mathbb{N}$-grading the "coarse grading". We say a polynomial ring has the "standard grading" when each variable has degree 1 with respect to the coarse grading.

IV.A.4. Depth

ection063021r

Assume R is a non-zero commutative ring with identity and let $\mathfrak{a} \lneq R$. In this chapter, we briefly recall the notion of depth, which is crucial for Cohen-Macaulayness.

defn200827e DEFINITION IV.A.4.1. A sequence $\mathbf{r} = r_1, \ldots, r_n \in \mathfrak{a}$ is <u>R-regular</u> if r_1 is a non-zero-divisor on R and r_k is a non-zero-divisor on $R/\langle r_1, \ldots, r_{k-1}\rangle$ for all $k = 2, 3, \ldots, n$. (Note that since $\mathfrak{a} \neq R$, we know $\langle \mathbf{r}\rangle \neq R$, so "weakly" R-regular implies R-regular in this setting.) The sequence $\mathbf{r}$ is a <u>maximal R-regular sequence in $\mathfrak{a}$</u> if it is R-regular and for all $s \in \mathfrak{a}$ the sequence $r_1, \ldots, r_n, s$ is not R-regular, i.e., the sequence $\mathbf{r}$ cannot be extended to a longer R-regular sequence in $\mathfrak{a}$.

ex200827f EXAMPLE IV.A.4.2. (a) A field k has no regular sequence other than the empty sequence, since the only proper ideal of k is the zero ideal.

(b) In the ring $\mathbb{Z}$, if $\mathfrak{a} = \langle 15\rangle$, then a maximal regular sequence in $\mathfrak{a}$ is 15. Clearly 15 is a non-zero-divisor on $\mathbb{Z}$. Since the cardinality of $\mathbb{Z}/15\mathbb{Z}$ is 15, all non-zero-divisors on $\mathbb{Z}/15\mathbb{Z}$ are units and therefore are not in $\mathfrak{a}$.

(c) In $R = A[X]$ where $A \neq 0$ and $\mathfrak{a} = \langle X\rangle$, the monomial X^n is R-regular for all $n \geq 1$.

(d) In $R = A[\![X]\!]$ where $A \neq 0$ and $\mathfrak{a} = \langle X\rangle$, the monomial X^n is R-regular for all $n \geq 1$.

(e) In $R = A[\mathbf{X}]$ where $A \neq 0$ and $\mathfrak{a} = \langle X_1, \ldots, X_d\rangle$, the list $X_1^{a_1}, \ldots, X_d^{a_d}$ is R-regular for any $\underline{a} \in \mathbb{N}^d$ satisfying $a_i \geq 1$ for all i.

prop200827g PROPOSITION IV.A.4.3. *There exists a maximal R-regular sequence in $\mathfrak{a}$. Moreover any R-regular sequence in $\mathfrak{a}$ can be extended to a maximal R-regular sequence in $\mathfrak{a}$.*

PROOF (SKETCH). The essential point is that the chain of ideals

$$\langle r_1\rangle \subsetneq \langle r_1, r_2\rangle \subsetneq \cdots \subseteq \mathfrak{a} \neq R$$

must stabilize. □

thm200827h THEOREM IV.A.4.4. *Every maximal R-regular sequence in $\mathfrak{a}$ has the same length (because one can characterize the length of regular sequences in terms of Ext-vanishing).*

defn200827i DEFINITION IV.A.4.5. $\operatorname{depth}(\mathfrak{a}, R)$ is the length of a maximal R-regular sequence in $\mathfrak{a}$, also denoted as $\operatorname{depth}_{\mathfrak{a}}(R)$ or $\operatorname{depth}(\mathfrak{a})$, e.g., if $(R, \mathfrak{m})$ is local, then $\operatorname{depth}(R) := \operatorname{depth}(\mathfrak{m}, R)$, e.g., if R is standard-graded and $\mathfrak{m} = R_+$, then $\operatorname{depth}(R) := \operatorname{depth}(\mathfrak{m}, R)$.

thm200827j THEOREM IV.A.4.6. *If R is standard-graded, then* $\operatorname{depth}(R) = \operatorname{depth}(R_{\mathfrak{m}})$ *(also because of Ext-vanishing).*

defn200827k DEFINITION IV.A.4.7. An ideal $\mathfrak{p} \in \operatorname{Spec}(R) = \{\mathfrak{p} \leq R \text{ prime}\}$ is <u>associated</u> to R if there exists an element $r \in R$ such that

$$\mathfrak{p} = \operatorname{Ann}_R(r) = \{s \in R \mid sr = 0\},$$

i.e., it is a prime ideal in the set of all annihilator ideals of elements of R. We denote

$$\operatorname{Ass}(R) = \{\text{associated prime ideals of } R\}.$$

ex200827l EXAMPLE IV.A.4.8. (a) If R is an integral domain, then $\operatorname{Ann}_R(0) = R$ and $\operatorname{Ann}_R(r) = 0$ for non-zero $r \in R$. Therefore $\operatorname{Ass}(R) = \{0\}$.

(b) If $R = k[X, Y]/\langle XY\rangle$, then $\operatorname{Ass}(R) = \{\langle X\rangle, \langle Y\rangle\}$ with $\langle X\rangle = \operatorname{Ann}_R(Y)$ and $\langle Y\rangle = \operatorname{Ann}_R(X)$.

fact200827m FACT IV.A.4.9. *Set $S = k[X_1, \ldots, X_d]$ and let $I \lneq S$ be generated by monomials such that*

$$I = \bigcap_{i=1}^{m} \left\langle X_{i_1}^{a_{i_1}}, \ldots, X_{i_{p_i}}^{a_{i_{p_i}}} \right\rangle$$

(irredundantly), then

$$\operatorname{Ass}(S/I) = \left\{\left\langle X_{i_1}, \ldots, X_{i_{p_i}}\right\rangle\right\}.$$

ex200827n EXAMPLE IV.A.4.10. Let $S = k[X, Y]$ and define the ideal $I = \langle XY\rangle = \langle X\rangle \cap \langle Y\rangle$. Then for all positive integers a, b we have

$$\operatorname{Ass}(S/I) = \{\langle X\rangle, \langle Y\rangle\} = \operatorname{Ass}\left(S/\left\langle X^a Y^b\right\rangle\right),$$

where the second equality holds because $\left\langle X^a Y^b\right\rangle = \langle X^a\rangle \cap \left\langle Y^b\right\rangle$.

notn200901a NOTATION IV.A.4.11. We use the following notation.

$$\mathrm{ZD}_R(R) = \{\text{zero-divisors of } R\}$$
$$\mathrm{ZD}^0_R(R) = \mathrm{ZD}_R(R) \cup \{0\}$$

ex200901b EXAMPLE IV.A.4.12. Given the ring $R = k[X,Y]/\left\langle X^aY^b\right\rangle$ where $a,b \geq 1$ we have $\overline{f} \in \mathrm{ZD}^0_R(R)$ if and only if $\overline{X}|\overline{f}$ or $\overline{Y}|\overline{f}$, i.e.,

$$\mathrm{ZD}^0_R = \left\langle \overline{X}\right\rangle \cup \left\langle \overline{Y}\right\rangle.$$

The equalities ($\overline{X} \cdot \overline{X^{a-1}Y^b} = 0$ and $\overline{Y} \cdot \overline{X^aY^{b-1}} = 0$ justify one of the containments.) Note also that $\overline{X}, \overline{Y} \in \mathrm{ZD}^0_R(R)$, but $\overline{X} + \overline{Y} \notin \mathrm{ZD}^0_R(R)$, demonstrating that $\mathrm{ZD}^0_R(R)$ is not an ideal in general.

The next result shows that some of the properties in the previous example hold in general.

thm200901c THEOREM IV.A.4.13.

thm200901c.a (a) $\mathrm{ZD}^0_R(R) = \displaystyle\bigcup_{\mathfrak{p}\in\mathrm{Ass}(R)} \mathfrak{p}$

thm200901c.b (b) $|\operatorname{Ass}(R)| < \infty$

thm200901c.c (c) $\operatorname{Ass}(R) \supseteq \operatorname{Min}(R) \neq \emptyset$ *where* $\operatorname{Min}(R)$ *is the set of minimal elements of* $\operatorname{Spec}(R)$ *with respect to containment.*

ex200901d EXAMPLE IV.A.4.14. Consider the ring $R = k[X,Y]/\langle XY\rangle$ where $\mathrm{ZD}^0_R(R) = \left\langle \overline{X}\right\rangle \cup \left\langle \overline{Y}\right\rangle$ as in Example IV.A.4.12. We have the finite set

$$\operatorname{Ass}(R) = \{\left\langle \overline{X}\right\rangle, \left\langle \overline{Y}\right\rangle\} = \operatorname{Min}(R),$$

where the second equality holds because both $\left\langle \overline{X}\right\rangle$ and $\left\langle \overline{Y}\right\rangle$ are minimal. To see this let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p} \subseteq \left\langle \overline{X}\right\rangle$; to prove $\left\langle \overline{X}\right\rangle = \mathfrak{p}$ it suffices to show that $\overline{X} \in \mathfrak{p}$. Since $\overline{XY} = 0 \in \mathfrak{p}$, a prime ideal, we have either $\overline{X} \in \mathfrak{p}$ or $\overline{Y} \in \mathfrak{p}$. If $\overline{Y} \in \mathfrak{p} \subseteq \left\langle \overline{X}\right\rangle$, then we obtain a contradiction and thus we conclude $\overline{X} \in \mathfrak{p}$. The proof of the minimality of $\left\langle \overline{Y}\right\rangle$ is identical.

The next result is used over and over again in this area.

lem200901e LEMMA IV.A.4.15 (Prime Avoidance). *Let* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \operatorname{Spec}(R)$. *If* $I \leq R$ *is an ideal such that* $I \not\subseteq \mathfrak{p}_i$ *for all* i, *then* $I \not\subseteq \bigcup_{i=1}^n \mathfrak{p}_i$.

alg200901f ALGORITHM IV.A.4.16. Here we present an algorithm for finding maximal R-regular sequences in the case when $(R, \mathfrak{m})$ is local.

(0) If $\mathfrak{m} \in \operatorname{Ass}(R)$, then stop, because $\mathrm{ZD}^0_R(R) = \mathfrak{m}$ implies the empty sequence is a maximal R-regular sequence.

(1) Assume $\mathfrak{m} \notin \operatorname{Ass}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.
   (a) Since $\mathfrak{m} \not\subseteq \mathfrak{p}_i$ for all i, by Prime Avoidance we have $\mathfrak{m} \not\subseteq \bigcup_{i=1}^n \mathfrak{p}_i$ and therefore there exists an element $x_1 \in \mathfrak{m} \setminus (\bigcup_{i=1}^n \mathfrak{p}_i)$. Since $x \notin \bigcup_{i=1}^n \mathfrak{p}_i = \mathrm{ZD}^0_R(R)$ we know $x_1 \in \mathfrak{m}$ is a non-zero-divisor on R.
   (b) Set $R_1 = R/\langle x_1\rangle$ and $\mathfrak{m}_1 = \mathfrak{m}/\langle x_1\rangle$.
   (c) If $\mathfrak{m}_1 \in \operatorname{Ass}(R_1)$, then stop: x_1 is a maximal R-regular sequence in $\mathfrak{m}$.

(2) Assume $\mathfrak{m}_1 \notin \operatorname{Ass}(R_1)$.
   (a) As above there exists an element $\overline{x}_2 \in \mathfrak{m}_1 \setminus \mathrm{ZD}^0_R(R_1)$. Then $x_1, x_2 \in \mathfrak{m}$ is an R-regular sequence.
   (b) Set $R_2 = R/\langle x_1, x_2\rangle$ and $\mathfrak{m}_2 = \mathfrak{m}/\langle x_1, x_2\rangle$.
   (c) If $\mathfrak{m}_2 \in \operatorname{Ass}(R_2)$, then stop: x_1, x_2 is a maximal R-regular sequence in $\mathfrak{m}$.

Repeat this process until $\mathfrak{m}_d \in \operatorname{Ass}(R_d)$. Then $x_1, \ldots, x_d$ is a maximal R-regular sequence in $\mathfrak{m}$, where $d = \operatorname{depth}(R)$.

ex200901g EXAMPLE IV.A.4.17. (a) Set $R = k[\![X,Y]\!]/\langle XY\rangle$ for which we have $\operatorname{Ass}(R) = \{\left\langle \overline{X}\right\rangle, \left\langle \overline{Y}\right\rangle\}$. Note that $\overline{X} - \overline{Y} \notin \left\langle \overline{X}\right\rangle \cup \left\langle \overline{Y}\right\rangle$ (i.e., $\overline{X} - \overline{Y}$ is not in the union of the associated primes of R), so $\overline{X} - \overline{Y} \in \left\langle \overline{X}, \overline{Y}\right\rangle = \mathfrak{m}$ is R-regular. Observe also that $\mathfrak{m}$ is not an associated prime. Defining the ring $R_1 = R/\left\langle \overline{X} - \overline{Y}\right\rangle$ we find

$$R_1 = \frac{R}{\left\langle \overline{X} - \overline{Y}\right\rangle} \cong \frac{k[\![X,Y]\!]}{\langle XY, X - Y\rangle} \cong \frac{k[\![X]\!]}{\langle X^2\rangle}.$$

One can check that $\operatorname{Ann}_{R_1}(\overline{X}) = \left\langle \overline{X}\right\rangle = \mathfrak{m}_1$, implying $\mathfrak{m}_1 \in \operatorname{Ass}(R_1)$ and the algorithm ends. We conclude $\overline{X} - \overline{Y}$ is a maximal R-regular sequence in $\mathfrak{m}$ and $\operatorname{depth}(R) = 1$.

(b) Consider the ring

$$R = \frac{k[\![a,b,\alpha,\beta]\!]}{\langle ab, a\alpha, b\beta\rangle}.$$

(This is the ring determined by the edge ideal of the simple graph

$$\alpha \text{ —— } a \text{ —— } b \text{ —— } \beta.)$$

We claim $\overline{a}-\overline{\alpha}, \overline{b}-\overline{\beta}$ is a maximal R-regular sequence (and therefore $\operatorname{depth}(R)=2$).

(0) We decompose the (edge) ideal

$$\langle ab, a\alpha, b\beta\rangle = \langle a, b\rangle \cap \langle a, \beta\rangle \cap \langle \alpha, b\rangle$$

to find

$$\operatorname{Ass}(R) = \{\langle \overline{a}, \overline{b}\rangle, \langle \overline{a}, \overline{\beta}\rangle, \langle \overline{\alpha}, \overline{b}\rangle\},$$

so $\mathfrak{m} = \langle \overline{a}, \overline{b}, \overline{\alpha}, \overline{\beta}\rangle \notin \operatorname{Ass}(R)$.

(1) We see that $\overline{a}-\overline{\alpha} \notin \langle \overline{a}, \overline{b}\rangle \cup \langle \overline{\alpha}, \overline{b}\rangle \cup \langle \overline{a}, \overline{\beta}\rangle$, so $\overline{a}-\overline{\alpha} \in \mathfrak{m} \setminus \operatorname{ZD}^0_R(R)$. We also have the isomorphisms

$$\frac{R}{\langle \overline{a}-\overline{\alpha}\rangle} \cong \frac{k[\![a,b,\alpha,\beta]\!]}{\langle ab, a\alpha, b\beta, a-\alpha\rangle} \cong \frac{k[\![a,b,\beta]\!]}{\langle ab, a^2, b\beta\rangle}$$

and the decomposition $\langle ab, a^2, b\beta\rangle = \langle a, \beta\rangle \cap \langle a^2, b\rangle$, so

$$\operatorname{Ass}(R_1) = \{\langle \overline{a}, \overline{\beta}\rangle, \langle \overline{a}, \overline{b}\rangle\}.$$

Since $\mathfrak{m}_1 = \langle \overline{a}, \overline{b}, \overline{\beta}\rangle \notin \operatorname{Ass}(R_1)$ we continue with the algorithm.

(2) Since $\overline{b}-\overline{\beta} \in \mathfrak{m}_1 \setminus (\langle \overline{a}, \overline{\beta}\rangle \cup \langle \overline{a}, \overline{b}\rangle)$, we know $\overline{a}-\overline{\alpha}, \overline{b}-\overline{\beta}$ is an R-regular sequence. We compute the associate primes of R_2 to verify that the algorithm terminates. We compute

$$R_2 = \frac{R}{\langle \overline{a}-\overline{\alpha}, \overline{b}-\overline{\beta}\rangle} \cong \frac{k[\![a,b,\alpha,\beta]\!]}{\langle ab, a\alpha, b\beta, a-\alpha, b-\beta\rangle} \cong \frac{k[\![a,b]\!]}{\langle ab, a^2, b^2\rangle},$$

and the decomposition $\langle ab, a^2, b^2\rangle = \langle a, b^2\rangle \cap \langle a^2, b\rangle$, so we have

$$\operatorname{Ass}(R_2) = \{\langle \overline{a}, \overline{b}\rangle\} = \{\mathfrak{m}_2\}.$$

Therefore the algorithm terminates, so we conclude $\overline{a}-\overline{\alpha}, \overline{b}-\overline{\beta}$ is a maximal R-regular sequence and $\operatorname{depth}(R)=2$, as claimed.

(c) (Emmy ring) Set $R = k[\![X,Y]\!]/\langle X^2, XY\rangle$. We claim $\operatorname{depth}(R)=0$. We justify this two ways. First, observe that $\operatorname{Ann}_R(\overline{X}) = \langle \overline{X}, \overline{Y}\rangle = \mathfrak{m}$, so $\mathfrak{m} \in \operatorname{Ass}(R)$. Alternatively, we have the irredundant decomposition $\langle X^2, XY\rangle = \langle X\rangle \cap \langle X^2, Y\rangle$, so $\operatorname{Ass}(R) = \{\langle \overline{X}\rangle, \langle \overline{X}, \overline{Y}\rangle\}$, which contains $\mathfrak{m}$. In either case we conclude that the algorithm given in IV.A.4.16 terminates in step zero.

We continue with a discussion of the graded situation.

thm200903a

THEOREM IV.A.4.18. *Assume that R is standard graded, and let $\mathfrak{p} \in \operatorname{Ass}(R)$. Then $\mathfrak{p}$ is homogeneous (i.e.,generated by homogeneous elements of R) and $\mathfrak{p} = \operatorname{Ann}_R(r)$, where r is homogeneous.*

ex200903b

EXAMPLE IV.A.4.19. Let $R = k[X,Y]/\langle X^2, XY\rangle$, then $\operatorname{Ass}(R) = \{\langle \overline{X}\rangle, \langle \overline{X}, \overline{Y}\rangle\}$ and

$$\langle \overline{X}\rangle = \operatorname{Ann}_R(\overline{Y}),$$
$$\langle \overline{X}, \overline{Y}\rangle = \operatorname{Ann}_R(\overline{X}).$$

lem200903c

LEMMA IV.A.4.20 (Graded Prime Avoidance). *Let R be standard graded and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \operatorname{Spec}(R)$ be not necessarily homogeneous and let $I \lneq R$ be a homogeneous ideal such that $I \not\subseteq \mathfrak{p}_i$ for all $i = 1, \ldots, n$. Then there exists a homogeneous element $f \in (I \setminus \cup_{i=1}^n \mathfrak{p}_i) \cap R_+$ (where R_+ consists of all homogeneous elements of positive degree).*

cor200903d

COROLLARY IV.A.4.21. *In Algorithm IV.A.4.16, if R is standard graded, then the maximal R-regular sequence in $\mathfrak{m} = R_+$ can be chosen to consist of homogeneous elements of positive degree. If $|R_0| = \infty$, then the elements can be chosen to be in R_1.*

ex200903e

EXAMPLE IV.A.4.22.

ex200903e.a

(a) Let $R = k[a,\alpha,b,\beta]/\langle a\alpha, b\beta, ab\rangle$. The maximal R-regular sequence in R_+ is $\overline{a}-\overline{\alpha}, \overline{b}-\overline{\beta}$.

ex200903e.b (b) What happens in Corollary IV.A.4.21 if $|R_0| < \infty$? Let $R = \mathbb{F}_2[X,Y]/\langle XY(X+Y)\rangle$. Then X, Y, and $X+Y$ are all zero divisors on R, so you cannot start an R-regular sequence with homogeneous elements of degree 1. However, we have that

$$\operatorname{Ass}(R) = \left\{\left\langle\overline{X}\right\rangle, \left\langle\overline{Y}\right\rangle, \left\langle\overline{X}+\overline{Y}\right\rangle\right\}$$

and

$$\overline{X}^2 + \overline{XY} + \overline{Y}^2 \notin \left\langle\overline{X}\right\rangle \cup \left\langle\overline{Y}\right\rangle \cup \left\langle\overline{X}+\overline{Y}\right\rangle.$$

Therefore $\overline{X}^2 + \overline{XY} + \overline{Y}^2$ is a non-zero divisor on R and is homogeneous of degree 2, so a maximal R-regular sequence can be started with an element in R_2.

warn200903f WARNING IV.A.4.23. We need to be careful in the multi-graded setting.

arn200903f.a (a) Associated primes in the $\mathbb{N}^d$-graded setting will all be $\mathbb{N}^d$-graded. For example, in the monomial setting (like in Example IV.A.4.17), the associated primes will all be generated by monomials.

arn200903f.b (b) However, the associated primes in this case will not be annihilators of multi-graded homogeneous elements. What goes wrong? In the $\mathbb{N}$-graded setting, let $a \in R_i$ and $b \in R_j$ satisfy $i, j \neq 0$. Then we have $a^j + b^i \in R_{ij}$. However, in the $\mathbb{N}^d$-graded setting for $d \geq 2$, let $a \in R_{\mathbf{i}}$ and $b \in R_{\mathbf{j}}$ such that $\mathbf{i}, \mathbf{j} \neq \mathbf{0}$. Then one cannot define the sum $a^{\mathbf{j}} + b^{\mathbf{i}}$ to work as in the $\mathbb{N}$-graded case.

We conclude with a brief discussion of the issues behind permutations of regular sequences.

ex200903g EXAMPLE IV.A.4.24. Rearrangements of R-regular sequences need not be R-regular. For example, let $R = k[X,Y,Z]$. Then the sequence $X, Y(1-X), Z(1-X)$ is R-regular, but the rearranged sequence $Y(1-X), Z(1-X), X$ is not R-regular. However, the next result shows that this can't happen in the local and standard graded cases.

thm200903h THEOREM IV.A.4.25. *Assume that* $(R, \mathfrak{m})$ *is a local or standard graded ring and let* $\mathbf{x} = x_1, \ldots, x_n \in \mathfrak{m}$. *In the graded case, assume that the* x_i *are all homogeneous.*

thm200903h.a (a) *The following are equivalent:*
  (i) *The sequence* $\mathbf{x}$ *is* R*-regular.*
  (ii) $H_i(K^R(\mathbf{x})) = 0$ *for all* $i \geq 1$, *where* $K^R(\mathbf{x})$ *is the Koszul complex on* n *variables.*
  (iii) $H_1(K^R(\mathbf{x})) = 0$.

thm200903h.b (b) *If* $\mathbf{x}$ *is* R*-regular, then* $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$ *is* R*-regular for all* $\sigma \in S_n$.

SKETCH OF PROOF. (a) $\Rightarrow$ (b): For the Koszul complex, we have the property that

$$K^R(\mathbf{x}) \cong K^R(x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \qquad \square$$

ection063021s

IV.A.5. Monomial Ideals

Assume for this chapter that $S = k[X_1, \ldots, X_d]$ where k is a field. We flesh out some of the ideas used in preceding sections.

fact200903i FACT IV.A.5.1. *Let* I *be a monomial ideal in* S, *i.e.,* I *is generated by monomials* $\mathbf{X}^{\mathbf{a}} = X_1^{a_1} \cdots X_d^{a_d}$ *for some* $\mathbf{a} \in \mathbb{N}^d$. *Then* $\operatorname{depth}(S/I) = 0$ *if and only if* $\left\langle\overline{X}_1, \ldots, \overline{X}_d\right\rangle \in \operatorname{Ass}(S/I)$ *if and only if in the decomposition from Fact IV.A.4.9, an ideal of the form* $\langle X_1^{e_1}, \ldots, X_d^{e_d}\rangle$ *occurs for some* $e_i \geq 1$ *for all* $i = 1, \ldots, d$. *For example, continuing with Emmy's Ring from Example IV.A.4.17, let* $R = k[X,Y]/\left\langle X^2, XY\right\rangle$. *Then* $\left\langle X^2, XY\right\rangle = \langle X\rangle \cap \left\langle X^2, Y\right\rangle$ *and the second ideal in the intersection uses all variables, so* $\operatorname{depth}(R) = 0$.

The following ideals are used to describe the decompositions from Fact IV.A.5.1.

defn200903j DEFINITION IV.A.5.2. An ideal $I \lneq S$ is irreducible if it cannot be decomposed non-trivially as an intersection of ideals, i.e., for all $J, K \leq S$, if $I = J \cap K$, then $I = J$ or $I = K$. On the other hand, I is reducible if it is not irreducible.

ex200903k EXAMPLE IV.A.5.3.

ex200903k.a (a) $\langle XY\rangle = \langle X\rangle \cap \langle Y\rangle$, so $\langle XY\rangle$ is reducible.

ex200903k.b (b) $\langle a\alpha, b\beta, ab\rangle = \langle a, b\rangle \cap \langle a, \beta\rangle \cap \langle \alpha, b\rangle$, so $\langle a\alpha, b\beta, ab\rangle$ is reducible.

ex200903k.c (c) $\left\langle X^2, XY\right\rangle = \langle X\rangle \cap \left\langle X^2, Y\right\rangle$, so $\left\langle X^2, XY\right\rangle$ is reducible.

ex200903k.d (d) $0 \leq S$ is irreducible.

The next result allows us to identify easily the irreducible monomial ideals.

prop2009031 PROPOSITION IV.A.5.4. *Let $0 \neq I \lneq S$ be a monomial ideal. Then I is irreducible if and only if I is of the form $\langle X_{i_1}^{a_1}, \ldots, X_{i_n}^{a_n} \rangle$ for some $i_j, a_j \geq 1$.*

The preceding proposition shows that most monomial ideals are not irreducible. However, as we see next, every proper monomial ideal can be decomposed nicely as an intersection of irreducible ideals. One may think of this as a version of prime factorization of integers for monomial ideals.

defn200903m DEFINITION IV.A.5.5. An irreducible decomposition of an ideal $I \lneq S$ is a decomposition $I = Q_1 \cap \cdots \cap Q_n$ such that each Q_i is irreducible. The intersection is irredundant if $I \neq \bigcap_{i \neq j} Q_i$ for all j.

thm200903n THEOREM IV.A.5.6 (Emmy Noether). *Let $I \lneq S$ be an ideal. Then I has an irredundant irreducible decomposition.*

thm200903o THEOREM IV.A.5.7. *Let $I \lneq S$ be a monomial ideal. Then the irredundant irreducible decomposition of I is unique up to reording of the factors. The factors in the irredundant decomposition are all monomial ideals.*

The next result, in conjunction with Theorem IV.A.5.12 below, makes it routine to check irredundancy of irreducible decomposition.

thm200903p THEOREM IV.A.5.8. *Let $I \lneq S$ be a monomial ideal with irreducible decomposition $I = Q_1 \cap \cdots \cap Q_n$. Then the decomposition is irredundant if and only if $Q_i \not\subseteq Q_j$ for all $i \neq j$.*

Even though the decomposition in the next result is not generally finite, it can still be useful. See Theorem IV.A.5.21c for another version of this.

thm200910a THEOREM IV.A.5.9. *Let $I \leq S$ be a monomial ideal. Then*

$$I = \bigcap_{\substack{I \subseteq J \\ \text{monom. ideal} \\ \text{irreducible}}} J,$$

where the intersection is taken over all irreducible monomial ideals J such that $I \subseteq J$.

PROOF. Note that if $I = S$, then there are no such J's. Since I is the empty intersection in this case, the desired conclusion vacuously holds. Therefore, we can assume without loss of generality that $I \lneq S$. Set

$$K = \bigcap_{\substack{I \subseteq J \\ \text{monom. ideal} \\ \text{irreducible}}} J \supseteq I,$$

and note that $I \subseteq K$ by definition. Then we need to show that $K \subseteq I$. We have the decomposition $I = Q_1 \cap \cdots \cap Q_n$ by Theorem IV.A.5.8, where each Q_i is an irreducible monomial ideal containing $Q_1 \cap \cdots \cap Q_n = I$. Therefore, each Q_i occurs in the intersection defining K, so K is obtained by intersecting a set of ideals including $Q_1, \ldots, Q_n$. Therefore, $K \subseteq Q_1 \cap \cdots \cap Q_n = I$. □

In part to make our use of Theorem IV.A.5.8 easier, we next describe how to detect containments between monomial ideals.

notn200910b NOTATION IV.A.5.10. Let $[\![S]\!]$ be the set of all monomials in S. For all ideals $I \leq S$, set $[\![I]\!] = I \cap [\![S]\!]$.

Our test for containment of monomial ideals uses finite monomial generating sequences which are guaranteed to exist by the following application of the noetherian property or, if one prefers, by Dickson's Lemma.

fact200910c FACT IV.A.5.11. *If $I \leq S$ is a monomial ideal, then I is generated by a finite list of monomials in I, i.e., there exist $f_1, \ldots, f_m \in [\![I]\!]$ such that $I = \langle f_1, \ldots, f_m \rangle$.*

In general, it is difficult to check containments between two ideals. However, the next result shows that it is quite easy in the monomial case.

thm200910d THEOREM IV.A.5.12. *Let $f_1, \ldots, f_m, g_1, \ldots, g_n \in [\![S]\!]$. Set $I = \langle f_1, \ldots, f_m \rangle$ and $J = \langle g_1, \ldots, g_n \rangle$.*

thm200910d.a (a) *We have $f_1 \in J$ if and only if $f_1 \in \langle g_i \rangle$ for some i if and only if $g_i \mid f_1$ for some i.*

thm200910d.b (b) *The following are equivalent.*

(i) $I \subseteq J$

(ii) $f_1, f_2, \ldots f_m \in J$

(iii) *for all j, there is some ij such that $f_j \in \langle g_{ij} \rangle$*
(iv) *for all j, there is some ij such that $g_{ij} \mid f_j$*
(v) $[\![I]\!] \subseteq [\![J]\!]$

Here is an example showing how this result can be used to verify irreduncancy of decompositions.

ex200910e EXAMPLE IV.A.5.13. Let $I = \langle X \rangle$ and $J = \langle X^2, Y \rangle$. Then $I \not\subseteq J$ because $X^2 \nmid X$ and $Y \nmid X$. Also, $I \not\supseteq I$ because $X \nmid Y$. Thus the decomposition $\langle X^2, XY \rangle = \langle X \rangle \cap \langle X^2, Y \rangle$ is irredundant by Theorem IV.A.5.8.

defn200910f DEFINITION IV.A.5.14. Let $f_1, \ldots, f_m \in [\![S]\!]$ and let $I = \langle f_1, \ldots, f_m \rangle$. Then the list $f_1, \ldots, f_m$ is <u>redundant</u> if $I = \langle f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m \rangle$ for some i. The list is <u>irredundant</u> if it is not redundant, i.e., for all $i = 1, \ldots, m$, we have $I \supsetneq \langle f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m \rangle$.

By definition, irredundant irreducible decompositions are optimal decompositions. Next, we discuss optimal choices for generators of monomial ideals.

thm200910g THEOREM IV.A.5.15. *Let $\mathbf{f} = f_1, \ldots, f_m \in [\![S]\!]$.*

thm200910g.a (a) *Then $\mathbf{f}$ is irredundant if and only if $f_i \nmid f_j$ for all $i \neq j$.*

thm200910g.b (b) *Every monomial ideal in S has an irredundant monomial generating sequence and such generating sequences are unique up to reordering.*

ex200910h EXAMPLE IV.A.5.16. Note that $I = \langle ab, a\alpha, b\beta \rangle$ is irredundant because none of the generating monomials divide any of the other ones, i.e.,$ab \nmid a\alpha$ and $a\alpha \nmid ab$ and $ab \nmid b\beta$ and $b\beta \nmid ab$ and $a\alpha \nmid b\beta$ and $b\beta \nmid a\alpha$.

Next, we describe how to how do we compute irreducible decompositions.

thm200910i THEOREM IV.A.5.17 ("Splitting generators"). *Let $f_1, \ldots, f_m \in [\![S]\!]$ and $I = \langle f_1, \ldots, f_m \rangle$. Suppose that $f_1 = X_{i_1}^{a_1} \cdots X_{i_p}^{a_p}$ is such that $a_i \in \mathbb{N}$ and $i_1 < i_2 < \cdots < i_p$. Then*

$$I = \bigcap_{j=1}^{p} \left\langle X_{i_j}^{a_j}, f_2, \ldots, f_m \right\rangle.$$

ex200910j EXAMPLE IV.A.5.18. Consider $I = \langle ab, a\alpha, b\beta \rangle$. Then by splitting generators and removing redundancies we have

$$\begin{aligned} I &= \langle a, \cancel{a\alpha}, b\beta \rangle \cap \langle b, a\alpha, \cancel{b\beta} \rangle \\ &= \langle a, b\beta \rangle \cap \langle b, a\alpha \rangle \\ &= \langle a, b \rangle \cap \langle a, \beta \rangle \cap \cancel{\langle b, a \rangle} \cap \langle b, \alpha \rangle \\ &= \langle a, b \rangle \cap \langle a, \beta \rangle \cap \langle b, \alpha \rangle . \end{aligned}$$

Therefore the irredundant irreducible decomposition of I is

$$I = \langle a, b \rangle \cap \langle a, \beta \rangle \cap \langle b, \alpha \rangle .$$

The remainder of this chapter deals with some families of monomial ideals where one can obtain high-level algebraic information visually.

defn200910k DEFINITION IV.A.5.19. A monomial f is <u>square-free</u> if it is not divisible by any squares of variables. A monomial ideal is <u>square-free</u> if it can be generated by square-free monomials.

ex200910l EXAMPLE IV.A.5.20.

(a) The ideal $\langle ab, a\alpha, b\beta \rangle$ is square-free.
(b) The ideal $\langle X^2, XY \rangle$ is not square-free.

thm200910m THEOREM IV.A.5.21. *Let $I \nleq S$ be a monomial ideal.*

thm200910m.a (a) *The following condition are equivalent:*
 (i) *I is square-free.*
 (ii) *The irredundant monomial generating sequence for I consists of square-free monomials.*
 (iii) *I is a finite intersection of irreducible square-free monomial ideals.*

thm200910m.b (b) *Non-zero irreducible square-free monomial ideals are generated by lists of variables, so they are prime.*

thm200910m.c (c) *If I is square-free, then $I = \bigcap J$, where the intersection is taken over all irreducible square-free monomial ideals containing I.*

thm200910m.d (d) *If I is square-free, then the intersection in part c can be taken over all minimal elements in the set $\{$irreducible square-free monomial ideals containing $I\}$. Moreover, the resulting decomposition is irredundant.*

ex200910n EXAMPLE IV.A.5.22. Let $I = \langle ab, a\alpha, b\beta\rangle$. Let J be generated by variables in the ring $S = k[a, b, \alpha, \beta]$ such that $J \supseteq I$ and observe that J is prime. Then $ab \in I \subseteq J$ implies that $a \in J$ or $b \in J$. Similarly, we have

$$\begin{aligned} a\alpha \in I \subseteq J \quad &\Rightarrow \quad a \in J \text{ or } \alpha \in J, \\ b\beta \in I \subseteq J \quad &\Rightarrow \quad b \in J \text{ or } \beta \in J. \end{aligned}$$

Therefore, $J \supseteq \langle a, b\rangle$ or $J \supseteq \langle a, \beta\rangle$ or $J \supseteq \langle \alpha, b\rangle$.

Edge Ideals. For this section, let G be a finite simple graph with vertex set $V = \{v_1, \ldots, v_d\}$. Recall that a simple graph is one with undirected edges and no multiple edges.

ex200910o EXAMPLE IV.A.5.23.

ex200910o.a (a) The 3-path or path with 3 edges, denoted P_3, can be represented as follows:

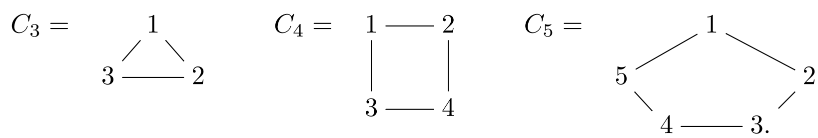
$$\alpha \text{ —— } a \text{ —— } b \text{ —— } \beta.$$

Furthermore, the path with n edges, denoted P_n, can be presented in either of the following two ways:

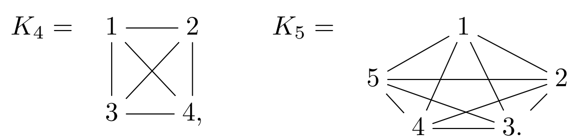
$$v_0 \text{ —— } v_1 \text{ —— } \cdots \text{ —— } v_n,$$

$$0 \text{ —— } 1 \text{ —— } \cdots \text{ —— } n.$$

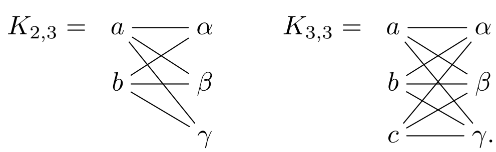
ex200910o.b (b) We denote the n-cycle as C_n. For example, we have



ex200910o.c (c) We denote the complete graph on n vertices or the n-clique as K_n. For example, we have



ex200910o.d (d) We denote the complete bipartite graph between m vertices and n vertices as $K_{m,n}$. For example, we have



The focus of this section is the following construction which takes our graph G and outputs a square-free monomial ideal.

defn200910p DEFINITION IV.A.5.24. The edge ideal associated to G is the ideal generated by the edges of G, i.e.

$$I(G) = I_G = \langle X_i X_j \mid v_i v_j \text{ is an edge in } G\rangle .$$

ex200910q EXAMPLE IV.A.5.25. The edge ideal of the 3-path from Example IV.A.5.23a is

$$I(P_3) = \langle \alpha a, ab, b\beta\rangle = \langle ab, a\alpha, b\beta\rangle .$$

The edge ideal of the complete bipartite graph $K_{2,3}$ from Example IV.A.5.23d is

$$I(K_{2,3}) = \langle a\alpha, a\beta, a\gamma, b\alpha, b\beta, b\gamma\rangle .$$

note200910r NOTE IV.A.5.26. The field that encompasses this construction is called combinatorial commutative algebra. In this field, one uses combinatorial objects to create algebraic objects. Then one uses combinatorial properties of the initial combinatorial object to understand algebraic properties of the algebraic object constructed. For instance, as we shall see next, the combinatorial properties of our graph G help us to understand the irreducible decompositions of the edge ideal $I(G)$.

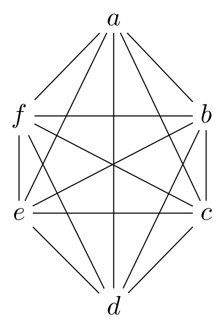
defn200915a DEFINITION IV.A.5.27. A vertex cover of a graph $G = (V, E)$ is a subset $W \subseteq V$ such that every edge is incident to an element of W. A minimal vertex cover of G is a vertex cover W such that for all $w \in W$ the set $W \setminus \{w\}$ is not a vertex cover of G.

ex200915b EXAMPLE IV.A.5.28. (a) We consider the 3-path P_3:

$$\alpha \text{ --- } a \text{ --- } b \text{ --- } \beta.$$

Since $\{a, b\}$ covers all the edges of P_3, it is a vertex cover of P_3. Moreover, it is minimal since neither $\{a\}$ nor $\{b\}$ is a vertex cover of P_3. The other minimal vertex covers of P_3 are $\{a, \beta\}$ and $\{\alpha, b\}$. Since the grouping symbols are superfluous we will often write these vertex covers as ab, $a\beta$, and αb.

(b) Now consider the complete graph on six vertices K_6:



We observe that since every pair of vertices are adjacent, any subset $W \subseteq V = \{a, b, c, d, e, f\}$ is a vertex cover of K_6 if and only if W contains at least one out of every pair of vertices in V. Hence any subset $W \subseteq V$ satisfying $|W^C| \geq 2$ is not a vertex cover of K_6. It follows that the minimal vertex covers of K_6 are exactly the subsets $W \subseteq V$ of size five. For instance, $abcde$ is a minimal vertex cover for K_6. In general, the subsets $W \subseteq \{v_1, \ldots, v_n\}$ of size $n - 1$ are precisely the minimal vertex covers for K_n.

The next result shows how combinatorial information about G, namely the (minimal) vertex covers, describes algebraic properties of the edge ideal $I(G)$, i.e., (irredundant) irreducible decompositions.

thm200915c THEOREM IV.A.5.29. *If G is a finite simple graph, then the edge ideal can be decomposed as*

$$I(G) = \bigcap_{\substack{W \subseteq V \\ W \text{ a vertex} \\ \text{cover}}} \langle W \rangle = \bigcap_{\substack{W \subseteq V \\ W \text{ a minimal} \\ \text{vertex cover}}} \langle W \rangle ,$$

where the first intersection is taken over all vertex covers of G and the second intersection is taken over all minimal vertex covers of G. The second decomposition is also irredundant.

ex200915d EXAMPLE IV.A.5.30. (a) The edge ideal of P_3 can be decomposed irredundantly as

$$I(P_3) = \langle a\alpha, ab, b\beta \rangle = \langle a, b \rangle \cap \langle a, \beta \rangle \cap \langle \alpha, b \rangle ,$$

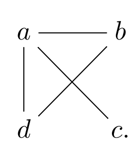
using the three minimal vertex covers ab, $a\beta$, and αb found in Example IV.A.5.28(a).

(b) The edge ideal of K_4 can be decomposed as

$$I(K_4) = \langle ab, ac, ad, bc, bd, cd \rangle = \langle a, b, c \rangle \cap \langle a, b, d \rangle \cap \langle a, c, d \rangle \cap \langle b, c, d \rangle$$

using the minimal vertex covers for K_n described in Example IV.A.5.28(b).

(c) Every square free monomial ideal generated by degree-2 monomials is an edge ideal and therefore can be decomposed in this way, e.g., the ideal $I = \langle ab, ac, ad, bd \rangle$ is the edge ideal of the simple graph



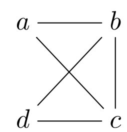
The minimal vertex covers of this graph are ab, ad, and bcd, and therefore by Theorem IV.A.5.29 the irredundant irreducible decomposition of I is

$$I = \langle a, b\rangle \cap \langle a, d\rangle \cap \langle b, c, d\rangle .$$

We can check this decomposition of I as in Example IV.A.5.18.

$$\begin{aligned} I &= \langle a, \cancel{ac}, \cancel{ad}, bd\rangle \cap \langle b, ac, ad, \cancel{bd}\rangle \\ &= \langle a, b\rangle \cap \langle a, d\rangle \cap \langle b, a, \cancel{ad}\rangle \cap \langle b, c, ad\rangle \\ &= \langle a, b\rangle \cap \langle a, d\rangle \cap \cancel{\langle b, a\rangle} \cap \cancel{\langle b, c, a\rangle} \cap \langle b, c, d\rangle \\ &= \langle a, b\rangle \cap \langle a, d\rangle \cap \langle b, c, d\rangle \end{aligned}$$

(d) In contrast with the previous part, we can compute an irredundant decomposition of an edge ideal in order to find all the minimal vertex covers of the corresponding graph. For the simple graph G given by



the edge ideal $I(G)$ can be decomposed as follows.

$$\begin{aligned} \langle ab, ac, bc, bd, cd\rangle &= \langle a, \cancel{ac}, bc, bd, cd\rangle \cap \langle b, ac, \cancel{bc}, \cancel{bd}, cd\rangle \\ &= \langle a, b, \cancel{bd}, cd\rangle \cap \langle a, c, bd, \cancel{cd}\rangle \cap \langle b, a, cd\rangle \cap \langle b, c, \cancel{cd}\rangle \\ &= \cancel{\langle a, b, c\rangle} \cap \langle a, b, d\rangle \cap \cancel{\langle a, c, b\rangle} \cap \langle a, c, d\rangle \cap \cancel{\langle b, a, c\rangle} \cap \cancel{\langle b, a, d\rangle} \cap \langle b, c\rangle \\ &= \langle a, b, d\rangle \cap \langle a, c, d\rangle \cap \langle b, c\rangle \end{aligned}$$

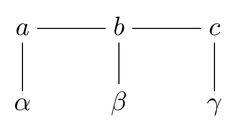
Next, we are interested in the problem of finding graphs whose edge ideals are Cohen-Macaulay. The next graphs will be key for this.

defn200915e

Definition IV.A.5.31. Let G be a finite simple graph with vertex set $V = \{v_1, \ldots, v_d\}$. The suspension of G (also known as the K_1-corona of G) is a new graph ΣG with vertex set $V(\Sigma G) = \{v_1, \ldots, v_d, w_1, \ldots, w_d\}$ and edge set $E(\Sigma G) = E(G) \cup \{v_1w_1, v_2w_2, \ldots, v_dw_d\}$.

ex200915f

Example IV.A.5.32. Set G to be the path $P_2 = (\ a \text{ ------ } b \text{ ------ } c\ )$, which has vertex covers b, ac, ab, bc, and abc. The suspension ΣG given by



has minimal vertex covers that we obtain from "filling out" the vertex covers of G with elements from $V(\Sigma G) \setminus V(G)$:

$$\begin{array}{ccc} \text{vertex covers of } G & & \text{minimal vertex covers of } \Sigma G \\ abc & \longrightarrow & abc \\ ab & \longrightarrow & ab\gamma \\ ac & \longrightarrow & a\beta c \\ bc & \longrightarrow & \alpha bc \\ b & \longrightarrow & \alpha b\gamma. \end{array}$$

Hence the irredundant irreducible decomposition of $I(\Sigma G)$ is

$$I(\Sigma G) = \langle a, b, c\rangle \cap \langle a, b, \gamma\rangle \cap \langle a, \beta, c\rangle \cap \langle \alpha, b, c\rangle \cap \langle \alpha, b, \gamma\rangle .$$

Since every irreducible component of $I(\Sigma G)$ has the same number of generators, we say that $I(\Sigma G)$ is "unmixed". Moreover, we shall see that the ring $S/I(\Sigma G)$ is Cohen-Macaulay, where $S = k[a, b, c, \alpha, \beta, \gamma]$.

thm200915g THEOREM IV.A.5.33. *In the notation of Definition IV.A.5.31, we have*

$$\operatorname{depth}(S/I(\Sigma G)) = d$$

where a maximal $S/I(\Sigma G)$-regular sequence is $v_1 - w_1, v_2 - w_2, \ldots, v_d - w_d$.

Simplicial Complexes and Stanley-Reisner Rings. We have seen that the edge ideal construction allows one to decompose square-free quadratic monomial ideals combinatorially. Next, we do this for arbitrary square-free monomial ideals, using simplicial complexes.

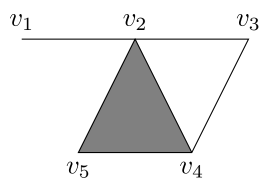
defn200915i DEFINITION IV.A.5.34. A simplicial complex with vertex set $V = \{v_1, v_2, \ldots, v_d\}$ is a collection $\Delta \neq \emptyset$ of subsets of V closed under subsets, i.e., $\emptyset \neq \Delta \subseteq \mathcal{P}(V)$ is a "downset". The elements of Δ are the faces of Δ and the maximal faces with respect to containment are the facets of Δ. The simplicial complex $\Delta = \mathcal{P}(V) = \Delta_{d-1}$ is the $(d-1)$-simplex.

Our definition of a simplicial complex is a set-theoretic one, but often it is helpful to think in terms of a more geometric representation.

ex200915j EXAMPLE IV.A.5.35. (a) The simplicial complex

$$\Delta = \{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_2, v_4, v_5\}\}$$

has the following geometric realization.



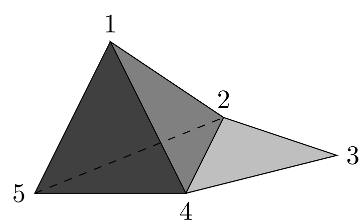
Alternatively, we can drop the grouping symbols from the set-theoretic representation to get

$$\Delta = \{\emptyset, v_1, v_2, v_3, v_4, v_5, v_1v_2, v_1v_3, v_2v_4, v_2v_5, v_3v_4, v_4v_5, v_2v_4v_5\}.$$

Moreover, since simplicial complexes are closed under taking subsets, they are completely determined by their facets, and we can write

$$\Delta = \langle v_1v_2, v_2v_3, v_3v_4, v_2v_4v_5\rangle.$$

ex200915j.b (b) The simplicial complex $\Delta = \langle 1245, 234\rangle$ has two facets: the solid tetrahedron $\langle 1245\rangle$ and the shaded triangle $\langle 234\rangle$. This has the following geometric realization.



fact200915k FACT IV.A.5.36. *Let Δ be a simplicial complex on $V = \{v_1, v_2, \ldots, v_d\}$.*

fact200915k.a (a) *Every face of Δ is a subset of a facet of Δ (because $|V| < \infty$).*

fact200915k.b (b) *Δ has a facet (because $\Delta \neq 0$).*

Next, we use simplicial complexes to construct more square-free monomial ideals.

defn200915l DEFINITION IV.A.5.37. The Stanley-Reisner ideal associated to Δ is the ideal of $S = k[X_1, \ldots, X_d]$ generated by the non-faces of Δ:

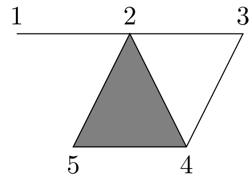
$$J(\Delta) = J_\Delta = \left\langle \mathbf{X}^W \;\middle|\; W \subseteq V \text{ and } W \notin \Delta \right\rangle$$

where $\mathbf{X}^W = \prod_{v_i \in W} X_i$. This is a square free monomial ideal.

note200917a NOTE IV.A.5.38. By definition, we have that J_Δ is a square-free monomial ideal.

ex200917b EXAMPLE IV.A.5.39.

ex200917b.a (a) Consider the simplicial complex Δ from Example IV.A.5.35(a) with the following geometric realization.



We can construct J_Δ by looking at all subsets of the vertex set and removing those which appear as faces in Δ.

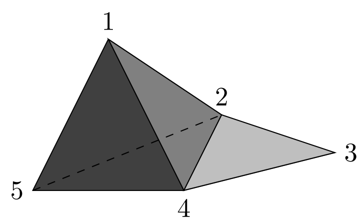
$$\begin{aligned}\text{Subsets not in } \Delta\text{:}\ & 12345, 1234, 1235, 1245, 1345, 2345,\\ & 123, 124, 125, 134, 135, 145, 234, 235, \cancel{245}, 345,\\ & \cancel{12}, 13, 14, 15, \cancel{23}, \cancel{24}, \cancel{25}, \cancel{34}, 35, \cancel{45}, \cancel{1}, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{5}.\end{aligned}$$

Then removing redundancies, we obtain the irredundant generators of the corresponding Stanley-Reisner ideal:

$$\begin{aligned}J_\Delta &= \langle 13, 14, 15, 35, \cancel{123}, \cancel{124}, \cancel{125}, \cancel{134}, \cancel{135}, \cancel{145}, 234, \cancel{235}, \cancel{345}, \cancel{1234}, \cancel{1235}, \cancel{1245}, \cancel{1345}, \cancel{2345}, \cancel{12345}\rangle\\ &= \langle 13, 14, 15, 35, 234\rangle\,.\end{aligned}$$

In general, the irredundant monomial generators of J_Δ are the minimal non-faces of Δ.

ex200917b.b (b) Consider the simplicial complex Δ from Example IV.A.5.35(b) with the following geometric realization.



Then the minimal non-faces of Δ are 13 and 35, so we have

$$J_\Delta = \langle 13, 35\rangle = \langle 3\rangle \cap \langle 1, 5\rangle\,.$$

We shall see in Theorem IV.A.5.49 below that the ideal $\langle 3\rangle$ in this decomposition corresponds to the solid tetrahedron 1245 (which misses the vertex 3), and $\langle 1, 5\rangle$ corresponds to the shaded triangle 234 (which misses vertices 1 and 5).

note200917c NOTE IV.A.5.40. The definition of $J(\Delta)$ in terms of non-faces may seem counterintuitive at first. Why not use the faces instead? A big part of the answer is that we have a one-to-one correspondence

$$\{\text{non-zero square-free monomial in } S/J_\Delta\} \iff \Delta.$$

For the example in IV.A.5.39b, the triangle 145 is in Δ and the corresponding monomial $X_1X_4X_5$ is not in J_Δ. Therefore, 145 represents a non-zero element in S/J_Δ.

note200917d NOTE IV.A.5.41. It is straightforward to show that

$$\{\text{square-free monomial ideals in } S = k[X_1, \ldots, X_d]\} \iff \{\text{simplicial complexes on } V = \{v_1, \ldots, v_d\}\}.$$

In particular, if G is a graph on V, then there is a simplicial complex Δ such that $J_\Delta = I_G$. We next describe how to find Δ.

defn200917e DEFINITION IV.A.5.42. Let G be a graph on $V = \{v_1, \ldots, v_d\}$. A subset $U \subseteq V$ is <u>independent</u> if no two vertices in U are adjacent in G. An independent subset $U \subseteq V$ is <u>maximal</u> if for all $v \in V \setminus U$, the set $U \cup \{v\}$ is not independent. In other words, the set $U \cup \{v\}$ is maximal with respect to containment in the set of all independent subsets for G. Furthermore, let Δ_G denote the set of all independent subsets for G. Then Δ_G is the <u>independence complex</u> for G.

note200917f NOTE IV.A.5.43. Every singleton set $\{v_i\}$ is independent because v_i is not adjacent to v_i in G, i.e.,there are no loops in G. Furthermore, every subset of an independent subset is independent. In other words, Δ_G is closed under subsets, i.e., Δ_G is a simplicial complex.

ex200917g EXAMPLE IV.A.5.44.

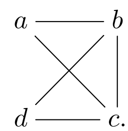
ex200917g.a (a) Consider the 3-path

$$\alpha \text{ —— } a \text{ —— } b \text{ —— } \beta.$$

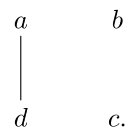
Independent sets for G include $\{\alpha, b\}$, $\{a, \beta\}$, and $\{\alpha, \beta\}$. It is straightforward to show that the independence complex of G is $\Delta_G = \langle \alpha b, a\beta, \alpha\beta \rangle$, which can be represented geometrically as follows:

$$b \text{ —— } \alpha \text{ —— } \beta \text{ —— } a.$$

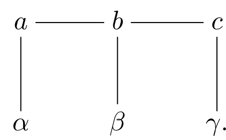
ex200917g.b (b) Consider the simple graph



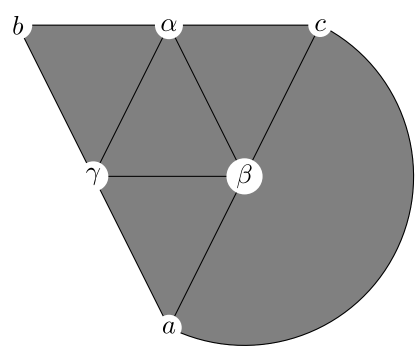
The maximal independent sets are ad, b, and c. Then Δ_G can be represented geometrically as follows:



ex200917g.c (c) Consider the suspension of P_2



The maximal independent sets are $\alpha\beta\gamma$, $a\beta\gamma$, $\alpha b\gamma$, $\alpha\beta c$, and $a\beta c$. Then Δ_G can be represented geometrically as follows:



thm200917h THEOREM IV.A.5.45. *The independence complex Δ_G is precisely the simplicial complex Δ such that $I_G = J_\Delta$, i.e.,$I_G = J_{\Delta_G}$.*

ex200917i EXAMPLE IV.A.5.46.

ex200917i.a (a) Consider the 3-path

$$G = \qquad \alpha \text{ —— } a \text{ —— } b \text{ —— } \beta.$$

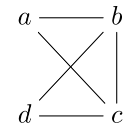
The edge ideal for G is $I_G = \langle a\alpha, ab, b\beta \rangle$. The independence complex Δ_G can be represented as the graph

$$\Delta_G = \qquad b \text{ —— } \alpha \text{ —— } \beta \text{ —— } a.$$

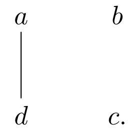
Then the Stanley-Reisner ideal for Δ_G is

$$\begin{aligned} J_{\Delta_G} &= \langle \text{non-edges of } \Delta_G, \text{ missing triangles, etc.} \rangle \\ &= \langle b\beta, a\alpha, ab \rangle \\ &= I_G. \end{aligned}$$

ex200917i.b (b) Consider the simple graph

$G =$ 

and its independence complex

$\Delta_G =$ 

Then $I_G = \langle ab, ac, bc, bd, cd\rangle = J_{\Delta_G}$.

Next, we show how the facets of Δ easily yield the irredundant irreducible decomposition of its Stanley-Reisner ideal.

defn200917j DEFINITION IV.A.5.47. For all $U \subseteq V = \{v_1, \ldots, v_d\}$, set $Q_U = \langle V \setminus U\rangle$.

ex200917k EXAMPLE IV.A.5.48. Consider the vertex set on $d = 6$ vertices. Then $Q_{135} = \langle X_2, X_4, X_6\rangle$.

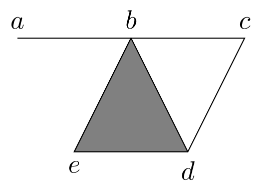
Here is the aforementioned decomposition result for Stanley-Reisner ideals.

thm200917l THEOREM IV.A.5.49. *If Δ is a simplicial complex on V, then we have irreducible decompositions*

$$J_\Delta = \bigcap_{F\in\Delta} Q_F = \bigcap_{\substack{F\in\Delta \\ F \text{ facet}}} Q_F,$$

where the second decomposition is irredundant.

ex200917m.a EXAMPLE IV.A.5.50. (a) Consider the simplicial complex

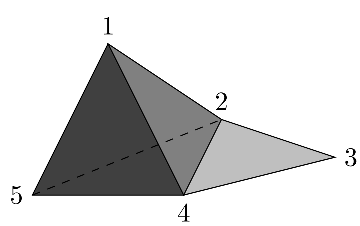


Then we have

$$J_\Delta = Q_{ab} \cap Q_{bc} \cap Q_{cd} \cap Q_{bde} = \langle c, d, e\rangle \cap \langle a, d, e\rangle \cap \langle a, b, e\rangle \cap \langle a, c\rangle\,.$$

One can verify this decomposition by splitting the generators from Example IV.A.5.39(a) as in Example IV.A.5.18.

ex200917m.b (b) Consider the simplicial complex

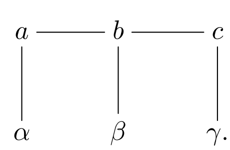


Then we have

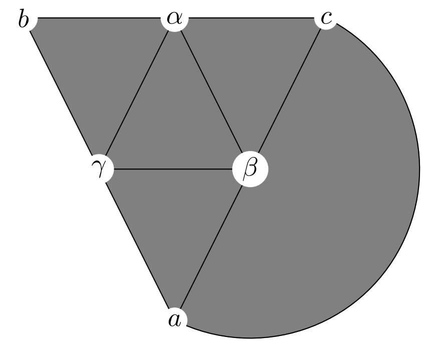
$$J_\Delta = Q_{abde} \cap Q_{bcd} = \langle c\rangle \cap \langle a, e\rangle\,.$$

One can verify this decomposition by splitting the generators from Example IV.A.5.39(b) as in Example IV.A.5.18.

ex200917m.c (c) Consider the graph G given by



The independence complex Δ_G for the above graph is given in Example IV.A.5.44(c) as



Then we have

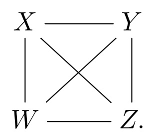
$$\begin{aligned} I_G = J_{\Delta_G} &= Q_{\alpha\beta\gamma} \cap Q_{a\beta\gamma} \cap Q_{\alpha b\gamma} \cap Q_{\alpha\beta c} \cap Q_{a\beta c} \\ &= \langle a, b, c \rangle \cap \langle \alpha, b, c \rangle \cap \langle a, \beta, c \rangle \cap \langle a, b, \gamma \rangle \cap \langle \alpha, b, \gamma \rangle . \end{aligned}$$

Compare this to the decomposition from Example IV.A.5.32.

ex200918a

EXAMPLE IV.A.5.51. We can use these techniques to decompose arbitrary square free monomial ideals.

(a) The ideal $I = \langle XY, XZ, XW, YZ, YW, ZW \rangle$ can be decomposed as the edge ideal of K_4 given by



Recalling that minimal vertex covers of K_n are the subsets of size $n-1$ we have

$$I = I(K_4) = \langle X, Y, Z \rangle \cap \langle X, Y, W \rangle \cap \langle X, Z, W \rangle \cap \langle Y, Z, W \rangle .$$

I is also a Stanley-Reisner ideal where the generators of I are the non-faces of Δ, so Δ has no edges and four vertices. Then Δ has the geometric realization

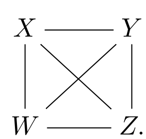
$$\begin{array}{cc} X & Y \\ \\ W & Z \end{array}$$

and using its facets we have

$$I = J(\Delta) = Q_X \cap Q_Y \cap Q_Z \cap Q_W = \langle Y, Z, W \rangle \cap \langle X, Z, W \rangle \cap \langle X, Y, W \rangle \cap \langle X, Y, Z \rangle ,$$

which is the same as above.

(b) The ideal $J = \langle XYZ, XYW, XZW, YZW \rangle$ is not an edge ideal, because the generators are of degree three. However, we can still write $J = J(\Delta)$ where XYZ, XYW, XWZ, and YZW are the minimal non-faces of Δ. Therefore Δ is missing all of the shaded triangles, but has all of the edges in X, Y, Z, and W, i.e., $\Delta = K_4$:



So Δ has six facets and thus the decomposition of J has six components:

$$\begin{aligned} J &= Q_{XY} \cap Q_{XZ} \cap Q_{XW} \cap Q_{YZ} \cap Q_{YW} \cap Q_{ZW} \\ &= \langle Z, W \rangle \cap \langle Y, W \rangle \cap \langle Y, Z \rangle \cap \langle X, W \rangle \cap \langle X, Z \rangle \cap \langle X, Y \rangle . \end{aligned}$$

defn200918b

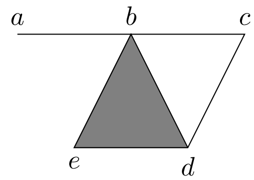
DEFINITION IV.A.5.52. A monomial ideal I is mixed if there exist ideals $p, p' \in \operatorname{Ass}(S/I)$ generated by different numbers of elements, i.e., if there exist ideals Q, Q' in an irredundant irreducible decomposition of I generated by different numbers of elements. I is unmixed if it is not mixed, i.e., all associated primes of S/I are generated by the same number of elements, i.e., all irreducible ideals in an irredundant irreducible decomposition of I are generated by the same number of elements.

ex200918c.a EXAMPLE IV.A.5.53. (a) The ideal $I = \langle XYZ, XYW, XZW, YZW\rangle$ is unmixed, because, in the decomposition

$$I = \langle Z, W\rangle \cap \langle Y, W\rangle \cap \langle Y, Z\rangle \cap \langle X, W\rangle \cap \langle X, Z\rangle \cap \langle X, Y\rangle$$

found in Example IV.A.5.51, we see that every irreducible component has two generators.

ex200918c.b (b) As computed above, the Stanley-Reisner ideal of the simplicial complex Δ with geometric realization



has the decomposition

$$J(\Delta) = \langle c, d, e\rangle \cap \langle a, d, e\rangle \cap \langle a, b, e\rangle \cap \langle a, c\rangle .$$

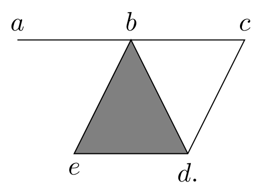
Note that we can see the fact that $J(\Delta)$ is mixed both from the decomposition as well as from the facets of differing sizes in the geometric realization.

note200918d NOTE IV.A.5.54. If $I = J_\Delta$, then I is unmixed if and only if all facets of Δ have the same size (dimension).

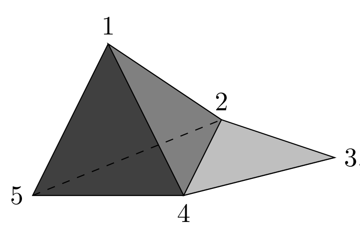
defn200918e DEFINITION IV.A.5.55. A simplicial complex is pure if all its facets have the same dimension, where for every face $F \in \Delta$ we define $\dim F = |F| - 1$. We also define the dimension of a simplicial complex to be the maximum dimension among the dimensions of its facets, i.e.,

$$\dim \Delta = \max\{\dim F \mid F \in \Delta\} = \max\{\dim F \mid F \in \Delta \text{ a facet}\}.$$

ex200918f EXAMPLE IV.A.5.56. (a) The simplicial complex given in Example IV.A.5.53(b) has dimension two, because its largest facet is a filled-in triangle:



(b) The simplicial complex given in Example IV.A.5.35(b) has dimension three, because its largest facet is the solid tetrahedron 1245:



ection063021t

IV.A.6. Krull Dimension

Throughout this chapter we assume R is a non-zero, noetherian, commutative ring with identity, and we assume k is a field. The subject of this chapter is the following measure of the size of R.

defn200918g DEFINITION IV.A.6.1. The Krull dimension of R is

$$\dim(R) = \sup\{n \geq 0 \mid \exists \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \operatorname{Spec}(R)\}.$$

ex200918h EXAMPLE IV.A.6.2. One has $\dim(k) = 0$ since $\operatorname{Spec}(k) = \{0\}$ implies any chain begins with $\mathfrak{p}_0 = 0$ and there exists no suitable $\mathfrak{p}_1$. One has $\dim(\mathbb{Z}) = 1$ because $\operatorname{Spec}(\mathbb{Z}) = \{0, 2\mathbb{Z}, 3\mathbb{Z}, 5\mathbb{Z}, \dots\}$ implies the chains of prime ideals of maximum length have the form $0 \subsetneq p\mathbb{Z}$. By similar reasoning one also has $\dim(k[X]) = 1$. Moreover, if R is a PID (and not a field), then $\dim(R) = 1$, since $\operatorname{Spec}(R) = \{0, pR \mid p \in R \text{ is irreducible}\}$ and every non-zero prime ideal is maximal in a PID.

Geometrically speaking, $\mathbb{R}[X]$ is essentially the 1-dimensional real line $\mathbb{R}^1$ and $k[X]$ is essentially a 1-dimensional line over k, denoted k^1. So, it makes sense for $k[X]$ to be 1-dimensional.

One might think that R will always have finite Krull dimension. However, we have the following:

ex200918i EXAMPLE IV.A.6.3 (Nagata). There exists a non-zero, noetherian, commutative ring with identity with infinite Krull dimension:

$$\begin{aligned} R &= U^{-1}k[X_1, X_2, X_3, \dots] \\ U &= k[X_1, X_2, X_3, \dots] \setminus \left(\bigcup_{N=1}^{\infty} \langle X_{2^{N-1}}, X_{2^{N-1}+1}, \dots, X_{2^N-1} \rangle \right) \\ &= k[X_1, X_2, X_3, \dots] \setminus (\langle X_1 \rangle \cup \langle X_2, X_3 \rangle \cup \langle X_4, X_5, X_6, X_7 \rangle \cup \cdots). \end{aligned}$$

For example, in this ring we can build a chain of length eight:

$$0 \subsetneq \langle X_8 \rangle \subsetneq \langle X_8, X_9 \rangle \subsetneq \cdots \subsetneq \langle X_8, \dots, X_{15} \rangle.$$

In fact, for any $n \in \mathbb{N}$ we can make a chain of length 2^n:

$$0 \subsetneq \langle X_{2^n} \rangle \subsetneq \langle X_{2^n}, X_{2^n+1} \rangle \subsetneq \cdots \subsetneq \langle X_{2^n}, \dots, X_{2^{n+1}-1} \rangle.$$

Hence $\dim(R) = \infty$.

On the other hand, we do have the following result which shows that local rings cannot exhibit the behavior from Example IV.A.6.3.

thm200918j THEOREM IV.A.6.4 (Krull). *If R is local, then $\dim(R) < \infty$, e.g., if S is a non-zero, noetherian, commutative ring with identity and $\mathfrak{p} \in \operatorname{Spec}(S)$, then $\dim(S_\mathfrak{p}) < \infty$, because $(S_\mathfrak{p}, \mathfrak{p}_\mathfrak{p})$ is local.*

Here are some other important properties of the Krull dimension.

It takes some work to show that R is noetherian, hinging on the fact that each localization $R_\mathfrak{m}$ is noetherian and each element $f \in R$ is in only finitely many maximal ideals.

thm200918k THEOREM IV.A.6.5.

thm200918k.a (a) $\dim(R[X]) = \dim(R) + 1 = \dim(R[\![X]\!])$

thm200918k.b (b) $\dim(k[X_1, \dots, X_d]) = d = \dim(k[\![X_1, \dots, X_d]\!])$

thm200918k.c (c) $\dim(U^{-1}R) \leq \dim(R)$ *for each multiplicatively closed subset* $U \subseteq R$.

thm200918k.d (d) *If* $I \lneq R$, *then* $\dim(R/I) \leq \dim(R)$.

thm200918k.e (e) *If* $I \lneq k[X_1, \dots, X_d] = S$, *then* $\dim(S/I) \leq \dim(S) = d$.

thm200918k.f (f) *We have the following equalities.*

$$\begin{aligned} \dim(R) &= \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\} \\ &= \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}(R)\} \\ &= \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Min}(R)\} \end{aligned}$$

Here are some computations showing how to use the various parts of Theorem IV.A.6.4 to calculate the Krull dimension of some rings we've been considering.

ex200918l EXAMPLE IV.A.6.6. (a) For the ideal

$$I = I(P_3) = \langle a\alpha, ab, b\beta \rangle \subsetneq k[a, \alpha, b, \beta] = S,$$

we have the decomposition $I = \langle a, b \rangle \cap \langle a, \beta \rangle \cap \langle \alpha, b \rangle$, and we observe that

$$\operatorname{Ass}(S/I) = \{\langle a, b \rangle, \langle a, \beta \rangle, \langle \alpha, b \rangle\}.$$

We also observe

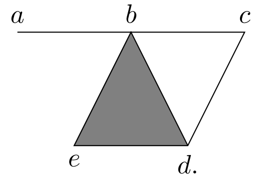
$$\frac{S/I}{\langle a, b \rangle} \cong \frac{k[a, \alpha, b, \beta]}{\langle a, b \rangle} \cong k[\alpha, \beta]$$

which implies

$$\dim\left(\frac{S/I}{\langle a, b \rangle}\right) = 4 - 2 = 2.$$

The rings $\frac{S/I}{\langle a,\beta \rangle}$ and $\frac{S/I}{\langle \alpha,b \rangle}$ have dimension two as well, so we have $\dim(S/I) = \max\{2\} = 2$.

(b) Recall again the simplicial complex Δ with geometric realization



and Stanley-Reisner ideal

$$J_\Delta = \langle c, d, e\rangle \cap \langle a, d, e\rangle \cap \langle a, b, e\rangle \cap \langle a, c\rangle \leq k[a, b, c, d, e] = S.$$

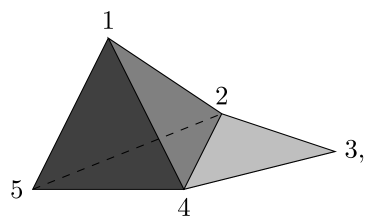
We have

$$\dim(S/J_\Delta) = \max\{5 - 3, 5 - 2\} = 3,$$

which corresponds to the fact that the maximal size among facets of Δ is 3. In general we have

$$\dim\left(\frac{k[X_1, \ldots, X_d]}{J_\Lambda}\right) = \dim\Lambda + 1.$$

For example, if Λ is the simplicial complex



then $\dim(S/J_\Lambda) = 4$.

(c) Given a complete graph K_d, we have $\dim(S/I(K_d)) = 1$. One can see this at least two ways. First, we have a bijection between the associated primes of the quotient ring $S/I(K_d)$ and the minimal vertex covers of K_d, which we recall have the form $\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}$. So the elements of $\operatorname{Ass}(S/I(K_d))$ all look like $P_i = \langle X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_d\rangle$. Hence

$$\dim\left(\frac{S/I(K_d)}{P_i}\right) = d - (d - 1) = 1.$$

On the other hand, we can consider the independence complex $\Delta(K_d)$ which has geometric realization

$$v_1 \qquad v_2 \qquad \cdots \qquad v_d.$$

It therefore has as its facets the d disjoint vertices of the complex. Hence

$$\dim(S/I(K_d)) = \dim(S/J(K_d)) = \max\{1, 1, \ldots, 1\} = 1.$$

(d) Considering the complete bipartite graph $K_{m,n}$, we have $\dim(S/I(K_{m,n})) = \max\{m, n\}$. Indeed , the two minimal vertex covers of $K_{m,n}$ have sizes m and n, respectively. Therefore the associate primes of $S/I(K_{m,n})$ are $P = \langle X_1, \ldots, X_m\rangle$ and $Q = \langle Y_1, \ldots, Y_n\rangle$, and we have

$$\dim\left(\frac{S/I(K_{m,n})}{P}\right) = (m + n) - m = n$$

and

$$\dim\left(\frac{S/I(K_{m,n})}{Q}\right) = (m + n) - n = m,$$

and therefore by Theorem IV.A.6.5(f) we have

$$\dim(S/I(K_{m,n})) = \max\{m, n\}.$$

(e) Given the suspension ΣG of a simple graph $G = (V, E)$ we have $\dim(S/I(\Sigma G)) = d$, where $d = |V|$. For instance, when $G = P_2$ is given by

$$a \text{ ------ } b \text{ ------ } c$$

and ΣG is given by

$$\begin{array}{ccccc} a & \text{---} & b & \text{---} & c \\ | & & | & & | \\ \alpha & & \beta & & \gamma, \end{array}$$

the minimal vertex covers all have size $3 = d = |\{a,b,c\}|$ and therefore

$$\dim(S/I(\Sigma G)) = 2d - d = 3.$$

As in the local case, each of the rings in Example IV.A.6.6 has finite Krull dimension. The next result explains why.

thm200922a THEOREM IV.A.6.7. *If $(R, \mathfrak{m})$ is standard graded, then $\dim(R) = \dim(R_\mathfrak{m})$.*

One of our goals is to justify the inequalities

$$\operatorname{depth}(R) \leq \dim(R) \leq \operatorname{edim}(R).$$

To do this, we give an alternate description of $\dim R$. To motivate this description, we consider the following computations.

ex200922b EXAMPLE IV.A.6.8. (a) The ring $R = k[X]$ is standard graded with graded components $R_n = kX^n$ of vector space dimension $\dim_k(R_n) = 1$ for all $n \geq 0$. We also have $\dim(R/\langle X\rangle^{n+1}) = n+1$ because a k-basis of $R/\langle X\rangle^{n+1}$ is $1, X, \ldots, X^n$.

(b) The ring $R = k[X,Y]$ is also standard graded with graded components $R_n = \operatorname{span}_k\left\{X^iY^j \mid i+j=n\right\}$ of vector space dimension $\dim_k(R_n) = n+1$ for all $n \geq 0$. We have

$$\dim\left(R/\langle X,Y\rangle^{n+1}\right) = 1+2+3+\cdots+(n+1) = \binom{n+2}{2} = \frac{1}{2}n^2 + \frac{3}{2}n + 1.$$

We observe that each of the polynomials in n computed above have degree equal to the number of variables in R, i.e., equal to the Krull dimension of R.

defn200922c DEFINITION IV.A.6.9. Assume $(R, \mathfrak{m}, k)$ is standard graded. Then $h_R(n)$ and $H_R(n)$ given below are Hilbert functions.

$$h_R(n) = \dim_k(R_n)$$

$$H_R(n) = \dim_k(R/R_{\geq n+1}) = \dim_k(R/\mathfrak{m}^{n+1}) = \sum_{i=0}^{n} \dim_k(R_i) = \sum_{i=0}^{n} h_R(i)$$

thm200922d THEOREM IV.A.6.10 (Hilbert). *Assume $(R, \mathfrak{m}, k)$ is standard graded. Then there exist polynomials $p_R(t)$ and $P_R(t)$ in $\mathbb{Z}[t]$ such that*

(1) *$h_R(n) = p_R(n)$ for all $n \gg 0$ (i.e., "the Hilbert function is eventually a polynomial"), where $\deg(p_R(t)) = \dim R - 1 = d - 1$, and*

$$p_R(t) = \frac{e}{(d-1)!}t^{d-1} + \text{ lower degree terms},$$

and $e \geq 1$;

(2) *$H_R(n) = P_R(n)$ for all $n \gg 0$, where $\deg(P_R(t)) = d$ and*

$$P_R(t) = \frac{e}{d!}t^d + \text{ lower degree terms}.$$

defn200922e DEFINITION IV.A.6.11. Both $p_R(t)$ and $P_R(t)$ are Hilbert polynomials of R and e is the degree/multiplicity of R.

ex200922f.a EXAMPLE IV.A.6.12. (a) If $R = k[X_1, \ldots, X_d]$, then

$$P_R(t) = \frac{1}{d!}t^d + \text{ lower degree terms},$$

where $d = \dim R$ and $e = 1$.

ex200922f.b (b) If $R = k[X,Y,Z]/\left\langle X^n - YZ^{n-1}\right\rangle$, then it is straightforward to show that

$$P_R(t) = \frac{n}{2!}t^2 + \text{ lower degree terms},$$

where $\dim R = 2$ and $e = n$.

note200922g NOTE IV.A.6.13. Note that $\dim R \leq 3$ in Example IV.A.6.12(b). In general, if

$$R = \frac{k[X_1, \ldots, X_m]}{\langle \text{homogeneous ideal} \rangle},$$

then $\dim R \leq m$. As with the Krull dimension of $k[X_1, \ldots, X_d]$, there is also geometric content in this inequality . Since

$$V(\text{homogeneous ideal}) \subseteq k^m = \mathbb{A}_k^m$$

and $\dim_k k^m = m$, we have $\dim_k V(\text{homogeneous ideal}) \leq m$, e.g., the variety $V(X^n - YZ^{n-1}) \subseteq \mathbb{R}^3$ is a two-dimensional surface in $\mathbb{R}^3$.

Hilbert polynomials are nice tools for the graded setting, but P. Samuel wanted a similar tool in the local setting.

defn200922h DEFINITION IV.A.6.14. Assume $(R, \mathfrak{m}, k)$ is local. We define the functions

$$h_R(n) = \dim\left(\frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}}\right)$$

and

$$H_R(n) = \sum_{i=0}^{n} h_R(n) \left(= \operatorname{len}\left(R/\mathfrak{m}^{n+1}\right)\right)$$

thm200922i THEOREM IV.A.6.15 (Samuel). *Assume $(R, \mathfrak{m}, k)$ is local. Then there exist polynomials $p_R(t)$ and $P_R(t)$ in $\mathbb{Z}[t]$ such that*

thm200922i.1 (1) *$h_R(n) = p_R(n)$ for all $n \gg 0$ (i.e., "the Hilbert function is eventually a polynomial"), where* $\deg(p_R(t)) = \dim R - 1 = d - 1$, *and*

$$p_R(t) = \frac{e}{(d-1)!} t^{d-1} + \text{ lower degree terms},$$

and $e \geq 1$;

thm200922i.2 (2) *$H_R(n) = P_R(n)$ for all $n \gg 0$, where* $\deg(P_R(t)) = d$ *and*

$$P_R(t) = \frac{e}{d!} t^d + \text{ lower degree terms}.$$

defn200922j DEFINITION IV.A.6.16. We say $h_R(n)$ and $H_R(n)$ are Hilbert-Samuel functions, $p_R(t)$ and $P_R(t)$ are Hilbert- Samuel polynomials, and e is the Hilbert-Samuel multiplicity of R. The embedding dimension of R is

$$\operatorname{edim} R = \dim_k(\mathfrak{m}/\mathfrak{m}^2) = h_R(1) = \text{ minimum number of generators of } \mathfrak{m} =: \beta_0(\mathfrak{m}).$$

Theorem IV.A.6.15 helps us to achieve one of the goals for this section next.

thm200922k THEOREM IV.A.6.17 (Samuel). *If R is local, then*

$$\dim R \leq \operatorname{edim} R.$$

PROOF. Theorem IV.A.6.152 gives the equality in the next display:

$$\dim R = \deg(P_R(t)) \leq \operatorname{edim} R.$$

The inequality follows from some other properties of Hilbert polynomials which we omit in the interest of time. □

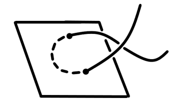
defn200922l DEFINITION IV.A.6.18. A local ring R is regular if $\dim R = \operatorname{edim} R$. This is another "niceness" condition. Geometrically speaking this is a smoothness condition, as $\operatorname{edim} R$ counts the number of independent tangent vector directions.

ex200922m EXAMPLE IV.A.6.19. (a) Given the local ring $R = k[\![X_1, \ldots, X_d]\!]$ with the maximal ideal $\mathfrak{m} = \langle X_1, \ldots, X_d \rangle$, the minimum number of generators of $\mathfrak{m}$ is $\operatorname{edim} R = d = \dim R$, so R is regular.

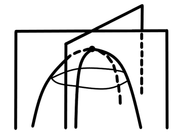
(b) The local quotient ring $R = k[\![X, Y, Z]\!]/\left\langle X^n - YZ^{n-1}\right\rangle$ (with $n \geq 2$) with the maximal ideal $\mathfrak{m} = \left\langle \overline{X}, \overline{Y}, \overline{Z}\right\rangle$ has $\operatorname{edim} R = 3$ and $\dim R = 2$. Since these are not equal, we conclude R is not regular.

note200924a NOTE IV.A.6.20. Consider a couple geometric ideas in $\mathbb{R}^3$.

(a) Consider the curve in $\mathbb{R}^3$ described by one parameter, so the curve has dimension 1. Intersect this curve with a plane in $\mathbb{R}^3$ to get a finite number of points.



(b) Consider a surface in $\mathbb{R}^3$ described by two parameters, so the surface has dimension 2. Intersect this surface with two planes in $\mathbb{R}^3$ to get a finite number of points.



defn200924b DEFINITION IV.A.6.21. Let $\mathfrak{m}$ be a maximal ideal in R and let $I \leq \mathfrak{m}$ be an ideal. Then I is $\mathfrak{m}$-primary if $\mathrm{rad}(I) = \mathfrak{m}$.

ex200924c EXAMPLE IV.A.6.22. (a) Let $R = k[X_1, \ldots, X_d]$ and $\langle X_1, \ldots, X_d\rangle = \mathfrak{m} \leq R$ be a maximal ideal. Consider $I = \langle X_1^{e_1}, \ldots, X_d^{e_d}\rangle$ for $e_i \geq 1$. Then $I \subseteq \mathfrak{m}$ because $X_i^{e_i} \in \langle X_i\rangle \subseteq \mathfrak{m}$ for all i. Therefore we have that $\mathrm{rad}(I) \supseteq \mathrm{rad}(\mathfrak{m}) = \mathfrak{m}$. Also, $x_i \in \mathrm{rad}(I)$ because $x_i^{e_i} \in I$ for all i. Therefore I is $\mathfrak{m}$-primary.

(b) More generally, let I be a monomial ideal and $I \neq S = k[X_1, \ldots, X_d]$. Then I is $\mathfrak{m}$-primary if and only if it contains a positive power of each variable in its generating sequence.

(c) Let $f \in S$ be a homogeneous non-constant polynomial, so $f \in \mathfrak{m}$. Let $R = S/\langle f\rangle$ and $\left\langle \overline{X}_1, \ldots, \overline{X}_d\right\rangle = \mathfrak{n} \in R$. Then $\left\langle \overline{X}_1^{e_1}, \ldots, \overline{X}_d^{e_d}\right\rangle$ for $e_i \geq 1$ is $\mathfrak{n}$-primary.

(d) Let $R = k[X,Y,Z]/\left\langle X^2 - YZ\right\rangle$ and $\mathfrak{m} = \left\langle \overline{X}, \overline{Y}, \overline{Z}\right\rangle$ and $I = \left\langle \overline{Y}, \overline{Z}\right\rangle$. Then I is $\mathfrak{m}$-primary because $\overline{X}^2 = \overline{YZ} \in I$.

thm200924d THEOREM IV.A.6.23. *Assume $(R, \mathfrak{m})$ is local or standard graded and set $d = \dim(R)$. In the graded case, assume that $f_1, \ldots, f_n \in \mathfrak{m}$ are homogeneous.*

(a) (Krull) *If $\langle f_1, \ldots, f_n\rangle$ is $\mathfrak{m}$-primary, then $d \leq n$.*

(b) *There exists and $\mathfrak{m}$-primary ideal I generated by exactly d elements.*

(c) $\dim(R) = \{n \geq 0 \mid R$ *has an $\mathfrak{m}$-primary ideal generated by n elements*$\}$.

defn200924e DEFINITION IV.A.6.24. Assume $(R, \mathfrak{m})$ is local or standard graded. A system of parameters for R is a sequence $f_1, \ldots, f_d \in \mathfrak{m}$ such that $\langle f_1, \ldots, f_d\rangle$ is $\mathfrak{m}$-primary. The number of elements in the generating sequence corresponds to the number of planes required to cut down a curve to a finite number of points in Note IV.A.6.20.

ex200924f EXAMPLE IV.A.6.25. (a) A system of parameters for $k[X_1, \ldots, X_d]$ is $X_1^{e_1}, \ldots, X_d^{e_d}$ such that $e_i \geq 1$.

(b) Two possible systems of parameters for $k[X,Y,Z]/\langle X_2 - YZ\rangle$ are $\overline{YZ}$ or $\overline{Y}^a\overline{Z}^b$ such that $a, b \geq 1$.

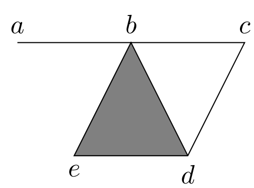
defn200924g DEFINITION IV.A.6.26. $\mathrm{Minh}(R) = \{\mathfrak{p} \in \mathrm{Min}(R) \mid \dim(R/\mathfrak{p}) = \dim(R)\}$.

ex200924h EXAMPLE IV.A.6.27. (a) Let $R = k[\alpha, \beta, a, b]/\langle ab, a\alpha, b\beta\rangle$ be the path P_3. Then

$$\mathrm{Ass}(R) = \left\{\left\langle \overline{a}, \overline{b}\right\rangle, \left\langle \overline{a}, \overline{\beta}\right\rangle, \left\langle \overline{b}, \overline{\alpha}\right\rangle\right\} = \mathrm{Min}(R)$$

because there are no containment relations between distinct associated primes. Also, $\dim(R/\mathfrak{p}) = 2$ for all $\mathfrak{p} \in \mathrm{Ass}(R)$. Therefore, $\mathrm{Minh}(R) = \mathrm{Min}(R) = \mathrm{Ass}(R)$ in this case.

(b) Consider $R = k[a, b, c, d, e]/J_\Delta$, where Δ is the simplicial complex



and $J_\Delta = \langle ac, ad, ae, ce, bcd\rangle$. Then $\mathrm{Ass}(R) = \left\{\left\langle \overline{c}, \overline{d}, \overline{e}\right\rangle, \left\langle \overline{a}, \overline{d}, \overline{e}\right\rangle, \left\langle \overline{a}, \overline{b}, \overline{e}\right\rangle, \langle \overline{a}, \overline{c}\rangle\right\}$. Also, we have $\dim(R/\langle \overline{a}, \overline{c}\rangle) = 3$ and $\dim(R/\mathfrak{p}) = 2$ for all $\mathfrak{p} \in \mathrm{Ass}(R) \setminus \langle \overline{a}, \overline{c}\rangle$. Then $\mathrm{Min}(R) = \mathrm{Ass}(R)$ but $\mathrm{Minh}(R) = \{\langle \overline{a}, \overline{c}\rangle\} \subsetneq \mathrm{Min}(R)$.

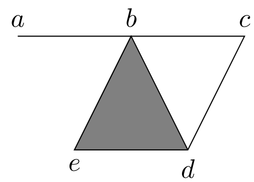
note200924i NOTE IV.A.6.28. $\mathrm{Minh}(R) \subseteq \mathrm{Min}(R) \subseteq \mathrm{Ass}(R)$.

alg200924j ALGORITHM IV.A.6.29. How do we find a system of parameters for a given R?
Step 0: If $\dim(R) = 0$, then stop because the system of parameters has no elements in it.

Step 1: If $\dim(R) \geq 1$, then use prime avoidance to find $x_1 \in \mathfrak{m}$ such that $x_1 \notin \bigcup_{\mathfrak{p}\in\mathrm{Minh}(R)} \mathfrak{p}$. Then set $R_1 = R/\langle x_1\rangle$.
Step 2: If $\dim(R_1) = 0$, then stop. A system of parameters if x_1. Otherwise, continue by finding $x_2 \in \mathfrak{m}$ such that $x_2 \notin \bigcup_{\mathfrak{p}\in\mathrm{Minh}(R_1)} \mathfrak{p}$.
Continue until you find $x_1, \ldots, x_d$.

ex200924k

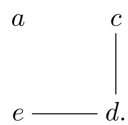
EXAMPLE IV.A.6.30. Consider $R = k[a,b,c,d,e]/J_\Delta$, where Δ is the simplicial complex



and $J_\Delta = \langle ac, ad, ae, ce, bcd\rangle$. From Example IV.A.6.27, we have that $\mathrm{Minh}(R) = \{\langle \overline{a}, \overline{c}\rangle\}$. By Algorithm IV.A.6.29, we can start the system of parameters using one of b or d or e. Since $\dim(R) = 3$, we are looking for a sequence of length 3.
Let $x_1 = \overline{b}$. Then

$$R_1 = R/\left\langle \overline{b}\right\rangle \cong \frac{k[a,b,c,d,e]}{\langle ac, ad, ae, ce, \cancel{bcd}, b\rangle} \cong \frac{k[a,c,d,e]}{\langle ac, ad, ae, ce\rangle}.$$

Also, the simplicial complex Δ_1 is



Then $\mathrm{Ass}(R_1) = \left\{\left\langle \overline{c}, \overline{d}, \overline{e}\right\rangle, \langle \overline{a}, \overline{c}\rangle, \langle \overline{a}, \overline{e}\rangle\right\}$ and $\mathrm{Minh}(R_1) = \{\langle \overline{a}, \overline{c}\rangle, \langle \overline{a}, \overline{e}\rangle\}$. Only d avoids these.
Let $x_2 = \overline{d}$. Then

$$R_2 = R_1/\left\langle \overline{d}\right\rangle \cong \frac{k[a,c,e]}{\langle ac, ae, ce\rangle}.$$

Also, the simplicial complex Δ_2 is

$a \qquad\qquad c$

$e.$

Then $\mathrm{Minh}(R_2) = \{\langle \overline{c}, \overline{e}\rangle, \langle \overline{a}, \overline{c}\rangle, \langle \overline{a}, \overline{e}\rangle\}$. None of a or e or c or $a - c$ or $a - e$ or $e - c$ avoid these generators, so we let $x_3 = \overline{a} + \overline{c} + \overline{e}$. Since $\dim(R) = 3$, we stop and the system of parameters is $\overline{b},\ \overline{d},\ \overline{a} + \overline{c} + \overline{e}$.

cor200924l

COROLLARY IV.A.6.31. (a) *Every regular sequence is part of a system of parameters.*
(b) $\mathrm{depth}(R) \leq \dim(R)$.

PROOF (SKETCH). (a) Algorithm IV.A.4.16 for finding regular sequences says to avoid associated primes at each step, so this also avoids Min and Minh primes and therefore satisfies Algorithm IV.A.6.29 for finding systems of parameters.
(b) Let $\delta = \mathrm{depth}(R)$. Then a maximal regular sequence $x_1, \ldots, x_\delta$ in $\mathfrak{m}$ is part of a system of parameters $x_1, \ldots, x_\delta, \ldots, x_d$. Therefore $\mathrm{depth}(R) = \delta \leq d = \dim(R)$.

□

defn200924m

DEFINITION IV.A.6.32. Assume $(R, \mathfrak{m})$ is local or standard graded. Then R is Cohen-Macaulay if $\mathrm{depth}(R) = \dim(R)$.

ex200924n

EXAMPLE IV.A.6.33. (a) Let $R = S/I(P_3)$. Then R is Cohen-Macaulay because $\mathrm{depth}(R) = 2 = \dim(R)$.
(b) Let $R = S/I(\Sigma G)$. Then R is Cohen-Macaulay because $\mathrm{depth}(R) = d = \dim(R)$.
(c) Let $R = S/I(K_n)$. First, $\mathrm{depth}(R) \leq \dim(R) = 1$. To show that R is Cohen-Macaulay, we want to show that $\mathrm{depth}(R) \geq 1$, i.e., we need to find at least one nonzero divisor. Then the associated primes for R are given by $\langle X_1, \ldots, X_{i-1}, \ldots, X_d\rangle$ for all i. Therefore $f_1 = \overline{X}_1 + \cdots + \overline{X}_d$ avoids all associated primes, so f_1 is a homogeneous nonzero divisor. Therefore R is Cohen-Macaulay.

ection063021u

IV.A.7. Completions

As usual, we will assume R is a non-zero, noetherian, commutative ring with identity, and we let k be a field.

mot200929a

MOTIVATION IV.A.7.1. (a) In real analysis we build $\mathbb{R}$ from $\mathbb{Q}$ in the following way. $\mathbb{Q}$ has a distance metric, so it has Cauchy sequences and we set $\mathbb{R} = \{\text{Cauchy sequences in } \mathbb{Q}\}/\sim$. The result of course is a field $\mathbb{R}$ which is complete and is equipped with a distance metric.

(b) In number theory, one encounters the p-adic completion of $\mathbb{Z}$.

thm200929b

THEOREM IV.A.7.2 (Krull's Intersection Theorem). *Let $I \lneq R$, If we have*

(1) $(R, \mathfrak{m})$ *is local,*
(2) $(R, \mathfrak{m})$ *is standard graded and I is homogeneous,*
(3) $I \subseteq \operatorname{Jac}(R)$, *or*
(4) R *is an integral domain,*

then

$$\bigcap_{n=0}^{\infty} I^n = 0.$$

ex200929c

EXAMPLE IV.A.7.3. (a) If $I = p\mathbb{Z} \lneq \mathbb{Z}$ for some prime p, then $m \in I^n$ if and only if $p^n | m$. Therefore $\cap_{n=0}^{\infty} I^n = 0$, which one can verify using properties of UFD's.

(b) Let $I = \langle X, Y \rangle \leq k[X, Y]$. Then $f \in I^n$ if and only if f has no terms of degree less than n. Hence $\cap_{n=0}^{\infty} I^n = 0$, which one can verify using a degree argument.

(c) Consider the ring $R = k \times k \gneq k \times 0 = I$ and observe that $I^n = I$ for all n, so $\cap_{n=0}^{\infty} I^n = I \neq 0$.

defn200929d

DEFINITION IV.A.7.4. Let $I \lneq R$. The <u>I-adic valuation</u> on R is the function

$$\nu_I : R \longrightarrow N \cup \{\infty\}$$

$$r \longmapsto \sup\{n \geq 0 \mid r \in I^n\}.$$

note200929e

NOTE IV.A.7.5. (a) If I satisfies the conclusion of Krull's Intersection Theorem, then for every non-zero $r \in R$ we have

$$\nu_I(r) = \max\{n \geq 0 \mid r \in I^n\}.$$

(b) In general we have $\cap_{n=0}^{\infty} I^n = \{r \in R \mid \nu_I(r) = \infty\}$.

(c) If $\nu_I(r)$ is large, then r is in high powers of I.

ex200929f

EXAMPLE IV.A.7.6. (a) If $I = p\mathbb{Z} \leq \mathbb{Z}$ and $r \in \mathbb{Z}$ is non-zero, then

$$\nu_I(r) = \max\{n \geq 0 \mid p|r\}.$$

(b) If $I = \langle X, Y \rangle \leq k[X, Y]$ and $r \in k[X, Y]$ is non-zero, then

$$\nu_I(r) = \min\{\text{degree of monomial terms of } r \text{ with non-zero coefficients}\}.$$

For instance, $\nu_I(X + 6Y) = 1$ and $\nu_I\left(X^3 + 8XY^{12}\right) = 3$.

(c) If $I = k \times 0 \lneq k \times k$, then $\nu_I(1, 0) = \infty$, but $\nu_I(1, 1) = 0$ because $(1, 1) \in I^0 \setminus I^1$.

defn200929g

DEFINITION IV.A.7.7. Let $I \lneq R$. The <u>I-adic norm</u> is

$$|\cdot| : R \longrightarrow \mathbb{Q}_{\geq 0}$$

$$r \longmapsto 2^{-\nu_I(r)}$$

where we set $2^{-\infty} = 0$.

note200929h

NOTE IV.A.7.8. (a) If I satisfies the conclusion of Krull's Intersection Theorem, then for any non-zero $r \in R$ we have

$$|r|_I = \min\left\{2^{-n} \mid r \in I^n\right\}.$$

(b) For every $r \in R$ we have

$$\bigcap_{n=0}^{\infty} I^n = \{r \in R \mid |r|_I = 0\}.$$

(c) If $|r|_I$ is small, then r is in high powers of I.

ex200929i EXAMPLE IV.A.7.9. (a) If $I = p\mathbb{Z} \lneq \mathbb{Z}$, then we have the following.

$$|12|_{2\mathbb{Z}} = 2^{-2} = \tfrac{1}{4} \qquad |400|_{2\mathbb{Z}} = \tfrac{1}{16} \qquad |400|_{5\mathbb{Z}} = \tfrac{1}{4}$$

(b) We have similar computations in $k[X,Y]$.

defn200929j DEFINITION IV.A.7.10. The <u>I-adic metric</u> on R is $\mathrm{dist}_I(r,s) = |r-s|_I$.

note200929k NOTE IV.A.7.11. (a) If $\mathrm{dist}_I(r,s)$ is small, then $r-s$ is in a high power of I, i.e., $r+I^n = s+I^n$ for some large n.

(b) We have $\mathrm{dist}_I(r,s) = 0$ if and only if $r - s \in \cap_{n=0}^{\infty} I^n$.

(c) If I satisfies the conclusion of Krull's Intersection Theorem, then $\mathrm{dist}_I(r,s) = 0$ if and only if $r = s$.

prop200929l PROPOSITION IV.A.7.12. dist_I *is a metric on* R *if and only if* $\cap_{n=0}^{\infty} I^n = 0$.

note200929m NOTE IV.A.7.13. (a) In general, dist_I is a pseudo-metric.

(b) The triangle inequality is very strong here: all triangles are isosceles.

(c) Operations on R are continuous under this metric, so R is a topological ring in this metric.

defn200929n DEFINITION IV.A.7.14. Let $\underline{r} = (r_0, r_1, r_2, \dots) \in \prod_{n=0}^{\infty} R = R^{\mathbb{N}}$. Then $\underline{r}$ is <u>I-adically Cauchy</u> if for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for every $m, n \geq N$ we have $\mathrm{dist}_I(r_m, r_n) < \varepsilon$. We also define

$$\mathrm{Cauchy}_I(R) = \{I\text{-adic Cauchy sequences in } R\}.$$

ex200929o.a EXAMPLE IV.A.7.15. (a) Set $R = \mathbb{Z}$ and $I = 5\mathbb{Z} \leq R$. A sequence $\underline{r}$ is 5-adically Cauchy if when one goes far out in the sequence, the terms are 5-adically close. For instance, the sequence

$$\underline{r} = (1, 5, 25, 75, 75+5, 75+5^2, 75+5^3, 75+5^4, \dots)$$

is 5-adically Cauchy:

$$\begin{aligned}
&\mathrm{dist}_I(5,25) = |5-25|_5 = |20|_5 = 2^{-1}\\
&\mathrm{dist}_I(25,75) = 2^{-2}\\
&\mathrm{dist}_I(5,75) = 2^{-1}\\
&\mathrm{dist}_I(75+5^m, 75+5^n) = |(75+5^m)-(75+5^n)|_5 = |5^m - 5^n|_5 = |5^m(1-5^{n-m})|_5 = 2^{-m}.
\end{aligned}$$

In the last line we assume $m < n$ and note that $\underline{r}$ actually converges to 75 in the 5-adic metric because $(75+5^n) - 75 = 5^n$.

What about a non-convergent example? The sequence $\underline{s}$ given by $s_n = 75 + \sum_{i=0}^{n} 5^i$ is 5-adically Cauchy since

$$s_m - s_n = \sum_{i=m+1}^{n} 5^i = 5^{m+1}\left(\sum_{i=0}^{n-m-1} 5^i\right).$$

ex200929o.b (b) If $\underline{t} \in \mathbb{R}^{\mathbb{N}}$ is eventually constant, then $\underline{t}$ is I-adically Cauchy because if m and n are sufficiently large then $t_m - t_n = 0 \in I^p$ for all $p \geq 0$.

ex200929o.c (c) Let $I = 0$, so $0^0 = R$ and $0^n = 0$ for all $n \geq 1$. Then

$$\nu_0(r) = \begin{cases} 0 & \text{if } r \neq 0 \\ \infty & \text{if } r = 0, \end{cases}$$

$$|r|_0 = \begin{cases} \frac{1}{2^0} = 1 & \text{if } r \neq 0 \\ \frac{1}{2^\infty} = 0 & \text{if } r = 0, \end{cases}$$

$$\mathrm{dist}_0(r,s) = |r-s|_0 = \begin{cases} 1 & \text{if } r \neq s \\ 0 & \text{if } r = s. \end{cases}$$

So $\underline{r}$ is 0-adically Cauchy if and only if $\underline{r}$ is eventually constant, i.e., $r_n = r_N$ for all $n \geq N$ for some fixed $N \in \mathbb{N}$.

ex200929o.d (d) Let $R = k[X]$ and $I = \langle X \rangle \leq R$ and

$$\underline{r} = \left(1, 1+X, 1+X+X^2, \dots, \sum_{i=0}^{n} X^i, \dots\right).$$

Then $r_m - r_n = \sum_{i=n+1}^{m} X^i \in I^{n+1}$ for $m > n$, so $\operatorname{dist}_I(r_m, r_n) = |r_m - r_n|_I = \frac{1}{2^{n+1}}$. Therefore the sequence of partial sums of infinite geometric series is Cauchy, so $\underline{r} \in \operatorname{Cauchy}_I(R)$.
More generally, $f \in k[\![X_1, \dots, X_d]\!]$ can be approximated by partial sums to get a Cauchy sequence.

prop201001x

Proposition IV.A.7.16. *$\operatorname{Cauchy}_I(R) \subseteq R^{\mathbb{N}}$ is a subring and the function*

$$\begin{aligned} R &\longrightarrow \operatorname{Cauchy}_I(R) \\ r &\longmapsto (r, r, r, \dots) \end{aligned}$$

is a ring monomorphism.

defn201001a

Definition IV.A.7.17. Let $\underline{r}$, $\underline{s} \in \operatorname{Cauchy}_I(R)$. Then $\underline{r} \sim \underline{s}$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$, then $\operatorname{dist}_I(r_n, s_n) < \varepsilon$.

ex201001b

Example IV.A.7.18. (a) Let $R = \mathbb{Z} \geq \langle 5 \rangle = I$ and $\underline{r} = \left\{75 + \sum_{i=0}^{n} 5^i\right\}$ and $\underline{s} = \left\{75 + \sum_{i=0}^{2n} 5^i\right\}$. Then $\underline{r} \sim \underline{s}$ because $s_n - r_n = \sum_{i=n+1}^{2n} 5^i \in I^{n+1}$.
(b) Let $\underline{r} = (r, r, r, \dots)$ be constant and $\underline{s} = (s_0, s_1, \dots, s_N, r, r, r, \dots)$ be eventually constant. Then $\underline{r} \sim \underline{s}$.
(c) If $I = 0$, then $\underline{r} \sim \underline{s}$ if and only if $r_n = s_n$ for all $n \gg 0$, i.e., $\underline{r}$ and $\underline{s}$ are eventually equal.
(d) Let $R = k[X] \geq \langle X \rangle = I$ and $\underline{r} = \left\{\sum_{i=0}^{n} X^i\right\}$ and $\underline{s} = \left\{\sum_{i=0}^{n} X^i + \sum_{i=1}^{5n} 3^i X^{7i+6n}\right\}$. Then $\underline{r} \sim \underline{s}$.

prop201001c

Proposition IV.A.7.19. *$\sim$ is an equivalence relation on $\operatorname{Cauchy}_I(R)$.*

defn201001d

Definition IV.A.7.20. The <u>I-adic completion</u> of R is $\widehat{R}^I = \operatorname{Cauchy}_I(R)/\sim$. If $(R, \mathfrak{m})$ is local or standard graded, then $\widehat{R} = \widehat{R}^{\mathfrak{m}}$. Frequently, we will use $\widehat{R} = \widehat{R}^I$ in literature.

notn201001e

Notation IV.A.7.21. For $\underline{r} \in \operatorname{Cauchy}_I(R)$, we say $\lim_{n\to\infty} r_n = 0$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$, then $\operatorname{dist}_I(r_n, 0) < \varepsilon$. Alternatively, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$, then $r_n \in I^N$. Let $\widetilde{I} = \{\underline{r} \in \operatorname{Cauchy}_I(R) \mid \lim_{n\to\infty} r_n = 0\}$.

prop201001f

Proposition IV.A.7.22. (a) *$\widetilde{I} \leq \operatorname{Cauchy}_I(R)$.*
(b) *$\underline{r} \sim \underline{s}$ if and only if $\underline{r} - \underline{s} \in \widetilde{I}$.*
(c) *$\widehat{R}^I = \operatorname{Cauchy}_I(R)/\widetilde{I}$. Therefore, $\widehat{R}^I$ is a commutative ring with identity.*
(d) *We have the following diagram.*

$$\begin{tikzcd} R \arrow[r, "\text{const}"] \arrow[dr, "\varepsilon"'] & \operatorname{Cauchy}_I(R) \arrow[d, "I"] \\ & \widehat{R}^I. \end{tikzcd}$$

Also, $\ker(\varepsilon) = \bigcap_{n=0}^{\infty} I^n$, so ε is a monomorphism if and only if $\bigcap_{n=0}^{\infty} I^n = 0$.

ex201001g.a ex201001g

Example IV.A.7.23. (a) Let $I = 0$, then we can show that the map $R \xrightarrow[\cong]{\varepsilon} \widehat{R}^0$ is an isomorphism. The key point here is that $\operatorname{dist}_0(r, s) < 1$ if and only if $r = s$, so Cauchy sequences are eventually constant. Furthermore, since $r_n \to 0$ sequences are eventually zero, we have that ε is injective. Therefore $\underline{r}$ is eventually constant, i.e., $r_n = r$ for all $n \geq N$. This implies that $\underline{r} \sim (r, r, r, \dots) = \operatorname{const}(r)$, so ε is surjective.

ex201001g.b

(b) Let $R = k[X] \geq \langle X \rangle = I$, then we can show that $\widehat{k[X]}^{\langle X \rangle} \cong k[\![X]\!]$ by considering the following commutative diagram:

$$\begin{tikzcd} R \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & k[X] \arrow[d, hook] \\ \widehat{R} \arrow[r, bend left, "\beta"] \arrow[r, phantom, "\cong"] & k[\![X]\!] \arrow[l, bend left, "\alpha"] \end{tikzcd}$$

The α map comes from Example IV.A.7.15(c). Approximate $f \in k[\![X]\!]$ by partial sums, so

$$\alpha\left(\sum_{i=0}^{\infty} a_i X^i\right) = \left\{\sum_{i=0}^{n} a_i X^i\right\}.$$

Here, $\overline{r} \in \widehat{R}$ corresponds to the power series $\sum_{i=0}^{\infty} a_i X^i$.
But what is a_i? Since $\underline{r}$ is Cauchy, we have that there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $r_m - r_n \in I^{i+1} = \langle X^{i+1} \rangle$, i.e., the terms of r_m and r_n agree up to degree i. So

$$\begin{aligned} r_m &= a_0 + a_1 X + \cdots + a_i X^i + \alpha_{i+1} X^{i+1} + \cdots \\ r_n &= a_0 + a_1 X + \cdots + a_i X^i + \beta_{i+1} X^{i+1} + \cdots . \end{aligned}$$

Therefore a_i is the stable value of the coefficients of X_i in r_n for $n \gg 0$, and we let $\beta(\overline{r}) = \sum_{i=0}^{\infty} a_i X^i$.

ex201001g.c (c) $\widehat{R[X_1, \ldots, X_d]}^{\langle X_1, \ldots, X_d \rangle} \cong R[\![X_1, \ldots, X_d]\!]$. More generally,

$$\widehat{\frac{k[X_1, \ldots, X_d]}{J}}^{\langle \overline{X}_1, \ldots, \overline{X}_d \rangle} \cong \frac{k[\![X_1, \ldots, X_d]\!]}{\langle J \rangle}.$$

thm201001h THEOREM IV.A.7.24. *Assume that* $I = \langle f_1, \ldots, f_m \rangle$.

thm201001h.a (a) $\widehat{R}^I \cong \dfrac{R[\![Y_1, \ldots, Y_m]\!]}{\langle Y_1 - f_1, \ldots, Y_m - f_m \rangle}$, *so* $\widehat{R}^I$ *is noetherian.*

thm201001h.b (b) *If* $(R, \mathfrak{m})$ *is local or standard graded, then* $\widehat{R}$ *is local. Moreover, if* I *is any maximal ideal (or if* $\operatorname{rad}(I)$ *is maximal), then* $\widehat{R}^I$ *is local.*

thm201001h.c (c) *If* $\bigcap_{n=0}^{\infty} I^n = 0$*, then* $R \to \widehat{R}^I$ *is flat, i.e.,* $\widehat{R}^I \oplus_R -$ *is exact.*

note201001i NOTE IV.A.7.25. Theorem IV.A.7.24(a) may make you a bit uncomfortable because we can draw the following diagram:

$$\begin{tikzcd} \frac{R[Y_1,\ldots,Y_m]}{\langle Y_1 - f_1,\ldots,Y_m - f_m \rangle} \arrow[r, phantom, "\cong"] & R \\ & R[Y_1, \ldots, Y_m], \arrow[ul, "\tau"] \arrow[u, "\pi"', "Y_i \mapsto f_i"] \end{tikzcd}$$

where $\ker(\pi) = \langle Y_i - f_i, \ldots, Y_m - f_m \rangle$.

Cohen Structure Theorem. Completion makes rings nicer. We can take homological features to the completion, then take homological conclusions found in the completion back to the original problem.

thm201001j THEOREM IV.A.7.26 (Cohen Structure Theorem I). *If* I *is maximal (or* $\operatorname{rad}(I)$ *is maximal), then there exists a regular local ring* S *and an ideal* $J \leq S$ *such that* $\widehat{R}^I \cong S/J$*. For example, if* $(R, \mathfrak{m})$ *is local or standard graded, then* $\widehat{R}$ *is a homomorphic image of a regular local ring.*

defn201006a DEFINITION IV.A.7.27. A local ring $(R, \mathfrak{m})$ is <u>complete</u> if $R \xrightarrow{\cong} \hat{R}$, i.e., if every $\mathfrak{m}$-adically Cauchy sequence in R converges.

thm201006b THEOREM IV.A.7.28 (Cohen Str. Theorem II). *Assume* $(R, \mathfrak{m}, k)$ *is local.*

(a) *[equicharacteristic case] Suppose* $\mathfrak{m} = \langle x_1, \ldots, x_d \rangle$*.If* R *contains a subfield (i.e., a subring that is also a field), then there exists an ideal* J *such that* $\hat{R} \cong [\![X_1, \ldots, X_d]\!]/J$.

(b) *[mixed characteristic case] If* R *does not contain a subfield, then* $\operatorname{char}(k) = p > 0$, $\mathfrak{m} = \langle p, x_1, \ldots, x_d \rangle$*, and there exists an ideal* J *such that* $\hat{R} \cong V[\![X_1, \ldots, X_d]\!]/J$ *and* (V, pV, k) *is a complete local PID (and therefore a DVR) not a field (e.g.,* $V = \widehat{\mathbb{Z}_{\langle p \rangle}}$ *and in general* V *is a ring of Witt vectors).*

note201006c NOTE IV.A.7.29. What if $(R, \mathfrak{m}, k)$ is standard graded? Then $R \cong k[X_1, \ldots, X_d]/J$ and therefore $\hat{R} \cong k[\![X_1, \ldots, X_d]\!]/\langle J \rangle$ by Example IV.A.7.23(c).

fact201016d FACT IV.A.7.30. *Assume* R *is local.*

fact201016d.a (a) $\operatorname{depth}(\hat{R}) = \operatorname{depth}(R)$

fact201016d.b (b) $\dim(\hat{R}) = \dim(R)$

fact201016d.c (c) $\operatorname{edim}(\hat{R}) = \operatorname{edim}(R) = \beta_0(\mathfrak{m})$

Exercises

For the following four exercises, let $R=\bigoplus_{n=0}^{\infty} R_n$ be a noetherian $\mathbb{N}$-graded commutative ring with non-zero identity. Let I be an ideal of R. For $n=0,1,2,\ldots$ set $J_n=I\cap R_n$.

exr210722k EXERCISE IV.A.7.31. Prove that for all $n=0,1,2,\ldots$ the subset J_n is an R_0-submodule of R_n.

exr210722l EXERCISE IV.A.7.32. Prove that $J=\bigoplus_{n=0}^{\infty} J_n$ is an ideal of R contained in I.

exr210722m EXERCISE IV.A.7.33. Prove that $R/J\cong\bigoplus_{n=0}^{\infty}(R_n/J_n)$ is a noetherian $\mathbb{N}$-graded commutative ring with identity.

exr210722n EXERCISE IV.A.7.34. Prove that the following conditions are equivalent.

(i) $I=J$, that is, $I=\bigoplus_{n=0}^{\infty} I\cap R_n$.
(ii) I is generated by finitely many homogeneous elements of R.

For the following eight exercises, let k be a field.

exr210722o EXERCISE IV.A.7.35. Set $I=\langle X_1X_2,X_2X_3\rangle\lneq k[X_1,X_2,X_3]=S$. Prove that $\operatorname{depth}(S/I,R)=1$.

exr210722p EXERCISE IV.A.7.36. Set $I'=\langle X_1X_2,X_2X_3,X_1Y_1,X_2Y_2,X_3Y_3\rangle\lneq k[X_1,X_2,X_3,Y_1,Y_2,Y_3]=S'$. Prove that $\operatorname{depth}(S'/I',R)=3$.

fact210722q FACT IV.A.7.37. *Let $I\lneq S$ be a square-free monomial ideal with irredundant irreducible decomposition $I=\bigcap_{j=1}^n Q_j$. Then*

$$I+\langle X_1^2,\ldots,X_i^2\rangle=\bigcap_{j=1}^n (Q_j+\langle X_1^2,\ldots,X_i^2\rangle)$$

is an irredundant irreducible decomposition.

ex210722r EXAMPLE IV.A.7.38. In $S=k[X_1,X_2,X_3]$ we have $\langle X_1X_2,X_2X_3\rangle=\langle X_1,X_3\rangle\cap\langle X_2\rangle$ and

$$\langle X_1X_2,X_2X_3,X_1^2\rangle=\langle X_1,X_3,X_1^2\rangle\cap\langle X_2,X_1^2\rangle=\langle X_1,X_3\rangle\cap\langle X_1^2,X_2\rangle.$$

exr210722s EXERCISE IV.A.7.39. Set $S=k[X_1,\ldots,X_d]$ and $S'=S[Y_1,\ldots,Y_d]=k[X_1,\ldots,X_d,Y_1,\ldots,Y_d]$. Let $f_1,\ldots,f_n$ be square-free monomials in S, and set $J=\langle X_1Y_1,\ldots,X_dY_d,f_1,\ldots,f_n\rangle\lneq S'$.

(a) Prove that the sequence $X_1-Y_1,\ldots,X_d-Y_d$ is (S'/J)-regular. (Hint: Split the first generator X_1Y_1 to start decomposing J, and use this to show that X_1-Y_1 is not in any associated prime of S'/J.)
(b) Prove that $\operatorname{depth}(S'/J,R)=d$.

exr210722t EXERCISE IV.A.7.40. Consider the polynomial ring $S=k[X_1,\ldots,X_d]$.

(a) Prove that the Hilbert function h_S is given by the formula $h_S(n)=\binom{n+d-1}{d-1}$.
(b) Use h_S to prove that $\dim(S)=d$ and the multiplicity of S is $e(S)=1$.

exr210722u EXERCISE IV.A.7.41. Consider the polynomial ring $S=k[X_1,\ldots,X_d]$. Fix a non-zero homogeneous polynomial $f\in S$ of degree m, and set $R=S/\langle f\rangle$.

(a) Prove that the Hilbert function h_R is given by the formula

$$h_R(n)=\begin{cases} h_S(n) & \text{for } n<m\\ h_S(n)-h_S(n-m) & \text{for } n\geq m.\end{cases}$$

(b) Use h_R to prove that $\dim(R)=d-1$ and the multiplicity of R is $e(R)=m$.

exr210722v EXERCISE IV.A.7.42. Prove that the power series ring $k[\![X_1,\ldots,X_d]\!]$ is Cohen-Macaulay.

exer4 EXERCISE IV.A.7.43. Consider the polynomial ring $S=k[a,b,c,d,e]$. Set $I=\langle ab,bc,cd,de,ae\rangle$ and $R=S/I$.

item1 (a) Split the generators of I to compute an irredundant irreducible decomposition of I.
(b) Use the decomposition from part (a) to compute $\dim(R)$ and determine whether I is unmixed or not.
item3 (c) Use a simplicial complex to re-derive the decomposition from part (a), to compute $\dim(R)$, and to determine whether I is unmixed or not.
(d) Use a graph to re-derive the decomposition from part (a).

Justify all your answers.

exr210722w EXERCISE IV.A.7.44. Repeat Exercise IV.A.7.43, parts (a)–(c) for the ring $S=k[a,b,c,d,e]$ and the ideal $J=\langle de,abcd,abce\rangle$.

CHAPTER IV.B

Regular Rings

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IV.B.1. Foundational Properties

Assume $(R, \mathfrak{m}, k)$ is local.

rec201006e RECALL IV.B.1.1. We have the compound inequality

$$\operatorname{depth}(R) \leq \dim(R) \leq \operatorname{edim}(R).$$

R is a regular local ring if $\dim(R) = \operatorname{edim}(R)$, i.e., it has large Krull dimension or small embedding dimension.

ex201006f.a EXAMPLE IV.B.1.2. (a) By Theorem IV.A.6.5, both $k[\![X_1, \dots, X_d]\!]$ and $k[X_1, \dots, X_d]_{\langle X_1,\dots,X_d\rangle}$ are regular local rings of dimension d. Thus we have

$$\beta_0(\mathfrak{m}) \leq d = \dim(k[X_1, \dots, X_d]) \leq \beta_0(\mathfrak{m})$$

and therefore have equality at every step. We also have

$$d \leq \dim\left(k[X_1, \dots, X_d]_{\langle X_1,\dots,X_d\rangle}\right) \leq \dim(k[X_1, \dots, X_d]) = d,$$

because there exists a chain of primes of length d:

$$0 \subsetneq \langle X_1\rangle \subsetneq \langle X_1, X_2\rangle \subsetneq \cdots \subsetneq \langle X_1, \dots, X_d\rangle .$$

Hence

$$\beta_0(\mathfrak{m}) \leq d = \dim\left(k[X_1, \dots, X_d]_{\langle X_1,\dots,X_d\rangle}\right) \leq \beta_0(\mathfrak{m})$$

and we have equality at every step.

ex201006f.b (b) Let $p \in \mathbb{Z}_{>0}$ be prime and let (V, pV, k) be a complete local PID that is not a field, i.e., a complete local p-ring. Then $R = V[\![X_1, \dots, X_d]\!]$ is a regular local ring of dimension $d+1$. One can prove this as above by Theorem IV.A.6.5 using Example IV.A.6.2 and the fact that $\dim V = 1$. So $\dim(V[\![X_1, \dots, X_d]\!]) = d+1$, $\mathfrak{m} = \langle p, X_1, \dots, X_d\rangle$, and one proceeds as in (a). The argument is similar for $V[X_1, \dots, X_d]_{\langle p,X_1,\dots,X_d\rangle}$.

ex201006f.c (c) $R = \widehat{\mathbb{Z}_{\langle p\rangle}}[\![X]\!]/\left\langle p - X^2\right\rangle$ is a regular local ring of dimension 1 with $\mathfrak{m} = \left\langle p, \overline{X}\right\rangle = \left\langle \overline{X}\right\rangle$. Therefore $\beta_0(\mathfrak{m}) \leq 1$ and we know $\dim(R) \leq \beta_0(\mathfrak{m})$, so we want to show $1 \leq \dim(R)$. For the sake of contradiction, suppose $\dim(R) < 1$, i.e., $\dim(R) = 0$. Since $\dim\left(\widehat{\mathbb{Z}_{\langle p\rangle}}[\![X]\!]\right) = 2$, we know a system of parameters for $\widehat{\mathbb{Z}_{\langle p\rangle}}[\![X]\!]$ has length 2. However, if $\dim(R) = 0$, then $\mathfrak{m} = 0$. Therefore $p - X^2$ is a system of parameters for $\widehat{\mathbb{Z}_{\langle p\rangle}}[\![X]\!]$, which is too short, because the dimension is 2. The fact $p = X^2 \in \mathfrak{m}^2$ says R is a <u>ramified</u> regular local ring.

Our goal is to show that regular local rings must be Cohen-Macaulay.

prop201006g PROPOSITION IV.B.1.3. *Let* $\mathbf{f} = f_1, \dots, f_n \in \mathfrak{m}$. *Then* $\dim(R/\langle \mathbf{f}\rangle) \geq \dim R - n$.

PROOF. Argue as in Example IV.B.1.2(c). If $\dim(R/\langle \mathbf{f}\rangle) < \dim R - n$, then generate an $\mathfrak{m}$-primary ideal by fewer then d elements, which is a contradiction since d gives the smallest number of generators of an $\mathfrak{m}$-primary ideal. □

thm201006h THEOREM IV.B.1.4. *If R is a regular local ring and $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ (i.e., f is a minimal generator of $\mathfrak{m}$), then $\overline{R} = R/\langle f\rangle$ is a regular local ring and $\dim \overline{R} = d - 1$, where $d = \dim(R)$.*

PROOF. Set $\overline{\mathfrak{m}} = \mathfrak{m}/\langle f\rangle$, the maximal ideal of $\overline{R}$. We have $\operatorname{edim}(R) = \beta_0(\overline{\mathfrak{m}}) \leq \beta_0(\mathfrak{m}) - 1$ since $f \in m \setminus \mathfrak{m}^2$. Then $\beta_0(\mathfrak{m}) - 1 = \dim R - 1$ because R is a regular local ring, so we can use Proposition IV.B.1.3 to get that $\dim R - 1 \leq \dim(\overline{R}) \leq \operatorname{edim}(\overline{R})$. Then

$$\operatorname{edim}(R) = \beta_0(\overline{\mathfrak{m}}) \leq \beta_0(\mathfrak{m}) - 1 = \dim R - 1 \leq \dim(\overline{R}) \leq \operatorname{edim}(\overline{R}),$$

so we have equality at every step. □

note201006i

NOTE IV.B.1.5. The conclusion of Theorem IV.B.1.4 fails if $f \in \mathfrak{m}^2$, because $\beta_0(\overline{\mathfrak{m}}) = \beta_0(\mathfrak{m})$ and

$$\dim(\overline{R}) \overset{*}{=} \dim R - 1 = \operatorname{edim} R - 1 = \operatorname{edim} \overline{R} - 1$$

in this case. Equality * will follow once we prove that regular local rings are integral domains.

defn201006j

DEFINITION IV.B.1.6. Assume R is a regular local ring of dimension d. Any sequence $\mathbf{x} = x_1, \ldots, x_d$ that generates $\mathfrak{m}$ is a regular system of parameters.

ex201006k

EXAMPLE IV.B.1.7. (a) The sequence $\mathbf{X} = X_1, \ldots, X_d$ is a regular system of parameters for the ring $k[\![X_1, \ldots, X_d]\!]$ as well as the ring $k[X_1, \ldots, X_d]_{\langle \mathbf{X} \rangle}$.

(b) The sequence $p, X_1, \ldots, X_d$ is a regular system of parameters of the ring $R = V[\![X_1, \ldots, X_d]\!]$ where V is a complete p-ring.

(c) The sequence $\overline{X}$ is a regular system of parameters of $R = \widehat{\mathbb{Z}_{\langle p \rangle}}[\![X]\!]/\left\langle p - X^2 \right\rangle$.

prop201008a

PROPOSITION IV.B.1.8. *Every regular system of parameters is a system of parameters.*

PROOF. Assume that R is a regular local ring and that $\mathbf{x} = x_1, \ldots, x_d \in \mathfrak{m}$ is a regular system of parameters, i.e., $d = \dim(R)$ and $\mathfrak{m} = \langle \mathbf{x} \rangle$. First, $\mathbf{x}$ has the correct number of elements for a system of parameters ($d = \dim(R)$). Second, $\langle \mathbf{x} \rangle = \mathfrak{m} = \operatorname{rad}(\mathfrak{m})$, so $\mathbf{x}$ generates an $\mathfrak{m}$-primary ideal. Therefore, $\mathbf{x}$ is a system of parameters. $\square$

thm201008b

THEOREM IV.B.1.9. *Every regular local ring is an integral domain and every regular system of parameters is R-regular.*

PROOF. Set $d = \dim(R)$ and assume that R is a regular local ring with regular system of parameters $\mathbf{x} = x_1, \ldots, x_d \in \mathfrak{m}$. We first claim that R is an integral domain.

PROOF. We induct on d.

$d = 0$: Here $\mathbf{x} = \emptyset$, so $\mathfrak{m} = 0$. Therefore R is a field, and hence an integral domain (this idea also shows that every field is a regular local ring).

$d \geq 1$: Let $\operatorname{Min}(R) = \{\mathfrak{p}, \ldots, \mathfrak{p}_n\}$. Note that $\mathfrak{m} \not\subseteq \mathfrak{p}_i$ for any i because $d \geq 1$, so $\mathfrak{m}$ is not minimal. Also $\mathfrak{m} \not\subseteq \mathfrak{m}^2$ because of Nakiyama's Lemma (by contradiction, if $\mathfrak{m} \subseteq \mathfrak{m}^2$, then $\mathfrak{m} = \mathfrak{m}^2$, so $\mathfrak{m} = 0$ by Nakiyama's Lemma, which implies $d = 0$). By prime avoidance, we then have that $\mathfrak{m} \subsetneq \mathfrak{m}^2 \cup \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$, so there exists $f \in \mathfrak{m} \setminus \left(\mathfrak{m}^2 \cup \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n\right)$. Since $f \in \mathfrak{m} \setminus \mathfrak{m}^2$, then $\overline{R} = R/\langle f \rangle$ is a regular local ring and $\dim(\overline{R}) = d - 1$ by Theorem IV.B.1.4. The inductive hypothesis states that $\overline{R}$ is an integral domain. Therefore f is prime in R, so there exists some minimal prime ideal which is a subset of $\langle f \rangle$. Without loss of generality, let $\mathfrak{p}_1 \subseteq \langle f \rangle$, so $f \notin \mathfrak{p}_1$. We claim that $\mathfrak{p}_1 = 0$.

PROOF. By Nakayama's Lemma, it suffices to show that $\mathfrak{p}_1 = f\mathfrak{p}_1$. $\mathfrak{p}_1 \supseteq f\mathfrak{p}_1$ is clear. Let $y \in \mathfrak{p}_1 \subseteq \langle f \rangle$, so $y = fz$ for some $z \in R$. But $f \notin \mathfrak{p}_1$, so $z \in \mathfrak{p}_1$, so $y = fz \inf \mathfrak{p}_1$. ✓

Now we can conclude that 0 is prime in R. Therefore R is an integral domain. ✓

Second, we claim that $\mathbf{x}$ is R-regular.

PROOF. We again induct on d.

$d = 0$: $\mathbf{x} = \emptyset$ is vacuously regular.

$d \geq 1$: $x_1 \neq 0$ (because otherwise we would have that $\mathfrak{m} = \langle x_1, \ldots, x_d \rangle = \langle x_2, \ldots, x_d \rangle$ is generated by $d - 1$ elements, which is too few). Since R is an integral domain and $x_1 \in \mathfrak{m}$, then x_1 is a non-unit. Therefore x_1 is a non zero-divisor on R. By Theorem IV.B.1.4, $\overline{R} = R/\langle x_1 \rangle$ is an regular local ring and $\dim(\overline{R}) = d - 1$ with regular system of parameters $x_2, \ldots, x_d$. By the inductive hypothesis, $x_2, \ldots, x_d$ is $\overline{R}$-regular, so $\mathbf{x} = x_1, \ldots, x_d$ is R-regular. ✓

$\square$

cor201008c

COROLLARY IV.B.1.10. *Every regular local ring is Cohen-Macaulay.*

Proof. Let $d = \dim(R)$. We have $d \leq \operatorname{depth}(R)$ by Theorem IV.B.1.9, so

$$d = \dim(R) \geq \operatorname{depth}(R) \geq d.$$

$\square$

prop201008d

Proposition IV.B.1.11. *Let R be a local integral domain, e.g., a regular local ring. Let $0 \neq f \in \mathfrak{m}$ and $d = \dim(R)$ and $\overline{R} = R/\langle f \rangle$. Then $\dim(\overline{R}) = d - 1$.*

Proof. Let $\overline{\mathfrak{p}}_n \supsetneq \cdots \supsetneq \overline{\mathfrak{p}}_0$ be a chain of primes in $\overline{R}$ of maximal length, where $n = \dim(\overline{R})$ and $\overline{\mathfrak{p}}_i = \mathfrak{p}_i/\langle f \rangle$ and $\mathfrak{p}_n \supsetneq \cdots \supsetneq \mathfrak{p}_0 \supseteq \langle f \rangle \supsetneq 0$. Then

$$d = \dim(R) \geq \mathfrak{n} + 1 = \dim(\overline{R}) + 1,$$

so $d - 1 \geq \dim(\overline{R})$.
Now let $\overline{f}_1, \ldots, \overline{f}_n \in \overline{\mathfrak{m}}$ be a system of parameters for $\overline{R}$. Then $\overline{\mathfrak{m}} = \operatorname{rad}(\langle \overline{f}_1, \ldots, \overline{f}_n \rangle)$ implies that $\mathfrak{m} = \operatorname{rad}(\langle f_1, \ldots, f_n, f \rangle)$ and $d = \dim(R) \leq n + 1$. Therefore $d - 1 \leq n = \dim(\overline{R})$. $\square$

thm201008e

Theorem IV.B.1.12. *The following are equivalent.*

thm201008e.i (i) *R is a regular local ring.*
hm201008e.ii (ii) *$\widehat{R}$ is a regular local ring.*
m201008e.iii (iii) *$R[\![X]\!]$ is a regular local ring.*
hm201008e.iv (iv) *$R[\![X_1, \ldots, X_n]\!]$ is a regular local ring for some $n \geq 1$.*
thm201008e.v (v) *$R[\![X_1, \ldots, X_n]\!]$ is a regular local ring for all $n \geq 1$.*
hm201008e.vi (vi) *$R[X]_{\langle \mathfrak{m}, X \rangle}$ is a regular local ring.*
m201008e.vii (vii) *$R[X_1, \ldots, X_n]_{\langle \mathfrak{m}, X_1, \ldots, X_n \rangle}$ is a regular local ring for some $n \geq 1$.*
n201008e.viii (viii) *$R[X_1, \ldots, X_n]_{\langle \mathfrak{m}, X_1, \ldots, X_n \rangle}$ is a regular local ring for all $n \geq 1$.*

Proof. First we show that (i) if and only if (ii). By Fact IV.A.7.30, $\dim(\widehat{R}) = \dim(R)$ and $\operatorname{edim}(\widehat{R}) = \operatorname{edim}(R)$. Therefore whenever $\dim(\widehat{R}) = \operatorname{edim}(\widehat{R})$, we have $\dim(R) = \operatorname{edim}(R)$ and vice versa.

Next, we prove that (i) implies (vi). Assume that R is a regular local ring. Set $R_6 = R[X]_{\langle \mathfrak{m}, X \rangle}$ and $\mathfrak{m}_6 = \langle \mathfrak{m}, X \rangle$. Then

$$\begin{aligned} \operatorname{edim}(R_6) &= \beta_0(\mathfrak{m}_6) \\ &\leq \beta_0((\mathfrak{m}) + 1 \\ &= \operatorname{edim}(R) + 1 \\ &= \dim(R) + 1 \\ &\overset{?}{\leq} \dim(R_6) \\ &\leq \operatorname{edim}(R_6). \end{aligned}$$

If we can show the inequality in the penultimate line, then we will have equality at every step. It suffices to show that there exists a chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_{d+1}$ in $\operatorname{Spec}(R_6)$. Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d$ be a chain in $\operatorname{Spec}(R)$ and consider the chain

$$\begin{array}{ccccccccc} P_0 & & P_1 & & & & P_d & & P_{d+1} \\ \| & & \| & & & & \| & & \| \\ \langle \mathfrak{p}_0 \rangle & \subsetneq & \langle \mathfrak{p}_1 \rangle & \subsetneq & \cdots & \subsetneq & \langle \mathfrak{p}_d \rangle & \subsetneq & \langle \mathfrak{p}_d, X \rangle \\ \cap\!| & & \cap\!| & & & & \cap\!| & & \cap\!| \\ R_6 & & R_6 & & & & R_6 & & R_6. \end{array}$$

Then P_i is prime for all $i = 0, \ldots, d$ because

$$R_6/\langle \mathfrak{p}_i \rangle = \frac{R[X]_{\langle \mathfrak{m}, X \rangle}}{\langle \mathfrak{p}_i \rangle} \cong \left(\frac{R}{\mathfrak{p}_i} \left[\overline{X} \right] \right)_{\langle \overline{\mathfrak{m}}, \overline{X} \rangle}$$

and

$$R_6/\langle \mathfrak{p}_d, X \rangle = \frac{R[X]_{\langle \mathfrak{m}, X \rangle}}{\langle \mathfrak{p}_d, X \rangle} \cong \left(\frac{\left(\frac{R}{\mathfrak{p}_d} \left[\overline{X} \right] \right)}{\langle \overline{X} \rangle} \right)_{\langle \overline{\mathfrak{m}}, \overline{X} \rangle},$$

which is isomorphic to a localization of the integral domain $R/\mathfrak{p}_d$.

Proving (vi) implies (vii) and (vii) implies (viii) is routine. To show (viii) implies (i), assume $n \geq 1$ such that $R' = R[X_1, \ldots, X_n]_{\langle \mathfrak{m}, X_1, \ldots, X_n \rangle}$ is a regular local ring. Then since $\dim R' = \dim R + n$ and $\operatorname{edim} R' = \operatorname{edim} R + n$ we have

$$\begin{aligned} R \text{ is a regular local ring} &\iff \dim R = \operatorname{edim} R \\ &\iff \dim R + n = \operatorname{edim} R + n \\ &\iff \dim R' = \operatorname{edim} R' \\ &\iff R' \text{ is a regular local ring.} \end{aligned}$$

Finally, one proves that

$$\text{(i)} \implies \text{(iii)} \implies \text{(iv)} \implies \text{(v)} \implies \text{(i)}$$

as above. $\square$

ex201013a EXAMPLE IV.B.1.13. If R is a local PID, then R is a regular local ring.

(1) If R is a field, then R is a regular local ring of dimension 0.
(2) If R is not a field, then $\beta_0(\mathfrak{m}) = 1$ since it is a PID of dimension one. (See Example IV.A.6.2)

que201013b QUESTION IV.B.1.14. In general $\dim R_{\mathfrak{p}}$ is easy to control, but $\operatorname{edim} R_{\mathfrak{p}}$ is more difficult to control, because $\beta_0(\mathfrak{p}_{\mathfrak{p}})$ can be much smaller than $\beta_0(\mathfrak{p})$. Hence one might ask: if R is a regular local ring and $\mathfrak{p} \in \operatorname{Spec}(R)$, must $R_{\mathfrak{p}}$ be a regular local ring as well?

ection063021w

IV.B.2. Homological Properties

Assume $(R, \mathfrak{m}, k)$ is a regular local ring.

defn201013c DEFINITION IV.B.2.1. An R-module M has finite projective dimension if there exists an exact sequence

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that each P_i is projective over R. We then define the projective dimension of M to be the infimum of the lengths of such chains, i.e.,

$$\operatorname{pd}_R(M) = \inf \{ n \geq 0 \mid \exists \text{ an exact sequence } 0 \to P_n \to \cdots \to P_0 \to M \to 0 \}.$$

ex201013d EXAMPLE IV.B.2.2. (a) Assume R is a local PID. Then every submodule of a free R-module is free. Therefore for every R-module M there is a surjection $F_0 \xrightarrow{\tau} M$ such that F_0 is free. Hence $\operatorname{Ker}(\tau)$ is free and therefore projective, so we have a short exact sequence

$$0 \longrightarrow \operatorname{Ker}(\tau) \longrightarrow F_0 \xrightarrow{\tau} M \longrightarrow 0,$$

with projective R-modules $\operatorname{Ker}(\tau)$ and F_0. Thus $\operatorname{pd}_R(M) \leq 1$.

(b) If $\mathbf{x} = x_1, \ldots, x_n \in \mathfrak{m}$ is R-regular, then the Koszul complex $K^R(\mathbf{x})$ is a free resolution of $R/\langle \mathbf{x} \rangle$, so $\operatorname{pd}_R(R/\langle \mathbf{x} \rangle) \leq n$. For instance, if $R = k[\![X_1, \ldots, X_n]\!]$ or $R = k[X_1, \ldots, X_n]_{\langle \mathbf{X} \rangle}$, where $\mathbf{X} = X_1, \ldots, X_n$, then $\mathbf{X}$ is R-regular. Therefore

$$\operatorname{pd}_R(k) = \operatorname{pd}_R(R/\langle \mathbf{X} \rangle) \leq n$$

since $k \cong R/\langle \mathbf{X} \rangle$.

(c) Over the ring $R = k[\![X]\!]/\langle X^2 \rangle$, the field k has infinite projective dimension. Observe that k has the projective resolution

$$\cdots \xrightarrow{X \cdot} R \xrightarrow{X \cdot} R \xrightarrow{X \cdot} R \xrightarrow{X \cdot} k \longrightarrow 0$$

and one can additionally show that there are no shorter resolutions for k when working over R. One can argue similarly for $R = k[\![X, Y]\!]/\langle XY \rangle$.

thmABF THEOREM IV.B.2.3 (Auslander-Buchsbaum formula). *If M is a finitely generated R-module of finite projective dimension, then* $\operatorname{pd}_R(M) = \operatorname{depth}(R) - \operatorname{depth}(M)$ *(where* $\operatorname{depth}(M)$ *is the length of maximal M-regular sequences in $\mathfrak{m}$).*

ABS THEOREM IV.B.2.4 (Auslander-Buchsbaum-Serre). *The following are equivalent.*

(i) *R is a regular local ring.*
(ii) *Every R-module has finite projective dimension over R.*
(iii) *Every finitely generated R-module has finite projective dimension over R.*
(iv) *The residue field of R has finite projective dimension over R.*

cor201013e

COROLLARY IV.B.2.5. *If R is a regular local ring and $\mathfrak{p} \in \operatorname{Spec}(R)$, then $R_{\mathfrak{p}}$ is a regular local ring.*

PROOF. We know $R_{\mathfrak{p}}$ is local with maximal ideal $\mathfrak{p}_{\mathfrak{p}}$ and residue field $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \cong (R/\mathfrak{p})_{\mathfrak{p}}$. By Auslander-Buchsbaum-Serre, to show that $R_{\mathfrak{p}}$ is a regular local ring, it suffices to show that $\operatorname{pd}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}})$ is finite. Working over R, Auslander-Buchsbaum-Serre implies that $\operatorname{pd}_R(R/\mathfrak{p}) < \infty$. So there exists an exact sequence

$$0 \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow R/\mathfrak{p} \longrightarrow 0$$

such that each Q_i is projective over R. Since localization is exact the sequence

$$\begin{tikzcd} 0 \arrow[r] & (Q_n)_{\mathfrak{p}} \arrow[r] & \cdots \arrow[r] & (Q_0)_{\mathfrak{p}} \arrow[r] & (R/\mathfrak{p})_{\mathfrak{p}} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\ & & & & R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} & \end{tikzcd}$$

is exact and furthermore one can show that each $(Q_i)_{\mathfrak{p}}$ is projective over $R_{\mathfrak{p}}$ ($(Q_i)_{\mathfrak{p}} = \left(R^{\beta_i}\right)_{\mathfrak{p}} = (R_{\mathfrak{p}})^{\beta_i}$). Therefore $\operatorname{pd}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}) \leq n < \infty$, as desired. $\square$

cor201013f

COROLLARY IV.B.2.6. *If $p \in \operatorname{Spec}(k[X_1, \ldots, X_d])$, then $k[X_1, \ldots, X_d]_p$ is a regular local ring.*

thmAB

THEOREM IV.B.2.7 (Auslander-Buchsbaum). *Regular local rings are unique factorization domains. (For instance, $\mathbb{Z}_{\langle p \rangle}[X]/\left\langle X^2 - p \right\rangle$ is a UFD.)*

Exercises

Let k be a field.

exr2107222x

EXERCISE IV.B.2.8. Let $(R, \mathfrak{m})$ be a local ring and $\mathbf{x} = x_1, \ldots, x_n \in \mathfrak{m}$ an R-regular sequence. Prove that if $R/\langle \mathbf{x} \rangle$ is a regular local ring, then R is a regular local ring and $\mathbf{x}$ is part of a regular system of parameters for R.

CHAPTER IV.C

Complete Intersection Rings

pter093021ak

As we noted in IV.B.1.5, if Q is a regular local ring and $\mathbf{x}$ is a Q-regular sequence, then $Q/\langle \mathbf{x}\rangle$ may not be a regular local ring.

ction063021x

IV.C.1. Foundational Properties

defn201013g

Definition IV.C.1.1.

(a) A local ring R is a natural complete intersection if there exists a regular local ring Q and a Q-regular sequence $\mathbf{x}$ such that $R \cong Q/\langle \mathbf{x}\rangle$.
(b) A local ring R is a formal complete intersection if $\hat{R}$ is a natural complete intersection.
(c) A standard graded ring R is a geometric complete intersection if there exists a polynomial ring $S = k[X_1, \ldots, X_d]$ and a homogeneous S-regular sequence $\mathbf{f}$ such that $R \cong S/\langle \mathbf{f}\rangle$.

ex201013h

Example IV.C.1.2. (a) The ring $k[X_1, \ldots, X_d]/\langle X_1^{a_1}, \ldots, X_n^{a_n}\rangle$, where $a_i \geq 1$, is a geometric complete intersection.
(b) The ring $k[\![X_1, \ldots, X_d]\!]/\langle X_1^{a_1}, \ldots, X_n^{a_n}\rangle$, where $a_i \geq 1$, is a natural complete intersection.
(c) Regular local rings are natural complete intersections, which in-turn are formal complete intersections, i.e.,

$$\text{regular local ring} \implies \text{natural complete intersection} \implies \text{formal complete intersection.}$$

The converses of the above fail, in general. One can see that the converse of the first implication fails by our motivating example. The converse of the second implication fails is a relatively new result by Heitmann and Jorgenson.
(d) If R is a standard graded geometric complete intersection, then $R_{\mathfrak{m}}$ is a natural complete intersection (because localizations of polynomial rings are regular local rings).

prop201013i

Proposition IV.C.1.3. *If $(R, \mathfrak{m})$ is a natural complete intersection and $\mathfrak{p} \in \operatorname{Spec}(R)$, then $R_{\mathfrak{p}}$ is a natural complete intersection.*

Proof. Since R is a natural complete intersection, there exists a regular local ring Q and a Q-regular sequence $\mathbf{f}$ such that $R \cong Q/\langle \mathbf{f}\rangle$. Then for some $P \in \operatorname{Spec}(Q)$ satisfying $\mathbf{f} \in P$ we have $\mathfrak{p} = P/\langle \mathbf{f}\rangle \in \operatorname{Spec}(R)$ and $R_{\mathfrak{p}} \cong Q_P/\langle \mathbf{f}\rangle$. Moreover, $R_{\mathfrak{p}}$ is therefore a natural complete intersection, since Q a regular local ring implies Q_P is a regular local ring, and $\mathbf{f}$ a Q-regular sequence implies $\mathbf{f}$ is a Q_P-regular sequence (note $\mathbf{f} \in P$). $\square$

prop201015a

Proposition IV.C.1.4. *If $(R, \mathfrak{m})$ is standard graded geometric complete intersection and $\mathfrak{p} \in \operatorname{Spec}(R)$, then $R_{\mathfrak{p}}$ is a natural complete intersection. (Note that $R_{\mathfrak{p}}$ is not generally a geometric complete intersection because $R_{\mathfrak{p}}$ is not usually standard graded.*

Proof. Since R is a geometric complete intersection, there exists a polynomial ring S and a homogeneous S-regular sequence $\mathbf{f}$ such that $R \cong S/\langle \mathbf{f}\rangle$. Prime correspondence tells us that $\mathfrak{p} = P/\langle \mathbf{f}\rangle$ for some $P \in \operatorname{Spec}(S)$ such that $\mathbf{f} \in P$. Therefore $R_{\mathfrak{p}} \cong S_P/\langle \mathbf{f}\rangle$. We see that $\mathbf{f}$ is S_P-regular because $\mathbf{f}$ is S-regular and $\mathbf{f} \in P$, and furthermore that S_P is a regular local ring by Corollary IV.B.2.6. $\square$

Our goal going forward is to show that any formal complete intersection is Cohen-Macaulay. Then we will have shown the following sequence of implications:

$$\begin{aligned} \text{field} &\Rightarrow \text{regular local ring} \\ &\Rightarrow \text{natural complete intersection} \\ &\Rightarrow \text{formal complete intersection} \\ &\Rightarrow \text{Cohen-Macaulay.} \end{aligned}$$

prop201015b PROPOSITION IV.C.1.5. *Assume that* $(R,\mathfrak{m})$ *is local and let* $\mathbf{x} = x_1,\ldots,x_n \in \mathfrak{m}$ *be an* R*-regular sequence. Set* $\overline{R} = R/\langle \mathbf{x}\rangle$.

rop201015b.a (a) $\operatorname{depth}(\overline{R}) = \operatorname{depth}(R) - n$
rop201015b.b (b) $\dim(\overline{R}) = \dim(R) - n$
rop201015b.c (c) $\overline{R}$ *is Cohen-Macaulay if and only if* R *is Cohen-Macaulay if and only if* $\widehat{R}$ *is Cohen-Macaulay.*

PROOF. (a) Let $\mathbf{y} = y_1,\ldots,y_m \in \mathfrak{m}$ be such that $\overline{\mathbf{y}} = \overline{y_1},\ldots,\overline{y_m} \in \overline{R}$ is a maximal $\overline{R}$-regular sequence. Then $\mathbf{x},\mathbf{y}$ is a maximal R-regular sequence, so

$$\operatorname{depth}(R) = n + m = n + \operatorname{depth}(\overline{R}).$$

(b) We induct on n. The important case is when $n = 1$. Since $x = x_1$ is R-regular, then $x \notin \bigcup_{\mathfrak{p}\in\operatorname{Ass}(R)} \mathfrak{p}$, i.e., $x \notin \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Ass}(R)$. Therefore $x \notin \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Min}(R)$. Let $\overline{\mathfrak{p}}_0 \subsetneq \cdots \subsetneq \overline{\mathfrak{p}}_q$ be a maximal chain of prime ideals in $\operatorname{Spec}(\overline{R})$. Here, each $\overline{\mathfrak{p}}_i = \mathfrak{p}_i/\langle x\rangle$ for $\mathfrak{p}_i \in \operatorname{Spec}(R)$ and $x \in \mathfrak{p}_i$ and $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_q$, where $q = \dim(\overline{R})$. Since $x \in \mathfrak{p}_0$ and $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Min}(R)$, then $\mathfrak{p}_0 \notin \operatorname{Min}(R)$. Therefore there exists $\mathfrak{p} \in \operatorname{Min}(R)$ such that $\mathfrak{p} \subsetneq \mathfrak{p}_0 \cdots \subsetneq \mathfrak{p}_q$ is a chain of length $q+1$ in $\operatorname{Spec}(R)$. Therefore $\dim(R) \geq q + 1 = \dim(\overline{R}) + 1$. Finish the proof as in Proposition IV.B.1.11.

(c) We have the following sequence of statements:

$$\begin{aligned}
\overline{R} \text{ is Cohen-Macaulay} &\iff \operatorname{depth}(\overline{R}) = \dim(\overline{R})\\
&\iff \operatorname{depth}(R) - n = \dim(R) - n\\
&\iff \operatorname{depth}(R) = \dim(R)\\
&\iff R \text{ is Cohen-Macaulay}\\
&\iff \operatorname{depth}(\widehat{R}) = \dim(\widehat{R})\\
&\iff \widehat{R} \text{ is Cohen-Macaulay.}
\end{aligned}$$

□

thm201015c THEOREM IV.C.1.6. *Every formal complete intersection is Cohen-Macaulay.*

PROOF. Assume that R is a formal complete intersection, so $\widehat{R}$ is a natural complete intersection, i.e., there exists a regular local ring Q and a Q-regular sequence $\mathbf{f}$ such that $\widehat{R} \cong \overline{Q} = Q/\langle \mathbf{f}\rangle$. Then Corollary IV.B.1.10 implies that Q is Cohen-Macaulay, which means that $\overline{Q}$ is Cohen-Macaulay by Proposition IV.C.1.5(c). Then $\widehat{R} \cong \overline{Q}$, so Proposition using IV.C.1.5(c) again gives us that R is Cohen-Macaulay. □

thm201015d THEOREM IV.C.1.7. *Assume that* $(R,\mathfrak{m})$ *is local and* $\mathbf{x} = x_1,\ldots,x_n \in \mathfrak{m}$ *is an* R*-regular sequence and set* $\overline{R} = R/\langle \mathbf{x}\rangle$.

thm201015d.a (a) *If* R *is a natural complete intersection, then* $\overline{R}$ *is a natural complete intersection.*
thm201015d.b (b) *If* R *is a formal complete intersection, then* $\overline{R}$ *is a formal complete intersection.*

PROOF. (a) Since R is a natural complete intersection, there exists a regular local ring Q and $\mathbf{y}$ is a Q-regular sequence such that $R \cong Q/\langle \mathbf{y}\rangle$. Let $\widetilde{x}_i \in Q$ be such that $\overline{\widetilde{x}}_i = x_i$ for all i. Then $\mathbf{y},\widetilde{\mathbf{x}}$ is a Q-regular sequence such that $Q/\langle \mathbf{y},\widetilde{\mathbf{x}}\rangle \cong (Q/\langle \mathbf{y}\rangle)/\langle \mathbf{x}\rangle \cong R/\langle \mathbf{x}\rangle = \overline{R}$. Since $\overline{R}$ looks like (regular local ring) / (regular sequence), $\overline{R}$ is a natural complete intersection.

(b) Since R is a formal complete intersection, then $\widehat{R}$ is a natural complete intersection. Then

$$\widehat{\overline{R}} = \widehat{R/\langle \mathbf{x}\rangle} \cong \widehat{R}/\langle \mathbf{x}\rangle = (\text{natural complete intersection})/(\text{regular sequence}).$$

By part (a), $\widehat{\overline{R}}$ is a natural complete intersection, so $\overline{R}$ is a formal complete intersection. □

ques201015e QUESTION IV.C.1.8. Do the converses of the statements in Theorem IV.C.1.7 hold?

prop201015f PROPOSITION IV.C.1.9. *If* R *is standard graded and a geometric complete intersection and* $\mathbf{x}$ *is a homogeneous* R*-regular sequence, then* $\overline{R} = R/\langle \mathbf{x}\rangle$ *is a geometric complete intersection.*

PROOF. This proof is similar to that of Theorem IV.C.1.7(a), using $S = k[X_1,\ldots,X_d]$ in place of Q. □

thm201015g THEOREM IV.C.1.10. *Assume that* R *is local. The following are equivalent.*

thm201015g.i (i) R *is a natural complete intersection.*
hm201015g.ii (ii) $R[\![X]\!]$ *is a natural complete intersection.*
m201015g.iii (iii) $R[\![X_1,\ldots,X_d]\!]$ *is a natural complete intersection for all* $d \in \mathbb{N}_+$.

thm201015g.iv (iv) *$R[\![X_1, \ldots, X_d]\!]$ is a natural complete intersection for some $d \in \mathbb{N}_+$.*
thm201015g.v (v) *$R[X]_{\langle \mathfrak{m}, X\rangle}$ is a natural complete intersection.*
thm201015g.vi (vi) *$R[X_1, \ldots, X_d]_{\langle \mathfrak{m}, X_1, \ldots, X_d\rangle}$ is a natural complete intersection for all $d \in \mathbb{N}_+$.*
thm201015g.vii (vii) *$R[X_1, \ldots, X_d]_{\langle \mathfrak{m}, X_1, \ldots, X_d\rangle}$ is a natural complete intersection for some $d \in \mathbb{N}_+$.*

Proof. The implications (ii)⇒ (iii) ⇒ (iv) are straightforward. We first show that (i) implies (ii). Let $R \cong Q/\langle \mathbf{f}\rangle$ be such that Q is a regular local ring and $\mathbf{f}$ is a Q-regular sequence. Since $R[\![X]\!] \cong Q[\![X]\!]/\langle \mathbf{f}\rangle$ and $\mathbf{f}$ is Q-regular, then $\mathbf{f}$ is also $Q[\![X]\!]$-regular. Since Q is a regular local ring, then $Q[\![X]\!]$ is also a regular local ring. Therefore $R[\![X]\!]$ is a natural complete intersection.

We next show that (iv) implies (i). Assume that $R[\![X_1, \ldots, X_d]\!]$ is a natural complete intersection. The sequence $X_1, \ldots, X_d$ is regular for this ring, so by Theorem IV.C.1.7, this implies that

$$R[\![X_1, \ldots, X_d]\!]/\langle X_1, \ldots, X_d\rangle \cong R$$

is a natural complete intersection.

Finally, the proofs for

$$\text{(i)} \Rightarrow \text{(v)} \Rightarrow \text{(vi)} \Rightarrow \text{(vii)} \Rightarrow \text{(i)}$$

are done similarly. □

thm201015h Theorem IV.C.1.11. *Assume that R is local. The following are equivalent.*

(i) *R is a formal complete intersection.*
(ii) *$R[\![X]\!]$ is a formal complete intersection.*
(iii) *$R[\![X_1, \ldots, X_d]\!]$ is a formal complete intersection for all $d \in \mathbb{N}_+$.*
(iv) *$R[\![X_1, \ldots, X_d]\!]$ is a formal complete intersection for some $d \in \mathbb{N}_+$.*
(v) *$R[X]_{\langle \mathfrak{m}, X\rangle}$ is a formal complete intersection.*
(vi) *$R[X_1, \ldots, X_d]_{\langle \mathfrak{m}, X_1, \ldots, X_d\rangle}$ is a formal complete intersection for all $d \in \mathbb{N}_+$.*
(vii) *$R[X_1, \ldots, X_d]_{\langle \mathfrak{m}, X_1, \ldots, X_d\rangle}$ is a formal complete intersection for some $d \in \mathbb{N}_+$.*

Proof. Consider the isomorphisms

$$\widehat{R[\![X_1, \ldots, X_d]\!]} \cong \widehat{R}[\![X_1, \ldots, X_d]\!] \cong \widehat{R[X_1, \ldots, X_d]_{\langle \mathfrak{m}, X_1, \ldots, X_d\rangle}}.$$

Apply Theorem IV.C.1.10 using these isomorphisms. □

thm201015i Theorem IV.C.1.12. *Assume that R is standard graded. The following are equivalent.*

(i) *R is a geometric complete intersection.*
(ii) *$R[X]$ is a geometric complete intersection.*
(iii) *$R[X_1, \ldots, X_d]$ is a geometric complete intersectionfor all $d \in \mathbb{N}_+$.*
(iv) *$R[X_1, \ldots, X_d]$ is a geometric complete intersectionfor some $d \in \mathbb{N}_+$.*

Proof. This proof is similar to the proof of Theorem IV.C.1.10. □

defn201020a Definition IV.C.1.13. Assume R is a local formal complete intersection. The codimension of R is

$$\operatorname{codim}(R) = \min\left\{ n \geq 0 \;\middle|\; \exists \text{ RLR } Q \text{ and } Q\text{-regular sequence } \mathbf{x} = x_1, \ldots, x_n \text{ s.t. } \hat{R} \cong Q/\langle \mathbf{x}\rangle \right\}.$$

R is a hypersurface if $\operatorname{codim}(R) \leq 1$. R is a proper hypersurface if $\operatorname{codim}(R) = 1$.

note201020b Note IV.C.1.14. Assume R is a local formal complete intersection.

(a) Since $\operatorname{codim}(R) = 0$ if and only if R is a regular local ring, all regular local rings are hypersurfaces. ($\operatorname{codim}(R) = 0$ if and only if there exists a regular local ring Q and a Q-regular sequence $\mathbf{x}$ with zero elements such that $\hat{R} \cong Q/\langle \mathbf{x}\rangle \cong Q$, i.e., $\hat{R}$ is a regular local ring if and only if R is a regular local ring.)
(b) We have

$$\begin{aligned}\operatorname{codim}(R) &= \min\left\{ n \geq 0 \;\middle|\; \exists \text{ RLR } (Q, \eta) \text{ and } Q\text{-reg. seq. } \mathbf{x} = x_1, \ldots, x_n \in \eta^2 \text{ s.t. } \hat{R} \cong Q/\langle \mathbf{x}\rangle \right\} \\ &\overset{(1)}{=} \operatorname{edim}(R) - \dim(R).\end{aligned}$$

Since $\widehat{R} \cong Q/\langle \mathbf{x} \rangle$ and Q is a regular local ring, and since $\mathbf{x} \in \eta^2$, we have

$$\begin{aligned} n &= \dim(Q) - \dim(Q/\langle \mathbf{x} \rangle) \\ &= \operatorname{edim}(Q) - \dim(\hat{R}) \\ &= \operatorname{edim}(\hat{R}) - \dim(\hat{R}) \\ &= \operatorname{edim}(R) - \dim(R), \end{aligned}$$

so the equality (1) is shown.

(c) R is a proper hypersurface if and only if $\operatorname{edim}(R) = \dim(R) + 1$.

IV.C.2. Homological Properties

Assume $(R, \mathfrak{m}, k)$ is local.

Definition IV.C.2.1. Let M be a finitely generated R-module. The n[th] Betti number of M is

$$\beta_n^R(M) = \min \left\{ b_n \;\middle|\; \exists \text{ exact sequence } \cdots \to R^{b_n} \to \cdots \to R^{b_1} \to R^{b_0} \to M \to 0 \right\}.$$

Example IV.C.2.2. (a) Let $R = k[\![X_1, \ldots, X_d]\!]$ and let $M = k = R/\langle \mathbf{X} \rangle$ be an R-module, where $\mathbf{X} = X_1, \ldots, X_d$. Then the Koszul complex on the list of variables is

$$0 \longrightarrow R^1 \longrightarrow R^d \longrightarrow R^{\binom{d}{d-2}} \longrightarrow \cdots \longrightarrow R^{\binom{d}{2}} \longrightarrow R^d \longrightarrow R \longrightarrow R/\langle \mathbf{X} \rangle \longrightarrow 0$$

So we see that $\beta_n^R(k) \leq \binom{d}{n}$. (These are actually equal. See note below.) More generally this works for any local ring R and $M = R/\langle \mathbf{x} \rangle$, where $\mathbf{x} = x_1, \ldots, x_d$ is an R-regular sequence.

(b) Consider the ring $R = k[\![X]\!]/\langle X^2 \rangle \cong k[X]/\langle X^2 \rangle$ and the R-module $M = k = R/\langle \overline{X} \rangle = R/\langle x \rangle$ (set $x = \overline{X} \in R$). Then we have the infinite resolution

$$\cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow k \longrightarrow 0,$$

so $\beta_n^R(k) \leq 1$ (and these are in fact equal).

(c) Consider the ring $R = k[\![X, Y]\!]/\langle X, Y \rangle^2 \cong k[X, Y]/\langle X, Y \rangle^2$ and the R-module $M = k = R/\langle x, y \rangle$, where we set $x = \overline{X}$ and $y = \overline{Y}$. Then we have the following resolution.

$$\cdots \longrightarrow R^8 \xrightarrow[\partial_3]{\left(\begin{smallmatrix} \partial_2 & 0 \\ 0 & \partial_2 \end{smallmatrix}\right)} R^4 \xrightarrow[\partial_2 = \left(\begin{smallmatrix} \partial_1 & 0 \\ 0 & \partial_1 \end{smallmatrix}\right)]{\left(\begin{smallmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{smallmatrix}\right)} R^2 \xrightarrow[\partial_1]{(x\ y)} R \longrightarrow k \longrightarrow 0$$

So we see that $\beta_n^R(k) \leq 2^n$ for all $n \geq 0$ (and these are in fact once again equal). One should also note that in this context $\mathfrak{m}^2 = 0$ and $x^2 = y^2 = xy = 0$.

(d) It is a fact that every finitely generated R-module M has a minimal free resolution, which we can denote

$$\cdots \xrightarrow[\partial_{n+1}]{} R^{\alpha_n} \xrightarrow[\partial_n]{} \cdots \xrightarrow[\partial_2]{} R^{\alpha_1} \xrightarrow[\partial_1]{} R^{\alpha_0} \xrightarrow[\tau]{} M \longrightarrow 0.$$

Moreover, we can do this such that $\operatorname{Im} \partial_i \subseteq \mathfrak{m} \cdot R^{\alpha_{i-1}}$ for all $i \geq 1$, i.e., entries of matrices representing ∂_i are all in $\mathfrak{m}$ (e.g., each resolution in parts (a) through (c)). Then $\beta_n^R(M) = \alpha_n$ since

$$k^{\beta_n^R(M)} \cong \operatorname{Tor}_n^R(M, k) \cong k^{\alpha_n}.$$

(The above are also isomorphic to the Ext module $\operatorname{Ext}_R^n(M, k)$.)

(e) We have

$$\operatorname{pd}_R(M) < \infty \iff \beta_n(M) = 0, \quad \forall n \gg 0.$$

(f) This is another result by Auslander, Buchsbaum, and Serre.

$$\begin{aligned} R \text{ is a regular local ring} &\iff \forall \text{ finitely generated } M : \ \beta_n(M) = 0, \quad \forall n \gg 0 \\ &\iff \beta_n(k) = 0, \quad \forall n \gg 0 \\ &\iff \exists n \geq 0 \text{ s.t. } \beta_n(k) = 0 \\ &\iff \beta^R_{\operatorname{depth}(R)+1}(k) = 0 \\ &\iff \beta^R_{\dim(R)+1}(k) = 0 \\ &\iff \beta^R_{\operatorname{edim}(R)+1}(k) = 0 \end{aligned}$$

defn201020e

Definition IV.C.2.3 (Avramov). Let M be a finitely generated R-module. The complexity of M is

$$\operatorname{cx}_R(M) = \inf\left\{m \geq 0 \;\middle|\; \exists \alpha \in \mathbb{R}_+ \text{ s.t. } \forall n \geq 0 : \ \beta_n^R(M) \leq \alpha \cdot n^{m-1}\right\}.$$

(The degree of the zero polynomial is -1 here.)

ex201020f

Example IV.C.2.4. Let M be a finitely generated R-module.

(a) $\operatorname{cx}_R(M) = 0$ if and only if $\beta_n^R(M) = 0$ for all sufficiently large $n > 0$ if and only if $\operatorname{pd}_R(M) < \infty$.
(b) $\operatorname{cx}_R(M) = 1$ if and only if $\beta_R(M)$ is bounded and $\operatorname{pd}_R(M) = \infty$.
(c) For $R = k[\![\mathbf{X}]\!]$ and $M = k$, where $\mathbf{X} = X_1, \ldots, X_d$, we have $\beta_n^R(k) = 0$ for all $n > d$ and therefore $\operatorname{cx}_R(k) = 0$.
(d) For the ring $R = k[\![X]\!]/\left\langle X^2\right\rangle$ we have $\beta_n^R(k) = 1$ for all sufficiently large $n > 0$, so $\operatorname{cx}_R(k) = 1$.
(e) For the ring $R = k[\![X, Y]\!]/\left\langle X, Y\right\rangle^2$ we have $\beta_n^R(k) = 2^n$ for all $n \geq 0$, so $\operatorname{cx}_R(k) = \infty$.

note201020g

Note IV.C.2.5. $\operatorname{cx}_R(M) < \infty$ means $\{\beta_n^R(M)\}$ is bounded above by a polynomial in n.

Gull

Theorem IV.C.2.6 (Gulliksen). *The following are equivalent.*

(i) *R is a formal complete intersection.*
(ii) *$\operatorname{cx}_R(M) < \infty$ for all finitely generated M.*
(iii) *$\operatorname{cx}_R(k) < \infty$.*

cor201022a

Corollary IV.C.2.7 (Avramov). *If R is a formal complete intersection and $\mathfrak{p} \in \operatorname{Spec}(R)$, then $R_{\mathfrak{p}}$ is a formal complete intersection.*

Proof. We need to show that $\operatorname{cx}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}) < \infty$. So

$$\begin{aligned} \operatorname{cx}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}) &= \operatorname{cx}_{R_{\mathfrak{p}}}((R/\mathfrak{p})_{\mathfrak{p}}) \\ &\leq \operatorname{cx}_R(R/\mathfrak{p}) \\ &< \infty, \end{aligned}$$

where the last line follows by Theorem IV.C.2.6. □

cor201022b

Corollary IV.C.2.8. *Let $\mathbf{x} = x_1, \ldots, x_n \in \mathfrak{m}$ be an R-regular sequence and let $\overline{R} = R/\langle \mathbf{x}\rangle$.*

cor201022b.a (a) *R is a formal complete intersection if and only if $\overline{R}$ is a formal complete intersection.*

cor201022b.b (b) *R is a natural complete intersection if and only if there exists a regular local ring $Q \twoheadrightarrow R$ and $\overline{R}$ is a natural complete intersection.*

Proof. We first prove the backwards implication of part (b). Assume that there exists a regular local ring $Q \twoheadrightarrow R$ and $\overline{R}$ is a natural complete intersection. Argue by induction on n. The important case is for $n = 1$. Consider the resolution

$$\begin{tikzcd} & & 0 \arrow[dr] & & 0 & & \\ & & & \mathfrak{m} \arrow[ur] \arrow[dr] & & & \\ \cdots \arrow[r] & R^{b_2} \arrow[r] & R^{b_1} \arrow[ur] & & R \arrow[r] & k \arrow[r] & 0. \end{tikzcd}$$

Then $\beta_n^R(\mathfrak{m}) = \beta_{n-1}^R(k)$ for all $n \geq 1$, so $\operatorname{cx}_R(k) = \operatorname{cx}_R(\mathfrak{m})$. Next we know that $x = x_1$ is R-regular and $\mathfrak{m}$-regular because $\mathfrak{m} \subseteq R$. If F is a minimal R-free resolution of $\mathfrak{m}$, then $\overline{F}$ is a minimal $\overline{R}$-free resolution of $\overline{\mathfrak{m}} = \mathfrak{m}/x\mathfrak{m}$, so $\beta_n^R(\mathfrak{m}) = \beta_n^{\overline{R}}(\overline{\mathfrak{m}})$. Then $\operatorname{cx}_R(\mathfrak{m}) = \operatorname{cx}_{\overline{R}}(\overline{\mathfrak{m}}) < \infty$ by Theorem IV.C.2.6. Therefore R is a formal complete intersection and is a natural complete intersection because $Q \twoheadrightarrow R$.

We next prove the backwards implication of part (a). Assume $\overline{R}$ is a formal complete intersection, so $\widehat{\overline{R}} \cong \overline{\widehat{R}}$ is a natural complete intersection. By Theorem IV.A.7.26, $\widehat{R}$ is a homomorphic image of a regular local ring Q, i.e., $\widehat{R} \twoheadleftarrow Q$. By part (b), $\widehat{R}$ is a natural complete intersection, and therefore R is a formal complete intersection by definition. □

defn201022c DEFINITION IV.C.2.9. If M is a finitely generated R-module, the Poincaré series of M is the "generating function" for $\left\{\beta_n^R(M)\right\}$, i.e.,

$$P_M^R(t) = \sum_{n=0}^{\infty} \beta_n^R(M) t^n.$$

ex201022d EXAMPLE IV.C.2.10. (a) Let R be a regular local ring. Then $\beta_n^R(k) = \binom{d}{n}$, where $d = \dim(R)$, so by the binomial theorem,

$$P_k^R(t) = (1+t)^d.$$

(b) Let $R = k[\![X]\!]/\left\langle X^2\right\rangle$. Then $\beta_n^R(k) = 1$ for all n. Then the Poincaré series of k is a geometic series:

$$P_k^R(t) = 1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t}.$$

(c) Let $R[\![X,Y]\!]/\langle X, Y\rangle^2$. Then $\beta_n^R(k) = 2^n$ for all n, so

$$P_k^R(t) = 1 + 2t + 2^2t^2 + 2^3t^3 + \cdots = \frac{1}{1-2t}.$$

(d) The complexity of M is related to the Poincaré series of M.

thm201022e THEOREM IV.C.2.11. *If R is a formal complete intersection, then every Poincaré series over R is represented by a rational function with a common denominator.*

IDEA OF PROOF. Since M is finitely generated over R, we pass to the completion $P_{\widehat{M}}^{\widehat{R}}(t) = P_M^R(t)$, where $\widehat{R} \cong Q/\langle \mathbf{x}\rangle$ is a regular local ring over a Q-regular sequence. Since this has finite projective dimension over Q, then $P_{\widehat{M}}^{Q}(t)$ is a polynomial. We can use a Syzygy argument (replace $\widehat{M}$ with N, where

$$0 \longrightarrow N \longrightarrow \widehat{R}^b \longrightarrow M \longrightarrow 0$$

is exact) to show that $P_M^R(t)$ is a rational function related to $P_{\widehat{M}}^{Q}(t)$. $\square$

note201022f NOTE IV.C.2.12. The converse to Theorem IV.C.2.11 fails in general. A current hot research topic is to determine which rings satisfy the conclusion of Theorem IV.C.2.11 (L. Şega).

Resolutions over Hypersurfaces.

trck201022g PARLOR TRICK IV.C.2.13. Let $S = k[X, Y, Z, W]$ and

$$\Delta = XY - ZW = \begin{vmatrix} X & Z \\ W & Y \end{vmatrix}.$$

It is a fact that Δ is irreducible. Then

$$\begin{bmatrix} X & Z \\ W & Y \end{bmatrix} \begin{bmatrix} Y & -Z \\ -W & X \end{bmatrix} = \begin{bmatrix} XY - ZW & 0 \\ 0 & -ZW + XY \end{bmatrix}.$$

Cramer's Rule says that if $A \in M_{n\times n}(R)$, then $A \cdot \mathrm{Adj}(A) = |A| \cdot I_n = \mathrm{Adj}(A) \cdot A$.

cor201022h COROLLARY IV.C.2.14. *Assume R is a formal complete intersection such that* $\operatorname{codim}(R) \leq 1$, *i.e., R is a hypersurface. Then every finitely generated M has* $\mathrm{cx}_R(M) \leq 1$, *i.e.,* $\left\{\beta_n^R(M)\right\}$ *is bounded.*

thm201022i THEOREM IV.C.2.15. *Assume R is a formal complete intersection such that* $\operatorname{codim}(R) \leq 1$, *i.e., R is a hypersurface. Then for every finitely generated M,* $\left\{\beta_n^R(M)\right\}$ *is eventually constant. Moreover,* $\beta_n^R(M)$ *is constant for all* $n > \operatorname{depth}(R) - \operatorname{depth}(M, R) \geq 0$.

note201022j NOTE IV.C.2.16. Compare Theorem IV.C.2.15 to IV.B.2.3. If R is a regular local ring, then $\operatorname{pd}_R(M) = \operatorname{depth}(R) - \operatorname{depth}(M, R)$, so $\beta_n^R(M) = 0$ for all $n > \operatorname{depth}(R) - \operatorname{depth}(M, R) = \operatorname{pd}_R(M)$.

defn201022k DEFINITION IV.C.2.17. Let $f \in R$. A matrix factorization of f over R is a pair $A, B \in M_{n\times n}(R)$ such that $A \cdot B = f \cdot I_n = B \cdot A$.

ex201022l EXAMPLE IV.C.2.18. For every $A \in M_{n\times n}(R)$, $A, \mathrm{Adj}(A)$ is a matrix factorization of $|A|$ by Cramer's Rule.

thm201022m

THEOREM IV.C.2.19 (Eisenbud 1980). *Let $(S, \mathfrak{m}, k)$ be a regular local ring and let $0 \neq f \in \mathfrak{m}$ and let $R = S/fS$ be a natural hypersurface and let M be a finitely generated R-module and let $t = \mathrm{pd}_R(M) = \mathrm{depth}(R) - \mathrm{depth}(M, R)$. Then the minimal free resolution of M over R has the form*

$$\cdots \xrightarrow{\overline{\psi}} R^b \xrightarrow{\overline{\phi}} R^b \xrightarrow{\overline{\psi}} R^b \xrightarrow{\overline{\phi}} R^b \longrightarrow R^{\beta_t} \longrightarrow \cdots \longrightarrow R^{\beta_0} \longrightarrow M \longrightarrow 0,$$

where $\psi, \phi \in M_{b\times b}(S)$ form a matrix factorization of f.

Exercises

exr210722y

EXERCISE IV.C.2.20. Let R be a local ring and $\mathfrak{p} \in \mathrm{Spec}$. Prove or disprove each of the following.

(a) If the localization $R_{\mathfrak{p}}$ is Cohen-Macaulay, then R is Cohen-Macaulay.
(b) If the localization $R_{\mathfrak{p}}$ is regular, then R is regular.
(c) If the localization $R_{\mathfrak{p}}$ is a formal complete intersection, then R is a formal complete intersection.

thm210722z

THEOREM IV.C.2.21 (Grothendieck). *Set $S = k[X_1, \ldots, X_d]$, let $\underline{f} = f_1, \ldots, f_n$ be a list of non-constant homogeneous elements of S. Assume that $\underline{f}$ is a minimal generating sequence for $I = \langle \underline{f} \rangle$. Then S/I is a geometric complete intersection if and only if $\underline{f}$ is S-regular.*

exr210722aa

EXERCISE IV.C.2.22. Set $S = k[X_1, \ldots, X_d]$, let I be a monomial ideal in S, and set $R = S/I$.

(a) Let $\underline{f} = f_1, \ldots, f_n \in [\![S]\!]$ be a sequence of monomials in S. Prove that $\underline{f}$ is an S-regular sequence if and only if $\underline{f}$ is a pairwise relatively prime list, i.e., for all $i \neq j$ we have $\gcd(f_i, f_j) = 1$.
(b) Prove that R is a geometric complete intersection if and only if I is generated by a pairwise relatively prime list of monomials.

CHAPTER IV.D

Artinian Rings

Throughout this part we assume that R is a non-zero commutative ring with identity.

IV.D.1. Foundational Properties

PROPOSITION IV.D.1.1. *The following are equivalent.*

(i) *R satisfies the descending chain condition on ideals, i.e., every descending chain of ideals in R stabilizes, i.e., for every chain of ideals $R \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, there exists a natural number $n_0 \in \mathbb{N}$ such that $I_n = I_{n_0}$ for all $n \geq n_0$.*

(ii) *R satisfies the minimum condition of non-empty sets of ideals, i.e., every non-empty set of ideals of R $\Sigma \neq \emptyset$ has a minimal element with respect to containment, i.e., there exists an ideal $I_0 \in \Sigma$ such that for every ideal $I \in \Sigma$, if $I \subseteq I_0$, then $I = I_0$.*

DEFINITION IV.D.1.2. R is <u>artinian</u> if it satisfies the descending chain condition on ideals.

EXAMPLE IV.D.1.3. (a) Fields are artinian.

(b) $\mathbb{Z}$ is not artinian:

$$\mathbb{Z} \supsetneq 2\mathbb{Z} \supsetneq 4\mathbb{Z} \supsetneq 8\mathbb{Z} \supsetneq \cdots .$$

Similarly, $k[X]$ is not artinian:

$$\langle X \rangle \supsetneq \langle X^2 \rangle \supsetneq \langle X^4 \rangle \supsetneq \cdots .$$

(c) $k[X]/\langle X^2 \rangle$ is artinian, because it is a finite dimensional vector space over k.

PROPOSITION IV.D.1.4. *Assume $k \subseteq R$ is a subfield such that $\dim_k R < \infty$. Then R is artinian.*

PROOF. For the sake of contradiction, suppose we have the chain $R \supsetneq I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \cdots$. Since these I_i's are ideals, they are also R-submodules and therefore k-subspaces. Set $d = \dim_k R < \infty$. Then since the containments are proper in the chain we have $\dim_k I_n \leq d - n$ for $n = 1, 2, \ldots, d$. In particular this implies $\dim_k I_d = 0$, so we have $I_n = 0$ for all $n \geq d$. □

NOTE IV.D.1.5. If $k \subseteq R$ is a subfield such that $\dim_k R < \infty$, then R is noetherian. One argues as in the proof of Proposition IV.D.1.4 using an ascending chain instead. Alternatively, if $I \leq R$, then $\dim_k I \leq \dim_k R < \infty$. Therefore I has a finite spanning set over k. Moreover, this spanning set will also generate I over R, because $k \subseteq R$ implies k-linear combinations are also R-linear combinations.

THEOREM IV.D.1.6. *Every artinian ring is a noetherian ring.*

PROPOSITION IV.D.1.7. *Assume R is artinian.*

(a) *Every ideal $I \subseteq \mathrm{Jac}(R)$ is nilpotent, i.e., $I^n = 0$ for all sufficiently large n.*
(b) $\mathrm{Jac}(R)$ *is nilpotent.*
(c) $\mathrm{Jac}(R) = \mathrm{Nil}(R)$.
(d) *R is semilocal, i.e., it only has a finite number of maximal ideals.*
(e) *Every prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ is maximal.*
(f) $\dim R = 0$.

PROOF. (a) Let $I \subseteq \mathrm{Jac}(R)$. Then $I \supseteq I^2 \supseteq I^3 \supseteq \cdots$ must stabilize, i.e., we must have

$$I^n = I^{n+1} = I \cdot I^n \subseteq \mathrm{Jac}(R) \cdot I^n \subseteq I^n$$

for some $n \geq 1$. Therefore $I^n = \mathrm{Jac}(R) \cdot I^n$ for some $n \geq 1$ and Nakayama's Lemma (Fact IV.A.2.4 (d)) implies $I^n = 0$, since R noetherian implies I^n is finitely generated.

(b) Set $I = \mathrm{Jac}(R)$ and apply (a).

(c) We have $\mathrm{Nil}(R) \overset{1}{\subseteq} \mathrm{Jac}(R) \overset{2}{\subseteq} \mathrm{Nil}(R)$ and therefore have equality at every step, where 1 holds for all rings and 2 holds by (b).

(d) Suppose $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \ldots$ are all distinct maximal ideals. Then we create the descending chain $\mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \supseteq \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \supseteq \cdots$, and we claim $\mathfrak{m}_1 \cdots \mathfrak{m}_n \supsetneq \mathfrak{m}_1 \cdots \mathfrak{m}_n\mathfrak{m}_{n+1}$ for all n. For the sake of contradiction, suppose there exists an $n \geq 1$ such that $\mathfrak{m}_1 \cdots \mathfrak{m}_n = \mathfrak{m}_1 \cdots \mathfrak{m}_{n+1} \subseteq \mathfrak{m}_{n+1}$. Since $\mathfrak{m}_{n+1}$ is prime, this implies $\mathfrak{m}_i \subseteq \mathfrak{m}_{n+1}$ for some $i \leq n$. Both ideals are maximal, implying $\mathfrak{m}_{n+1} = \mathfrak{m}_i$, a contradiction.

(e) Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Therefore we have

$$\mathfrak{p} \supseteq \mathrm{Nil}(R) \overset{3}{=} \mathrm{Jac}(R) = \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n \supseteq \mathfrak{m}_1 \cdots \mathfrak{m}_n,$$

where 3 holds by c. Then $\mathfrak{p} \supseteq \mathfrak{m}_i$ for some i, and since $\mathfrak{m}_i$ is maximal and $\mathfrak{p} \neq R$, we have $\mathfrak{p} = \mathfrak{m}_i$ is maximal.

(f) Every prime ideal of R is maximal, so for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(R)$, the containment $\mathfrak{p} \subset \mathfrak{p}'$ is not proper, i.e., chains of primes in R are of at most length 0. $\square$

thm201027h

THEOREM IV.D.1.8. *The following are equivalent.*

thm201027h.i (i) *R is artinian.*
hm201027h.ii (ii) *R is noetherian and* $\dim R = 0$.
m201027h.iii (iii) *R is noetherian and every prime ideal in R is maximal.*
hm201027h.iv (iv) *R is noetherian and every prime ideal in R is minimal.*
thm201027h.v (v) *R is noetherian and* $|\mathrm{Spec}(R)| < \infty$.

cor201027i

COROLLARY IV.D.1.9. *Assume R is local or standard graded. Then if R is artinian, it is also Cohen-Macaulay.*

PROOF. We have

$$\mathrm{depth}(R) \leq \dim(R) = 0 \leq \mathrm{depth}(R),$$

where the equality follows from Theorem IV.D.1.8 (ii). Thus we have equality at every step. $\square$

prop201027j

PROPOSITION IV.D.1.10. *If R is non-negatively graded and artinian, then $R_i = 0$ for all $i \gg 0$ and R_0 is artinian. Moreover, all elements with the constant term 0 are nilpotent, i.e., $R_+ \subseteq \mathrm{Nil}(R)$.*

PROOF. Since R is artinian, the chain $R \supseteq R_+ \supseteq R_{\geq 2} \supseteq R_{\geq 3} \supseteq \cdots$ must stabilize, i.e.,

$$R_n \oplus R_{n+1} \oplus \cdots = R_{\geq n} = \mathbb{R}_{\geq n+1} = R_{n+1} \oplus R_{n+2} \oplus \cdots$$

for all $n \gg 0$. Let $R_0 \supsetneq I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \cdots$. Then $R_0 \supsetneq I_1 \oplus R_+ \supsetneq I_2 \oplus R_+ \supsetneq \cdots$, because R is artinian. [I DIDN'T FOLLOW THE ARGUMENT HERE.] $\square$

prop201027k

PROPOSITION IV.D.1.11. *Let $S = k[X_1, \ldots, X_d]$, let $I \subsetneq S$ be a monomial ideal, and set $R = S/I$. The following are equivalent.*

rop201027k.i (i) *R is artinian.*
rop201027k.ii (ii) *I contains a power of each variable, i.e., for every $i = 1, \ldots, d$ there exists some $n_i \in \mathbb{N}$ such that $X_i^{n_i} \in I$.*
p201027k.iii (iii) *Every monomial generating sequence for I contains a power of each variable.*
rop201027k.iv (iv) *There exists an irredundant monomial generating sequence $X_1^{e_1}, \ldots, X_d^{e_d}, f_1, \ldots, f_m$ for I.*
rop201027k.v (v) $\mathrm{rad}(R) = \langle \mathbf{X} \rangle$.

PROOF. (i) $\Longrightarrow$ (ii): Assume R is artinian. Then Proposition IV.D.1.10 implies there is some $n_0 \in \mathbb{N}$ such that $R_i = 0$ for all $i \geq n_0$. Therefore $\overline{X_j}^{n_0} = 0 \in R$ for all j, so $X_j^{n_0} \in I$ for all j.

(ii) $\Longrightarrow$ (iii): Assume (ii) holds and let $g_1, \ldots, g_p$ be a monomial generating sequence for I. By assumption, for every $i = 1, \ldots, d$ we have $X_i^{n_i} \in I = \langle g_1, \ldots, g_p \rangle$ for some $n_i \geq 1$, implying that $g_j | X_i^{n_i}$ for some $j = 1, \ldots, p$. Therefore $g_j = X_i^{m_i}$ for some $m_i \leq n_i$.

(iii) $\Longrightarrow$ (iv): This is evident.

(iv) $\Longrightarrow$ (v): Assume (iv) holds. Then $I \subsetneq S$ implies $I \subseteq \langle \mathbf{X} \rangle$. This justifies 1 in the statement

$$\langle \mathbf{X} \rangle = \langle X_1, \ldots, X_d \rangle \subseteq \operatorname{rad}(I) \overset{1}{\subseteq} \operatorname{rad}(\langle \mathbf{X} \rangle) = \langle \mathbf{X} \rangle$$

and we therefore have equality at every step.

(v) $\Longrightarrow$ (ii): If we assume $\operatorname{rad}(I) = \langle \mathbf{X} \rangle$, then for every $i = 1, \ldots, d$ we have $X_i \in \operatorname{rad}(I)$ and thus there exists some n_i such that $X_i^{n_i} \in I$.

(ii) $\Longrightarrow$ (i): Assume (ii). Then $\dim_k R \leq n_1 \cdots n_d < \infty$ and Proposition IV.D.1.4 implies R is artinian. Alternatively, one can use Theorem IV.D.1.8 since R is noetherian and $X_i^{n_i} \in I$ (for some n_i) implies $\overline{X_i}^{n_i} = 0 \in R$, so the unique prime ideal of R is $\langle \overline{X_1}, \ldots, \overline{X_d} \rangle$. □

ction063021aa

IV.D.2. Structural Properties

thm201029a

Theorem IV.D.2.1. *Let $(R, \mathfrak{m})$ be local and artinian. Then R is complete and therefore R is a homomorphic image of a regular local ring.*

Proof. We prove that R is complete, so we want to show that every $\mathfrak{m}$-adic Cauchy sequence $\{x_n\}$ in R converges. To show this, we claim that $\{x_n\}$ is eventually constant.

Proof. Since R is artinian, $\mathfrak{m}^n = 0$ for all $n \gg 0$, i.e., there exists $N_0 \in \mathbb{N}$ such that $\mathfrak{m}^n = 0$ for all $n \geq N_0$. Since $\{x_n\}$ is Cauchy, there exists $t \geq 0$ such that for every $i, j > t$, we have $x_i - x_j \in \mathfrak{m}^{N_0} = 0$, or $x_i = x_j$. Therefore not only is $\{x_n\}$ convergent, but it eventually converges to a constant value. ✓

Then by the Cohen Structure Theorem (Theorem IV.A.7.26), R is a homomorphic image of a regular local ring. □

thm201029b

Theorem IV.D.2.2. *Assume that R is artinian. Then $R \cong R_1 \times \cdots \times R_n$ such that each R_i is a complete local artinian ring. Also, $n = |\operatorname{Spec}(R)|$ and if $\operatorname{Spec}(R) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$, then $R_i = R_{\mathfrak{m}_i}$ for all $i = 1, \ldots, n$.*

Exercises

Let R be a non-zero commutative ring with identity. In the following five exercises, we prove Theorem IV.D.1.6 from earlier in this chapter:

If R is an artinian ring, then R is noetherian.

We accomplish this using the following results from Atiyah and Mac Donald. [Note to Keri: Work these results with proofs into an earlier section.]

Proposition 8.1. *In an artinian ring R, every prime ideal is maximal.*

Corollary 8.2. *In an artinian ring R, the nilradical is equal to the Jacobson radical.*

Proposition 8.3. *An artinian ring has only finitely many maximal ideals.*

Proposition 8.4. *In an artinian ring R, the nilradical is nilpotent.*

exr201031ab

Exercise IV.D.2.3. Let M be an R-module with a finite descending chain of submodules $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n = 0$ such that each quotient M_{i-1}/M_i is finitely generated. Prove that M is finitely generated.

defn201031ac

Definition IV.D.2.4. An R-module M is artinian if it satisfies the descending chain condition for submodules.

exer201031b

Exercise IV.D.2.5. Let V be a vector space over a field k. Prove that V is artinian as a k-module if and only if it is finite dimensional as a k-vector space.

exer201031c

Exercise IV.D.2.6. Let M be an artinian R-module, and let $N \subseteq M$ be a submodule. Prove that N and M/N are artinian.

exer201031d

EXERCISE IV.D.2.7. Let R be an artinian ring. To prove that R is noetherian, let I be an ideal, and show that I finitely generated as follows.

(a) Use the results from Exercise **??** show that there is a list of (not necessarily distinct) maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of R such that $\mathfrak{m}_1 \cdots \mathfrak{m}_n = 0$, so $I\mathfrak{m}_1 \cdots \mathfrak{m}_n = 0$.

(b) Prove that for each ideal J of R, the quotient $J/\mathfrak{m}_i J$ is finitely generated for all i. Conclude that $I\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1}/I\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1}\mathfrak{m}_i$ is finitely generated for all i.

(c) Use the descending chain $I \supseteq I\mathfrak{m}_1 \supseteq I\mathfrak{m}_1\mathfrak{m}_2 \supseteq \cdots$ to prove that I is finitely generated.

Cohen-Macaulay Rings

pter093021am

In this part, assume that $(R, \mathfrak{m}, k)$ is local or standard graded.

tion063021ab

IV.E.1. Foundational Properties

defn201029c DEFINITION IV.E.1.1. We say that R is unmixed if for all $\mathfrak{p} \in \operatorname{Ass}(R)$, we have $\dim(R/\mathfrak{p}) = \dim(R)$.

ex201029d EXAMPLE IV.E.1.2. (a) If R is an integral domain, then R is unmixed because every domain has that $\operatorname{Ass}(R) = \{0\}$. Since $R/0 \cong R$, then $\dim(R/0) = \dim(R)$.

(b) Let $R = k[X, Y, Z]/\langle XY, XZ\rangle$. Then $\operatorname{Ass}(R) = \{\langle \overline{X}\rangle, \langle \overline{Y}, \overline{Z}\rangle\}$, and

$$R/\langle \overline{X}\rangle \cong k[Y, Z] \qquad \therefore \dim(R/\langle \overline{X}\rangle) = 2,$$
$$R/\langle \overline{Y}, \overline{Z}\rangle \cong k[X] \qquad \therefore \dim(R/\langle \overline{Y}, \overline{Z}\rangle) = 1.$$

Since $\dim(R/\langle \overline{Y}, \overline{Z}\rangle) \neq \dim(R) = 2$, R is mixed.

(c) Let $R = k[X, Y, Z]/\langle XY, YZ, XZ\rangle$. Then $\operatorname{Ass}(R) = \{\langle \overline{X}, \overline{Y}\rangle, \langle \overline{X}, \overline{Z}\rangle, \langle \overline{Y}, \overline{Z}\rangle\}$, and for all $\mathfrak{p} \in \operatorname{Ass}(R)$, $R/\mathfrak{p} \cong k[T]$ is a polynomial ring, so $\dim(R/\mathfrak{p}) = 1$. Since $\dim(R) = 1$ as well, R is unmixed.

prop201029e PROPOSITION IV.E.1.3. *Let $S = k[X_1, \ldots, X_d]$ and let $I \subsetneq S$ be a monomial ideal and let $R = S/I$. Then R is unmixed if and only if I is unmixed.*

PROOF. Let $\mathfrak{p} = \langle \overline{x}_{i_1}, \ldots, \overline{x}_{i_m}\rangle$ such that $i_1 < \cdots < i_m$. Then $R/\mathfrak{p} \cong k[T_1, \ldots, T_{d-m}]$, so $\dim(R/\mathfrak{p}) = d - m$. $\square$

thm201029f THEOREM IV.E.1.4. *Every Cohen-Macualay ring is unmixed. Therefore, every mixed ring is not Cohen-Macaulay.*

ex201029g EXAMPLE IV.E.1.5. (a) $R = k[X, Y, Z]/\langle XY, XZ\rangle$ is mixed, so R is not Cohen-Macaulay.

(b) If Δ is the simplicial complex on $\{v_1, \ldots, v_d\}$, then $k[X_1, \ldots, X_d]/J_\Delta$ is unmixed if and only if J_Δ is unmixed if and only if Δ is pure. Therefore if Δ is not pure, then $k[\Delta]$ is not Cohen-Macaulay.

ex201029h EXAMPLE IV.E.1.6 (Converse of Theorem IV.E.1.4 fails). Consider $R = k[X, Y, Z, W]/\langle XY, YZ, ZW, WX\rangle$, and set $x = \overline{X}$, $y = \overline{Y}$, $z = \overline{Z}$, and $w = \overline{W}$. Then $\operatorname{Ass}(R) = \{\langle x, z\rangle, \langle y, w\rangle\}$ and $\dim(R) = 2$, so R is unmixed. We claim that $\operatorname{depth}(R) = 1$.

PROOF. Let $f = w - z \notin \langle x, z\rangle \cup \langle y, w\rangle = \operatorname{ZD}^0_R(R)$. Then f is R-regular, so $\operatorname{depth}(R) \geq 1$. Let $\overline{R} = R/\langle f\rangle \cong k[X, Y, Z]/\langle XY, YZ, Z^2, XZ\rangle$, and note that

$$\langle XY, YZ, Z^2, XZ\rangle = \langle X, Y, Z^2\rangle \cup \left\langle X, Z, \cancel{Z^2}\right\rangle \cup \left\langle Y, Z, \cancel{Z^2}\right\rangle = \langle X, Y, Z^2\rangle \cup \langle X, Z\rangle \cup \langle Y, Z\rangle.$$

Then $\operatorname{Ass}(\overline{R}) = \{\langle x, y, z\rangle, \langle x, z\rangle, \langle y, z\rangle\}$. Therefore $\overline{\mathfrak{m}} \in \operatorname{Ass}(\overline{R})$, so there is no $\overline{R}$-regular sequence. This means that f is a maximal R-regular sequence, so $\operatorname{depth}(R) = 1 < 2 = \dim(R)$. Therefore, R is not Cohen-Macaulay. ✓

Alternatively, since $\overline{R}$ is mixed, then $\overline{R}$ is not Cohen-Macaulay. By Proposition IV.C.1.5c, R is not Cohen-Macaulay.

prop201029i prop201029i.a PROPOSITION IV.E.1.7. (a) *If $\dim(R) = 0$, then R is Cohen-Macaulay.*

rop201029i.b (b) *If $\dim(R) \geq 1$ and R is unmixed, then $\operatorname{depth}(R) \geq 1$.*

rop201029i.c (c) *Assume that $\dim(R) = 1$. Then R is Cohen-Macaulay if and only if R is unmixed.*

rop201029i.d (d) *If $\dim(R) = 1$ and R is an integral domain, then R is Cohen-Macaulay.*

rop201029i.e (e) *If $\mathbf{f} = f_1, \ldots, f_n \in \mathfrak{m}$ is an R-regular sequence, each f_i is homogeneous, and $\overline{R} = R/\langle \mathbf{f}\rangle$ is mixed, then R is not Cohen-Macaulay.*

PROOF. (a) If R is noetherian and has $\dim(R) = 0$, then R is artinian, so R is Cohen-Macaulay by Corollary IV.D.1.9.

(b) By prime avoidance, we can find $\mathfrak{m} \not\subseteq \mathrm{ZD}_R^0(R)$. Then there is some $f \in \mathfrak{m} \setminus \mathrm{ZD}_R^0(R)$, so f is R-regular and therefore $\operatorname{depth}(R) \geq 1$.

(c) Assume that $\dim(R) = 1$ and that R is unmixed. Then part (b) implies that $\operatorname{depth}(R) \geq 1 = \dim(R) \geq \operatorname{depth}(R)$, so there is equality at every step. Therefore $\operatorname{depth}(R) = \dim(R)$, so R is Cohen-Macaulay.

(d) If R is an integral domain of dimension 1, then R is unmixed of dimension 1. Therefore R is Cohen-Macaulay by part (c).

(e) As in the alternative method in Example IV.E.1.6. □

ex201029j

EXAMPLE IV.E.1.8. (a) Let $R = k[X_1, \ldots, X_d]/I(k_d)$. Then $\operatorname{Ass}(R) = \left\{ \left\langle X_1, \ldots, \widehat{X}_i, \ldots, X_d \right\rangle \Big|_{i=1,\ldots,d} \right\}$. Therefore R is unmixed and $\dim(R) = 1$, so R is Cohen-Macaulay.

(b) (Numerical Semigroup Rings) Let $e_1, \ldots, e_t \in \mathbb{N}$ such that $1 \leq e_1 < e_2 < \cdots < e_t$ and $\gcd(e_1, \ldots, e_t) = 1$. Then $R = k[X^{e_1}, \ldots, X^{e_t}] \subseteq k[X]$ by an integral extension. As an example of an integral extension, we have

$$k[Y,Z]/\left\langle Y^4 - Z^3 \right\rangle \cong k[X^3, X^4] \ni a_0 + a_3X^3 + a_4X^4 + a_6X^6 + a_7X^7 + \ldots,$$

because $k[X_3, X^4]$ only contains elements with powers which are sums of multiples of 3 and 4. The integral extension of R implies that $\dim(R) = \dim(k[X]) = 1$. Since R is a domain, then R is Cohen-Macaulay.

(c) (Domain of Dimension 2 but not Cohen-Macaulay) Let $R = k[X^4, X^3Y, XY^3, Y^4] \subseteq k[X, Y]$. Then $\dim(R) = 2$ and R is an integral domain because the integral closure of R is $k[X, Y]$. We claim that $\operatorname{depth}(R) = 1$, so R is not Cohen-Macaulay.

PROOF. Since R is an integral domain and $\dim(R) \geq 2$, R is unmixed, so Proposition IV.E.1.7(b) implies that $\operatorname{depth}(R) \geq 1$. An integral domain actually implies that any non-zero non-unit is R-regular. We want to show that X^3Y is a maximal R-regular sequence. As a warning, we note that

$$R/\left\langle X^3Y \right\rangle = \frac{k[X^4, X^3Y, XY^3, Y^4]}{\langle X^3Y \rangle} \not\cong k[X^4, XY^3, Y^4],$$

since the ring $k[X^4, X^3Y, XY^3, Y^4]/\left\langle X^3Y \right\rangle$ is not an integral domain but $k[X^4, XY^3, Y^4]$ is a subring of $k[X, Y]$ and hence an integral domain. We can check that the surjection

$$\begin{aligned} k[T,U,V,W] &\xrightarrow{\tau} k[X^4, X^3Y, XY^3, Y^4] \\ T &\longmapsto X^4 \\ U &\longmapsto X^3Y \\ V &\longmapsto XY^3 \\ W &\longmapsto Y^4 \end{aligned}$$

is a well-defined ring epimorphism with

$$\operatorname{Ker}(\tau) = \left\langle TW - UV, TV^2 - U^2W, U^3 - T^2V, V^3 - W^2U \right\rangle.$$

From this, we conclude that

$$R \cong \frac{k[T,U,V,W]}{\langle TW - UV, TV^2 - U^2W, U^3 - T^2V, V^3 - W^2U \rangle}$$

and

$$\begin{aligned} \frac{R}{\langle X^3Y \rangle} &\cong \frac{k[T,U,V,W]}{\langle TW - UV, TV^2 - U^2W, U^3 - T^2V, V^3 - W^2U, U \rangle} \\ &\cong \frac{k[T,V,W]}{\langle TW, TV^2, T^2V, V^3 \rangle}. \end{aligned}$$

The ring in the previous line is mixed because

$$\left\langle TW, TV^2, T^2V, V^3 \right\rangle = \left\langle T, V^3 \right\rangle \cap \left\langle T^2, V^2, W \right\rangle \cap \langle V, W \rangle.$$

Therefore R is not Cohen-Macaulay. ✓

thm201105a THEOREM IV.E.1.9. *Assume R is Cohen-Macaulay and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $R_{\mathfrak{p}}$ is Cohen-Macaulay.*

There is some subtlety in this result. Since R is Cohen-Macaulay, we have $\dim(R) = \operatorname{depth}(R)$. But it is not immediately obvious how this implies that $\dim(R_{\mathfrak{p}}) = \operatorname{depth}(R_{\mathfrak{p}})$. It is possible in the non-Cohen-Macaulay case that $\dim(R_{\mathfrak{p}}) \leq \dim(R)$ but $\operatorname{depth}(R_{\mathfrak{p}})$ can be greater than $\operatorname{depth}(R)$, as in Example IV.E.1.10.

PROOF. Induct on $\dim(R_{\mathfrak{p}})$.
Base case: If $\dim(R_{\mathfrak{p}}) = 0$, then $R_{\mathfrak{p}}$ is Cohen-Macaulay by Proposition IV.E.1.7(a).
Inductive step: Assume $\dim(R_{\mathfrak{p}}) \geq 1$, so $\dim(R) \geq \dim(R_{\mathfrak{p}}) \geq 1$. Therefore R is Cohen-Macaulay and $\dim(R) \geq 1$, and $\mathfrak{p}$ is not a minimal prime of R. Since R is Cohen-Macaulay, then R is unmixed, and in particular we know that

$$\operatorname{Ass}(R) = \operatorname{Min}(R) \not\ni \mathfrak{p}.$$

In fact, $\mathfrak{p}$ is not contained in any minimal prime, so $\mathfrak{p}$ is also not contained in any associated prime. By prime avoidance, then

$$\mathfrak{p} \not\subseteq \bigcup_{\mathfrak{q} \in \operatorname{Ass}(R)} \mathfrak{q} = \operatorname{ZD}^0_R(R),$$

which implies that there is some $x \in \mathfrak{p} \setminus \operatorname{ZD}^0_R(R)$. In other words, there is some $x \in \mathfrak{p}$ that is R-regular. The ideal $x/1 \in R_{\mathfrak{p}}$ is $\mathfrak{p}_{\mathfrak{p}}$, so is not a unit. Also, $x/1$ is $R_{\mathfrak{p}}$-regular because

$$0 \longrightarrow R \xrightarrow{x} R \text{ exact} \implies 0 \longrightarrow R_{\mathfrak{p}} \xrightarrow{x/1} R_{\mathfrak{p}} \text{ exact.}$$

Set $\overline{R} = R/\langle x\rangle$ and $\overline{\mathfrak{p}} = \mathfrak{p}/\langle x\rangle \in \operatorname{Spec}(\overline{R})$ so that $\dim(\overline{R}_{\overline{\mathfrak{p}}}) = \dim(\overline{R_{\mathfrak{p}}}) = \dim(R_{\mathfrak{p}}) - 1$. Since R is Cohen-Macaulay, then $\overline{R}$ is Cohen-Macaulay. By the inductive hypothesis, we have that $\overline{R}_{\overline{\mathfrak{p}}}$ is Cohen-Macaulay, so $\overline{R_{\mathfrak{p}}}$ is Cohen-Macaulay. Since $x/1$ is $R_{\mathfrak{p}}$-regular, this implies that $R_{\mathfrak{p}}$ is Cohen-Macaulay. □

ex201105b EXAMPLE IV.E.1.10. Consider the ring $R = k[X,Y,Z]_{\langle X,Y,Z\rangle}/\langle XY, XZ, X^2\rangle$ which has $\dim(R) = 2$ and $\operatorname{depth}(R) = 0$, and consider the prime ideal $\mathfrak{p} = \langle \overline{X}, \overline{Y}\rangle$. Then

$$\begin{aligned} R_{\mathfrak{p}} &\cong \frac{k[X,Y,Z]_{\langle X,Y\rangle}}{\langle XY, XZ, X^2\rangle} \\ &\cong \frac{k[X,Y,Z]_{\langle X,Y\rangle}}{\left\langle \cancel{XY}, X, \cancel{X^2}\right\rangle} \\ &\cong k[Y,Z]_{\langle Y\rangle}. \end{aligned}$$

Then $R_{\mathfrak{p}}$ is an integral domain but not a field, so it has depth at least 1. In particular, we have

$$1 \leq \operatorname{depth}(R_{\mathfrak{p}}) \leq \dim(R_{\mathfrak{p}}) \leq \operatorname{edim}(R_{\mathfrak{p}}) = 1,$$

so $R_{\mathfrak{p}}$ is a regular local ring and therefore Cohen-Macaulay of dimension 1. In particular,

$$1 = \operatorname{depth}(R_{\mathfrak{p}}) > \operatorname{depth}(R) = 0,$$

so the depth of a ring can increase when we localize at $\mathfrak{p}$.

note201105c NOTE IV.E.1.11. Let $\mathbf{f} = f_1, \ldots, f_n \in \mathfrak{m}$. In the standard graded cases, assume that each f_i is homogeneous.

(a) If R is Cohen-Macaulay, then $R/\langle \mathbf{f}\rangle$ is unmixed for all R-regular sequences $\mathbf{f}$.
(b) If R is Cohen-Macaulay, then $R/\langle \mathbf{f}\rangle$ is artinian for some R-regular sequence $\mathbf{f}$.
(c) In the standard graded case, if R is Cohen-Macaulay, then $R/\langle \mathbf{f}\rangle$ is a finite dimensional vector space over $R_0 = k$ for some R-regular sequence $\mathbf{f}$.

IV.E.2. Homological Properties

tion063021ac

We will begin with a survey of Ext modules. Let (S, η, k) be local or standard graded, let $I \lneq S$ be a proper ideal (and homogeneous in the standard graded case), and set $R = S/I$.

defn201110a DEFINITION IV.E.2.1. A chain complex over S (or an S-complex) is a sequence of S-module homomorphisms

$$X = \qquad \cdots \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \xrightarrow{\partial^X_{i-1}} \cdots$$

such that $\partial^X_{i-1} \circ \partial^X_i = 0$ (and therefore $\operatorname{Ker}\partial^X_i \supseteq \operatorname{Img}\partial^X_{i+1}$) for all $i \in \mathbb{Z}$ and the i^{th} homology module of X is $H_i(X) = (\operatorname{Ker}\partial^X_i)/(\operatorname{Im}\partial^X_{i-1})$.

note201110b NOTE IV.E.2.2. Let X be an S-complex. Then S is exact at X_i if and only if $H_i(X) = 0$. Therefore $H_i(X)$ measures how far X is from being exact.

ex201110c EXAMPLE IV.E.2.3. (a) An augmented projective resolution of an S-module M is an exact sequence

$$P^+ = \qquad \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that each P_i is projective over S. The associated (truncated) projective resolution is

$$P = \qquad \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0.$$

Then P is an S-complex such that

$$H_i(P) \begin{cases} = 0 & i \neq 0 \\ \cong M & i = 0. \end{cases}$$

(b) Let $x_1, \ldots, x_n \in S$. The Koszul complex $K = K^S(\mathbf{x})$ is $K_i = S^{\binom{n}{i}}$ with basis

$$\{e_F \mid F \subsetneq [n] = \{1, \ldots, n\} \text{ and } |F| = i\}$$

and differential

$$\partial^K_i(e_F) = \sum_f \in F(-1)^{\sigma(f,F)} x_f e_{F\setminus\{f\}},$$

where $\sigma(f, F) = |\{g \in F \mid g < f\}|$. For instance

$$\partial^K_3(e_{135}) = x_1e_{35} - x_3e_{15} + x_5e_{13}.$$

So K has the form

$$K = \qquad 0 \longrightarrow S \longrightarrow S^n \longrightarrow \cdots \longrightarrow S^n \longrightarrow S \longrightarrow 0$$

and is an S-complex. If $\mathbf{x}$ is S-regular, then $H_i(K) = 0$ for all $i \neq 0$ (and therefore K is a a projective resolution of $S/\langle \mathbf{x}\rangle$). The converse holds if S is local or S is standard graded with homogeneous x_i and $\mathbf{x} \subset \eta$.

note201110d NOTE IV.E.2.4. Ext solves two problems.

(1) Ext fixes the lack of exactness of Hom.
(2) Ext gives a homological characterization of depth.

note201110d.1 (1) Given the short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ and an S-module M, the left-exactness of $\operatorname{Hom}_S(M, -)$ we have the exact sequence

$$0 \longrightarrow \operatorname{Hom}_s(M, A) \longrightarrow \operatorname{Hom}_S(M, B) \longrightarrow \operatorname{Hom}_S(M, C).$$

For instance, if $S = \mathbb{Z}$, and $M = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$, and we consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

then the functor $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, -)$ yields the exact sequence

$$0 \longrightarrow \underbrace{\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z})}_{=0} \xrightarrow{2\cdot} \underbrace{\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z})}_{=0} \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2)$$

which can also be written $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}_2$. Note that this would not be exact if we augmented it with an additional zero on the right. The long exact sequence in $\operatorname{Ext}_S(M, -)$ "fixes" this lack of exactness on the right.

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_S(N,A) \arrow[r] & \mathrm{Hom}_S(N,B) \arrow[r] & \mathrm{Hom}_S(N,C) \arrow[dll, rounded corners] \\
& \mathrm{Ext}^1_S(N,A) \arrow[r] & \mathrm{Ext}^1_S(N,B) \arrow[r] & \mathrm{Ext}^1_S(N,C) \arrow[dll, rounded corners] \\
& \mathrm{Ext}^2_S(N,A) \arrow[r] & \cdots &
\end{tikzcd}
$$

note201110d.2 (2) David Rees proved

$$
\begin{aligned}
\mathrm{depth}(I,S) &= \min\left\{ i \geq 0 \;\middle|\; \mathrm{Ext}^i_S(S/I,S) \neq 0 \right\} \\
&= \min\left\{ i \geq 0 \;\middle|\; \mathrm{Ext}^i_S(R,S) \neq 0 \right\}.
\end{aligned}
$$

con201110e CONSTRUCTION IV.E.2.5. Let M and N be S-modules and let P be a projective resolution of M. We have

$$\mathrm{Ext}^i_S(M,N) = H_{-i}(\mathrm{Hom}_S(P,N))$$

ex201110f EXAMPLE IV.E.2.6. (a) Let $S = \mathbb{Z}$ and let $M = \mathbb{Z}_2 = N$. We have the augmented projective resolution of M

$$P^+ = \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

and the corresponding truncated projective resolution

$$P = \qquad 0 \longrightarrow \underset{P_1}{\mathbb{Z}} \xrightarrow{2\cdot} \underset{P_0}{\mathbb{Z}} \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z}_2)$ to P we obtain

$$
\begin{aligned}
\mathrm{Hom}_{\mathbb{Z}}(P,\mathbb{Z}_2) = &\qquad 0 \longrightarrow \underbrace{\overset{\mathrm{Hom}(P_0,N)}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}_2)}}_{\cong \mathbb{Z}_2} \longrightarrow \underbrace{\overset{\mathrm{Hom}(P_1,N)}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}_2)}}_{\cong \mathbb{Z}_2} \longrightarrow 0 \\
\cong &\qquad 0 \longrightarrow \mathbb{Z}_2 \xrightarrow[=0]{2\cdot} \mathbb{Z}_2 \longrightarrow 0.
\end{aligned}
$$

Now we can compute the following Ext modules.

$$\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Z}_2) = \frac{\mathrm{Ker}\left(\mathbb{Z}_2 \xrightarrow[=0]{2\cdot} \mathbb{Z}_2\right)}{\mathrm{Im}\,(0 \longrightarrow \mathbb{Z}_2)} = \frac{\mathbb{Z}_2}{0} \cong \mathbb{Z}_2$$

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Z}_2) \cong \mathbb{Z}_2$$

The corresponding long exact sequence for $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2,-)$ is

$$
\begin{tikzcd}
0 \arrow[r] & \cancelto{=0}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Z})} \arrow[r] & \cancelto{=0}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Z})} \arrow[r] & \overbrace{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Z}_2)}^{\cong \mathbb{Z}_2} \arrow[dll, rounded corners] \\
& \overbrace{\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Z})}^{\cong \mathbb{Z}_2} \arrow[r] & \overbrace{\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Z})}^{\cong \mathbb{Z}_2} \arrow[r] & \overbrace{\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Z}_2)}^{\cong \mathbb{Z}_2} \arrow[dll, rounded corners] \\
& \cancelto{=0}{\mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Z})} \arrow[r] & \cdots ,
\end{tikzcd}
$$

which can be written

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \xrightarrow[=0]{2\cdot} \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

(b) Recall the Fundamental Theorem of Finitely Generated Abelian Groups. If $S = \mathbb{Z}$ and M is a finitely generated $\mathbb{Z}$-module (i.e., a finitely generated abelian group), then there exist positive integers $d_1, \ldots, d_p \in \mathbb{Z}$ and a non-negative integer $r \in \mathbb{Z}$ such that

$$M \cong \mathbb{Z}/\langle d_1 \rangle \oplus \cdots \oplus \mathbb{Z}/\langle d_p \rangle \oplus \mathbb{Z}^r.$$

Then an augmented projective resolution P^+ of M is

$$0 \longrightarrow \mathbb{Z}^p \xrightarrow[\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_p \\ & \mathbf{0} & \end{pmatrix}]{} \mathbb{Z}^{p+r} \longrightarrow M \longrightarrow 0.$$

To compute $\operatorname{Ext}^i_{\mathbb{Z}}(M, N) = H_{-i}(\operatorname{Hom}_{\mathbb{Z}}(P, N))$ we apply $\operatorname{Hom}_{\mathbb{Z}}(-, N)$ to P and obtain

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^{p+r}, N) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^p, N) \longrightarrow 0,$$

which can be written as

$$0 \longrightarrow N^{p+r} \xrightarrow[\begin{pmatrix} d_1 & & 0 & \\ & \ddots & & \mathbf{0} \\ 0 & & d_p & \end{pmatrix}]{} N^p \longrightarrow 0.$$

Hence we compute $\operatorname{Ext}^0_{\mathbb{Z}}(M, N) \cong N^r$, and $\operatorname{Ext}^1_{\mathbb{Z}}(M, N) \cong N/d_1N \oplus \cdots \oplus N/d_pN$, and $\operatorname{Ext}^i_{\mathbb{Z}}(M, N) = 0$ for all $i \neq 0, 1$. What if M is not finitely generated? The projective resolution instead looks like

$$P = \qquad 0 \longrightarrow \mathbb{Z}^{(Y)} \longrightarrow \mathbb{Z}^{(Z)} \longrightarrow 0$$

and similar vanishing results hold for any PID.

(c) Recall the Auslander, Buchsbaum, and Serre characterization of regular local rings. The ring S is a regular local ring if and only if $\operatorname{pd}_S M < \infty$ for all S-modules M. Since $\operatorname{pd}_S M < \dim S = d < \infty$, we therefore have $\operatorname{Ext}^i_S(M, S) = 0$ for all $i > d$.

(d) Let $S = k[\![X]\!]/\langle X^2 \rangle$ and $M = k$. Then we have the projective resolution

$$P = \qquad \cdots \xrightarrow{X\cdot} S \xrightarrow{X\cdot} S \xrightarrow{X\cdot} S \xrightarrow{X\cdot} S \longrightarrow 0.$$

To compute Ext modules $\operatorname{Ext}^i_S(k, k)$ we need the exact sequence $\operatorname{Hom}_S(P, k)$, which is isomorphic to

$$0 \longrightarrow k \xrightarrow[=0]{X\cdot} k \xrightarrow[=0]{X\cdot} k \xrightarrow[=0]{X\cdot} k \xrightarrow[=0]{X\cdot} \cdots.$$

Thus we have $\operatorname{Ext}^{(}_S k, k) \cong k \neq 0$ for all $i \geq 0$.

thm201110g

THEOREM IV.E.2.7. *Assume S is Cohen-Macaulay. Then we have*

$$\begin{aligned} \operatorname{depth}(I, S) &= \dim(S) - \dim(S/I) \\ &= \dim(S) - \dim(R) \\ &= \operatorname{depth}(S) - \dim(R) \\ &\leq \operatorname{depth}(S) - \operatorname{depth}(R) \end{aligned}$$

and therefore R is Cohen-Macaulay if and only if $\operatorname{depth}(I, S) = \operatorname{depth}(S) - \operatorname{depth}(R)$.

cor201110h

COROLLARY IV.E.2.8. *Assume S is Cohen-Macaulay and* $\operatorname{pd}_S R < \infty$ *(e.g., S is a regular local ring or $S = k[X_1, \ldots, X_d]$). Then R is Cohen-Macaulay if and only if* $\operatorname{depth}(I, S) = \operatorname{pd}_S R$.

PROOF. Note that by Auslander and Buchsbaum we have $\operatorname{pd}_S R = \operatorname{depth}(S) - \operatorname{depth}(R)$ and then apply Theorem IV.E.2.7. $\square$

thm201110i THEOREM IV.E.2.9. *Assume S is Cohen-Macaulay, and $p = \mathrm{pd}_S R < \infty$, and $q = \mathrm{depth}(I, S)$*

thm201110i.a (a) $\mathrm{Ext}^i_S(R, S) = 0$ *for all* $i > p$ *and* $\mathrm{Ext}^p_S(R, S) \neq 0$

thm201110i.b (b) $\mathrm{Ext}^i_S(R, S) = 0$ *for all* $i < q$ *and* $\mathrm{Ext}^p_S(R, S) \neq 0$

thm201110i.c (c) *The following are equivalent.*
  (i) *R is Cohen-Macaulay*
  (ii) $\mathrm{Ext}^i_S(R, S) = 0$ *for all* $i \neq p$
  (iii) $\mathrm{Ext}^i_S(R, S) = 0$ *for all* $i \neq q$
  (iv) $\mathrm{Ext}^i_S(R, S) = 0$ *for all* $i < p$
  (v) $\mathrm{Ext}^i_S(R, S) = 0$ *for all* $i > q$
  (vi) $\mathrm{Ext}^i_S(R, S) = 0$ *for all* $i \neq s$ *for some* s

PROOF. (a) The vanishing part is true by definition of projective dimension and by the construction of Ext. For the non-vanishing part, we first note there exists a "minimal" projective resolution

$$0 \longrightarrow R^{\beta_p} \xrightarrow{\partial^P_p} R^{\beta_{p-1}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \longrightarrow R \longrightarrow 0$$

of R over S, where the matrices representing the maps have all their entries in $\mathfrak{m}$. So we compute

$$\begin{array}{c} \mathrm{Ext}^p_S(R, S) \cong R^{\beta_p} / \operatorname{Im}\left(\partial^P_p\right)^T \\ \big\downarrow \\ R^{\beta_p}/\mathfrak{m} \cdot R^{\beta_p} \cong k^{\beta_p} \neq 0 \end{array}$$

where all the entries of $\operatorname{Im}\left(\partial^P_p\right)^T$ are in $\mathfrak{m}$. Note that $k^{\beta_p} \neq 0$ is justified, since $p = \mathrm{pd}_S R$ implies $\beta_p \neq 0$.

(b) This holds by a previous result by Rees (see Note IV.E.2.42).

(c) Assume R is Cohen-Macaulay. Then $\mathrm{depth}(I, S) = \mathrm{pd}_S R$, i.e., $q = p$. Then by parts a and b we have $\mathrm{Ext}^i_S(R, S) = 0$ for all $i > p = q$, and $\mathrm{Ext}^i_S(R, S) = 0$ for all $i < p = q$, and $\mathrm{Ext}^q_S(R, S) = \mathrm{Ext}^p_S(R, S) \neq 0$. On the other hand, if we instead assume that $s \in \mathbb{Z}$ such that $\mathrm{Ext}^i_S(R, S) = 0$ for all $i \neq s$, then parts a and b imply that $\mathrm{Ext}^p_S(R, S) \neq 0$ and $\mathrm{Ext}^q_S(R, S) \neq 0$, respectively. It follows that $p = s = q$. Moreover, $\mathrm{depth}(I, S) = \mathrm{pd}_S R$ now implies that R is Cohen-Macaulay. □

cor201117a COROLLARY IV.E.2.10. *Assume S is a regular local ring or a polynomial ring over k, and assume R is Cohen-Macaulay. Then for every $\mathfrak{p} \in \mathrm{Spec}(R)$, the ring $R_\mathfrak{p}$ is Cohen-Macaulay. (This is a special case of Theorem IV.E.1.9)*

PROOF. Recall the the following correspondence of prime ideals:

$$R = S/I \longleftarrow S$$

$$\mathfrak{p} = P/I \longleftarrow\!\shortmid P \in \mathrm{Spec}(S) \text{ s.t. } P \supseteq I.$$

Then we have

$$R_\mathfrak{p} \cong R_P \cong (S/I)_P \cong S_P/I_P \twoheadleftarrow S_P$$

where S_P is regular local ring. Therefore

$$\mathrm{Ext}^i_{S_P}(R_\mathfrak{p}, S_P) \cong \mathrm{Ext}^i_{S_P}(R_P, S_P) \cong \mathrm{Ext}^i_S(R, S)_P,$$

where the second isomorphism holds since S is noetherian and R is finitely generated over S. Moreover, by Theorem IV.E.2.9 there exists some $s \in \mathbb{Z}$ such that $\mathrm{Ext}^i_S(R, S)_P = 0$ for all $i \neq s$. Hence $\mathrm{Ext}^i_{S_P}(R_\mathfrak{p}, S_P) = 0$ for all $i \neq s$ and Theorem IV.E.2.9 implies $R_\mathfrak{p}$ is Cohen-Macaulay. □

note201117b NOTE IV.E.2.11. (a) The proof of Corollary IV.E.2.10 shows more than the stated result. If we use $s = p$, then $\mathrm{pd}_{S_P} R_\mathfrak{p} = p = \mathrm{pd}_S R$, which is surprising since usually projective dimension goes down under localization. Also, using $s = q$ yields $\mathrm{depth}(I, S_P) = q = \mathrm{depth}(I, S)$, which is again surprising since usually depth changes under localization. This is very special, and sometimes called "perfect".

(b) If A is a standard graded ring, then there exists a polynomial ring $T = k[Y_1, \ldots, Y_e] \twoheadrightarrow A$, so $A \cong T/J$ for some ideal $J \leq T$. Therefore Corollary IV.E.2.10 implies Cohen-Macaulayness localizes for A.

(c) If A is local and Cohen-Macaulay, then it may not be of the form T/J, where T is a regular local ring. So we do not immediately recover the full strength of Theorem IV.E.1.9 in the local case. We try to fix this using completion and the Cohen Structure Theorem (Theorem IV.A.7.26). Since A is Cohen-Macaulay, the completion $\hat{A}$ is Cohen-Macaulay as well. The Cohen Structure Theorem implies $\hat{A} \cong T/J$ for some ideal $J \leq T$, where T is a regular local ring. Corollary IV.E.2.10 implies $\hat{A}_Q$ is Cohen-Macaulay for any idea $Q \in \operatorname{Spec}(\hat{A})$. We want to show that A_q is Cohen-Macaulay for every $q \in \operatorname{Spec}(A)$, but how do we find Q related to $\mathfrak{q}$ such that this process might work? One could try using $\mathfrak{q}$, but this is an ideal of A, not necessarily $\hat{A}$. One could try $\mathfrak{q} \cdot \hat{A} = \hat{\mathfrak{q}}$ using the natural map $A \to \hat{A}$, but it may be the case that $\hat{\mathfrak{q}} \notin \operatorname{Spec}(\hat{A})$. One could try localizing $\hat{A}$ as an A-module with respect to $\mathfrak{q}$ to obtain $\left(\hat{A}\right)_{\mathfrak{q}}$, but it might not be local.

thm201117c THEOREM IV.E.2.12 (lying-over for completions). *Assume A is a local ring and $\mathfrak{q} \in \operatorname{Spec}(A)$. Then there exists and ideal $Q \in \operatorname{Spec}\left(\hat{A}\right)$ such that $Q \cap A = \mathfrak{q}$.*

$$\begin{tikzcd} A \arrow[r, hook] & \hat{A} \\ \mathfrak{q} & \exists Q \arrow[l, bend left, mapsto] \end{tikzcd}$$

note201117d NOTE IV.E.2.13. Could we use this to try to reprove the local case of Theorem IV.E.1.9? Since A is Cohen-Macaulay, we know $\hat{A}$ is Cohen-Macaulay and then the Cohen Structure Theorem implies $\hat{A} \cong T/J$ for some regular local ring T. Then Corollary IV.E.2.10 implies $(\hat{A})_Q$ is Cohen-Macaulay, where $Q \in \operatorname{Spec}(\hat{A})$ such that $Q \cap A = \mathfrak{q}$. Now, how do we conclude that $A_{\mathfrak{q}}$ is Cohen-Macaulay? We have to be cautious, since localization does not commute with completions, i.e., $(\hat{A})_Q \not\cong \widehat{A_{\mathfrak{q}}}$. The solution involves broadening our context to flat local ring homomorphisms.

defn201117e DEFINITION IV.E.2.14. Let A be a noetherian commutative ring with identity.

lefn201117e.a (a) An A-module N is <u>flat</u> if $N \otimes_A -$ is exact, i.e., for every exact sequence $N \otimes_A \mathcal{S}$ is exact (equivalently, for every short exact sequence $\mathcal{S}$), i.e., $\operatorname{Tor}_i^A(N, -) = 0$ for all $i \geq 1$, i.e., $\operatorname{Tor}_1^A(N, -) = 0$.

lefn201117e.b (b) A ring homomorphism $A \xrightarrow{\phi} B$ is <u>flat</u> if B is flat as an A-module by restriction of scalars: $a \cdot b = \phi(a)b$.

lefn201117e.c (c) Let $\mathfrak{q} \in \operatorname{Spec}(A)$ and let $\phi : A \to B$ be a ring homomorphism. The <u>fibre</u> of ϕ over $\mathfrak{q}$ is

$$B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} \cong (A/\mathfrak{q}) \otimes_A B_{\mathfrak{q}} \cong ((A/\mathfrak{q}) \otimes_A A_{\mathfrak{q}}) \otimes_A B.$$

lefn201117e.d (d) A <u>local</u> ring homomorphism is a ring homomorphism $A \xrightarrow{\phi} B$ between local rings $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ such that $\mathfrak{m}_A B \subseteq \mathfrak{m}_B$, i.e., $\phi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

ex201117f EXAMPLE IV.E.2.15. Let A be a noetherian commutative ring with identity.

(a) Free implies projective implies flat. A is flat over A, because $A \otimes_A - = \operatorname{id}$ is exact. Free implies flat, because $A^{(X)} \otimes_A - = \operatorname{id}^{(X)}$ is exact. Projective implies flat: if N is projective over A, then there exists some N' such that $N \oplus N'$ is free. This implies $N \oplus N'$ is flat, which implies N and N' are flat, because

$$(N \oplus N') \otimes_A - \cong (N \otimes_A -) \oplus (N' \otimes_A -).$$

(b) If A is local, then projective implies free. If A is local, then this implication fails, in general. Consider the ring $A = \mathbb{R}[X, Y, Z]/\left\langle X^2 + Y^2 + Z^2 - 1 \right\rangle$, which has a non-free finitely generated projective module that comes from the tangent bundle on $\mathbb{S}^2$ (which has no nowhere vanishing tangent vector fields).

(c) If N is finitely generated, then N flat implies N projective, but if N is not finitely generated, then flat no longer implies projective. For instance if $A = \mathbb{Z}$, then consider $N = \mathbb{Q} = \operatorname{Frac}(\mathbb{Z})$, which is flat but not free over $\mathbb{Z}$ and not projective. (One can make a local example as well: consider $\mathbb{Z}_{\langle p \rangle}$ and $N = \mathbb{Q} = \operatorname{Frac}(\mathbb{Z}_{\langle p \rangle})$ for some positive prime $p \in \mathbb{Z}$.) The same holds for any integral domain A that is not a field, e.g., $A = k[X_1, \ldots, X_d]$ or $A = k[\![X_1, \ldots, X_d]\!]$.

(d) The natural map $A \to A[X_1, \ldots, X_d]$ is flat, because it is free.

(e) The natural map $A \to A[\![X_1, \ldots, X_d]\!]$ is flat, because A is noetherian. (This uses Baer's criterion for flatness.)

(f) Any localization $A \to U^{-1}A$ for a multiplicatively closed set $U \subseteq A$ is flat, because $(U^{-1}A) \otimes_A - \cong U^{-1}(-)$ is flat, e.g., $A \to A_{\mathfrak{q}}$ is flat for every $\mathfrak{q} \in \operatorname{Spec}(A)$.

(g) If $\mathfrak{a} \lneq A$, then $A \to \hat{A}^{\mathfrak{a}}$ is flat, because A is noetherian, e.g., if A is local, then $A \to \hat{A}$ is flat.

(h) Assume A is local. Then $A \to A[\![X_1, \ldots, X_d]\!]$ and $A \to \hat{A}$ are both flat and local.

(i) Fibres contain geometric information.

thm201117g THEOREM IV.E.2.16. *Let $\varphi : A \to B$ be flat and local, and let $F = B/\mathfrak{m}_A B$ be the fibre of φ at $\mathfrak{m}_A$.*

thm201117g.a (a) $\dim B = \dim A + \dim F$

thm201117g.b (b) $\operatorname{depth}(B) = \operatorname{depth}(A) + \operatorname{depth}(F)$

thm201117g.c (c) *B is Cohen-Macaulay if and only if A and F are each Cohen-Macaulay.*

thm201117g.d (d) *B is artinian if and only if A and F are each artinian.*

thm201117g.e (e) *B is a formal complete intersection if and only if A and F are each formal complete intersections.*

thm201117g.f (f) *If A and F are regular local rings, then B is a regular local ring. If B is a regular local ring, then A is a regular local ring.*

PROOF. We leave parts (a), (b), (e), and (f) as black boxes.

(c) First, we know that B is Cohen-Macaulay if and only if $\operatorname{depth}(B) = \dim(B)$. Then by parts a and a, we get
$$\operatorname{depth}(A) + \operatorname{depth}(F) = \dim(A) + \dim(F).$$
In particular, we must have $\operatorname{depth}(A) = \dim(A)$ and $\operatorname{depth}(F) = \dim(F)$ because $\operatorname{depth}(A) \le \dim(A)$ and $\operatorname{depth}(F) \le \dim(F)$, which means that A and F are both Cohen-Macaulay.

(d) We have
$$\begin{aligned} B \text{ artinian} &\iff \dim(B) = 0 \\ &\iff \dim(A) + \dim(F) = 0 \\ &\iff \dim(A) = 0 = \dim(F) && (\because \dim(A), \dim(F) \ge 0) \\ &\iff A\&F \text{ artinian.} \end{aligned}$$
$\square$

ex201119a EXAMPLE IV.E.2.17. Let $\phi : A \to B$ be flat and local. Then B being a regular local ring does not imply that F is a regular local ring. Consider
$$\begin{tikzcd} A \arrow[r, phantom, "="] & k[\![X^2]\!] \arrow[r, hook] \arrow[d, phantom, "\wr\|"] & k[\![X]\!] \arrow[r, phantom, "="] & B. \\ & k[\![Y]\!] & & \end{tikzcd}$$
Here, A and B are both regular local rings. The inclusion above is flat because B is free as an A-module with basis $1, X$. But
$$F = B/\mathfrak{m}_A B = k[\![X]\!]/\left\langle X^2 \right\rangle$$
is not a regular local ring because $\dim(F) = 0 < 1 = \operatorname{edim}(F)$.

cor201119b COROLLARY IV.E.2.18. *Let A be local.*

cor201119b.a (a) *A is Cohen-Macaulay if and only if $\widehat{A}$ is Cohen-Macaulay if and only if $A[\![X]\!]$ is Cohen-Macaulay if and only if $A[\![X_1, \ldots, X_d]\!]$ is Cohen-Macaulay for all d or for some d.*

cor201119b.b (b) *Two similar results come from replacing "Cohen-Macaulay" in part (a) with either "formal complete intersection" or "regular local ring."*

cor201119b.c (c) *A is artinian if and only if $\widehat{A}$ is artinian.*

PROOF. Consider the maps $\alpha : A \to \widehat{A}$ and $\beta : A \to A[\![\mathbf{X}]\!]$. Both α and β are flat and local. Their fibres are
$$F(\alpha) = \widehat{A}/\mathfrak{m}_A\widehat{A} \cong \widehat{A}/\mathfrak{m}_{\widehat{A}} \cong k \tag{IV.E.2.18.1}$$
$$F(\beta) = A[\![X]\!]/\mathfrak{m}_A A[\![X]\!] \cong k[\![X]\!]. \tag{IV.E.2.18.2}$$
Note that $F(\alpha)$ is a field, so is a regular local ring and $F(\beta)$ is a power series ring, so is a regular local ring. Therefore (a) and (b) both follow from Theorem IV.E.2.16. Part (c) follows because k is artinian (but $k[\![X]\!]$ is not artinian unless $d = 0$). But A artinian implies that $\widehat{A} \cong A$, so $\widehat{A}$ is artinian. $\square$

prop201119c PROPOSITION IV.E.2.19. *Let $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ be ring homomorphisms and let $Q \in \operatorname{Spec}(B)$ and let $\mathfrak{q} = \phi^{-1}(Q) \in \operatorname{Spec}(A)$.*

prop201119c.a (a) *There is a map $\phi_Q : A_{\mathfrak{q}} \to B_Q$ given by $\frac{a}{t} \mapsto \frac{\phi(a)}{\phi(t)}$ is a well-defined local ring homomorphism.*

prop201119c.b (b) *If ϕ and ψ are flat, then $\psi \circ \phi$ is flat.*

prop201119c.c (c) *If ϕ is flat, then ϕ_Q is flat.*

prop201119c.d (d) *If ϕ is flat and B_Q is Cohen-Macaulay, then $A_{\mathfrak{q}}$ is Cohen-Macaulay.*

PROOF. (a) We use the universal mapping property.

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] \arrow[dr] & B \arrow[d] \\ A_{\mathfrak{q}} \arrow[r, dashed] & B_Q. \end{tikzcd}$$

(b) $C \otimes_A S \cong C \otimes_B B \otimes_A S$, using associativity of the tensor product.
(c) Black box.
(d) This follows from (c) and Theorem IV.E.2.16. □

note201119d NOTE IV.E.2.20. We give a complete alternate proof of the local case of Theorem IV.E.1.9. Let $Q \in \operatorname{Spec}(\widehat{A})$ such that $Q \cap A = \mathfrak{q}$. Then the map $A \to \widehat{A}$ is flat and A Cohen-Macaulay implies $\widehat{A}$ is Cohen-Macaulay, which implies $\widehat{A}_Q$ is Cohen-Macaulay. By Corollary d, then $A_{\mathfrak{q}}$ is Cohen-Macaulay. □

ex201119e EXAMPLE IV.E.2.21. Let $S = k[X, Y]$.

ex201119e.a (a) Let $I = \langle X^2, Y^2 \rangle$ and let $R = k[X, Y]/\langle X^2, Y^2 \rangle$. We want to calculate $\operatorname{Ext}^i_S(R, S)$. Theorem IV.E.2.9 guarantees $\operatorname{Ext}^i_S(R, S) = 0$ for all $i \neq p = q$, where $p = \operatorname{pd}_S(R)$ and $q = \operatorname{depth}(I, S) = 2$. To find p, resolve R over S using the Koszul complex:

$$K = \left(0 \longrightarrow S \xrightarrow{\begin{pmatrix} Y^2 \\ -X^2 \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} X^2 & Y^2 \end{pmatrix}} S \longrightarrow 0 \right)$$

$$K^* = \operatorname{Hom}_S(K, S) = \left(0 \longrightarrow S \xrightarrow{\begin{pmatrix} X^2 \\ Y^2 \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} Y^2 & -X^2 \end{pmatrix}} S \longrightarrow 0 \right)$$

We need to show $K^* \cong \Sigma^{-2} K$ is exact at each degree. But

$$\operatorname{Ext}^i_S(R, S) = H_{-i}(K^*) \cong H_{-i}(\Sigma^{-2} K) = H_{-i+2}(K) = \begin{cases} R & \text{if } -i+2 = 0 \\ 0 & \text{if } -i+2 \neq 0 \end{cases} = \begin{cases} R & \text{if } i = 2 \\ 0 & \text{if } i \neq 2. \end{cases}$$

ex201119e.b (b) Let $I = \langle X^2, XY, Y^2 \rangle$. Therefore $R = S/I$ is artinian, so R is Cohen-Macaulay of dimension 0, so R has depth 0. Also,

$$p = \operatorname{pd}_S(R) = \operatorname{depth}(S) - \operatorname{depth}(R) = 2 - 0 = 2,$$

using Auslander-Buchsbaum. The resolution is

$$F = \left(0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} Y & 0 \\ -X & Y \\ 0 & -X \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} X^2 & XY & Y^2 \end{pmatrix}} S \longrightarrow 0 \right)$$

$$F^* = \left(0 \longrightarrow S \xrightarrow{\begin{pmatrix} X^2 \\ XY \\ Y^2 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} Y & -X & 0 \\ 0 & Y & -X \end{pmatrix}} S^2 \longrightarrow 0 \right).$$

We can calculate Ext in each degree:

$$\begin{aligned}\operatorname{Hom}_S(R,S) \cong \operatorname{Ext}^0_S(R,S) &= H_0(F^*) = 0 && (\because S \text{ integral domain})\\ \operatorname{Ext}^1_S(R,S) &= H_{-1}(F^*) = 0\\ \operatorname{Ext}^2_S(R,S) &= H_{-2}(F^*) \cong \frac{S^2}{\left\langle \binom{Y}{0}, \binom{-X}{Y}, \binom{0}{X} \right\rangle}.\end{aligned}$$

By Nakiyama's Lemma, $\beta_0(\operatorname{Ext}^2_S(R,S)) = 2$.

ex201119e.c (c) Let $T = k[a,b,\alpha,\beta]$ and $J = \langle a\alpha, ab, b\beta \rangle$ and $A = T/J$. A is Cohen-Macaulay of dimension 2, so $\operatorname{pd}_T(A) = \operatorname{depth}(T) - \operatorname{depth}(A) = 4 - 2 = 2$. We should find that $\operatorname{Ext}^i_T(A,T) = 0$ for all $i \neq 2$. The free resolution of A over T looks like

$$G = \left(0 \longrightarrow T^2 \xrightarrow{\begin{pmatrix} b & 0 \\ -\alpha & \beta \\ 0 & -a \end{pmatrix}} T^3 \xrightarrow{\begin{pmatrix} a\alpha & ab & b\beta \end{pmatrix}} T \longrightarrow 0 \right)$$

$$G^* = \left(0 \longrightarrow T \xrightarrow{\begin{pmatrix} a\alpha \\ ab \\ b\beta \end{pmatrix}} T^3 \xrightarrow{\begin{pmatrix} b & -\alpha & 0 \\ 0 & \beta & -a \end{pmatrix}} T^2 \longrightarrow 0 \right).$$

We get that

$$\operatorname{Ext}^2_T(A,T) \cong \frac{T^2}{\left\langle \binom{b}{0}, \binom{-\alpha}{\beta}, \binom{0}{-a} \right\rangle}.$$

By Nakayama's Lemma, $\beta_0^T(\operatorname{Ext}^2_T(A,T)) = 2$.

thm201119f THEOREM IV.E.2.22. *Assume S is a regular local ring or $k[\mathbf{X}]$ and let F be a minimal S-free resolution of R. If R is Cohen-Macaulay, then $\Sigma^p F^*$ is a minimal S-free resolution of $\operatorname{Ext}^p_S(R,S)$. In particular,*

$$\begin{aligned}\beta_0^S(\operatorname{Ext}^p_S(R,S)) &= \beta_p^S(R), \text{ and}\\ \beta_i^S(\operatorname{Ext}^p_S(R,S)) &= \beta_{p-i}^S(R).\end{aligned}$$

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IV.E.3. The Type of a Cohen-Macaulay Ring

Assume S is a regular local ring or a polynomial ring $k[\mathbf{X}]$ and $I \leq S$ be an ideal (which is homoegeneous in the standard graded case) and $R = S/I$.

disc201119g DISCUSSION IV.E.3.1. Let $I = \bigcap_{i=1}^{n} Q_i$ be an irredundant irreducible decomposition. Then n, the number of ideals in the previous decomposition, is a measure of the complexity of I or R. A problem is that n might depend on the decomposition. However, everything is okay when I is a monomial ideal in $S = k[\mathbf{X}]$.

prop201119h PROPOSITION IV.E.3.2. *Assume that R is artinian and $I = \bigcap_{i=1}^n Q_i$ is an irredundant irreducible decomposition. Then $n = \dim_k \operatorname{Hom}_R(k,R)$. Therefore n is independent of choice of decomposition. Moreover, for $p = \operatorname{pd}_S(R)$,*

$$n = \beta_p^S(R) = \beta_0^S(\operatorname{Ext}^p_S(R,S)).$$

ex201119i EXAMPLE IV.E.3.3. Let $S = k[X,Y]$. We consider the same examples as in Example IV.E.2.21.

(a) Let $I = \langle X^2, Y^2 \rangle$. Then $n = 1$.

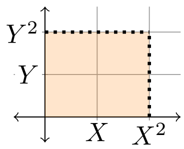
(b) Let $I = \langle X^2, XY, Y^2 \rangle = \langle X^2, Y \rangle \cap \langle X, Y^2 \rangle$. Then $n = 2$.

In each case, this agrees with the calculation of $\beta_p^S(R)$ and $\beta_0^S(\operatorname{Ext}^p_S(R,S))$ in Example IV.E.2.21.

(a) Check $\operatorname{Ext}^0_R(k,R) \overset{?}{\cong} k$. Alternately, we can check

$$k \overset{?}{\cong} \hom_R(k,R) = \operatorname{Hom}_R(R/\mathfrak{m}, R) \cong (0 :_R \mathfrak{m}) = \{r \in R \mid \mathfrak{m}r = 0\}.$$

A visualization of this is below:

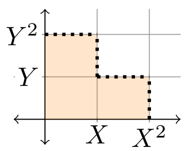


The region shaded in orange corresponds to R. We can find the elements annihilated by $\mathfrak{m}$ by looking at lattice points in the orange region which leave the region when multiplying by X or Y. Furthermore, the basis $\overline{XY} \in \mathbb{R}$ can be represented by the point $(1,1)$ in the graph above, and so R has vector space dimension 1.

(b) Check $\operatorname{Ext}^0_R(k,R) \overset{?}{\cong} k^2$.Alternately, we can check

$$k^2 \overset{?}{\cong} \hom_R(k,R).$$

A visualization of this is below:



The basis $\overline{X}, \overline{Y} \in \mathbb{R}$ can be represented by the points $(1,0)$ and $(0,1)$ in the graph above, and so R has vector space dimension 2.

thm201119j THEOREM IV.E.3.4. *Assume R is Cohen-Macaulay and $p = \operatorname{pd}_S(R)$. Set $\mu = \mu^p_R(R) = \dim_k(\operatorname{Ext}^p_R(k,R))$. Let $\mathbf{x} \in \mathfrak{m}_S$ be a maximal R-regular sequence (which is graded in the standard graded case). Set $J = I + \langle \mathbf{x} \rangle$, which satisfies $S/J \cong R/\langle \overline{\mathbf{x}} \rangle$ artinian. Then the number of ideals in an irredundant irreducible decomposition of J is independent of decomposition and independent of choice of $\mathbf{x}$, and equals*

$$\dim_k(\operatorname{Ext}^p_R(k,R)) = \beta^S_p(R) = \beta^S_0(\operatorname{Ext}^p_S(R,S)).$$

ex201119k EXAMPLE IV.E.3.5. Let $S = k[a,b,\alpha,\beta]$ and let $I = \langle a\alpha, ab, b\beta \rangle = \langle a,b \rangle \cap \langle a,\beta \rangle \cap \langle b,\alpha \rangle$ and let $R = S/I$. The maximal R-regular sequence in S is $a-\alpha, b-\beta$. Then

$$R/\left\langle \overline{a-\alpha}, \overline{b-\beta} \right\rangle \cong \frac{k[a,b,\alpha,\beta]}{\langle a\alpha, ab, b\beta, a-\alpha, b-\beta \rangle} \cong \frac{k[a,b]}{\langle a^2, ab, b^2 \rangle}.$$

In the last step, there are two irreducible factors, which is the same as $\beta^S_0(\operatorname{Ext}^2_S(R,S))$ by Example IV.E.2.21(c). Next, check $\operatorname{Ext}^2_R(k,R) \overset{?}{\cong} k^2$. Calculating Ext is messy, so we do this by showing $\operatorname{Ext}^2_R(k,R) \cong \operatorname{Hom}_{\overline{R}}(k,\overline{R})$, since we calculated $\operatorname{Hom}_{\overline{R}}(k,\overline{R}) \cong k^2$ in Example IV.E.3.3.

prop201119l PROPOSITION IV.E.3.6. *Let A be a non-zero, noetherian commutative ring with identity, and let M and N be finitely generated A-modules. Let $\mathbf{x} = x_1, \ldots, x_n \in A$ be an M-regular sequence such that $\mathbf{x}N = 0$. Set $\overline{A} = A/\langle \mathbf{x} \rangle$ and $\overline{M} = M/\langle \mathbf{x} \rangle$. Then*

$$\operatorname{Ext}^i_A(N,M) \cong \operatorname{Ext}^{i-n}_A(N,\overline{M})$$

for all $i \leq n$. In particular, if $i < n$, then $\operatorname{Ext}^i_A(N,M) = 0$, because $i - n < 0$ and

$$\operatorname{Ext}^n_A(N,M) \cong \operatorname{Ext}^0_A(N,\overline{M}) = \operatorname{Hom}_A(N,\overline{M}) = \operatorname{Hom}_{\overline{A}}(N,\overline{M}).$$

PROOF. We prove this by inducting on n. The base case is when $n = 1$. If x_1 is M-regular, then there is a short exact sequence

$$0 \longrightarrow M \xrightarrow{x_1 \cdot} M \longrightarrow \overline{M} \longrightarrow 0.$$

From this we obtain the following long exact sequence in Ext.

$$
\begin{aligned}
0 \longrightarrow \operatorname{Hom}_A(N,M) &\xrightarrow{x_1} \operatorname{Hom}_A(N,M) \longrightarrow \operatorname{Hom}_A(N,\overline{M}) \\
\longrightarrow \operatorname{Ext}^1_A(N,M) &\xrightarrow{x_1} \operatorname{Ext}^1_A(N,M) \longrightarrow \operatorname{Ext}^1_A(N,\overline{M}) \\
\longrightarrow \operatorname{Ext}^2_A(N,M) &\xrightarrow{x_1} \cdots
\end{aligned}
$$

Since x_1 annihilates N by assumption, the induced multiplication maps in the long exact sequence in the previous display are each the zero map. A short argument using the exactness of the sequence shows that $\operatorname{Hom}_A(N,M) = 0$ and yields the short exact sequences

$$0 \longrightarrow \operatorname{Ext}^i_A(N,M) \longrightarrow \operatorname{Ext}^i_A(N,\overline{M}) \longrightarrow \operatorname{Ext}^{i+1}_A(N,M) \longrightarrow 0$$

for every $i \geq 1$, and

$$0 \longrightarrow \operatorname{Hom}_A(N,\overline{M}) \xrightarrow[\therefore\cong]{} \operatorname{Ext}^1_A(N,M) \longrightarrow 0.$$

In the inductive step the argument in similar. □

NOTE. If $x \in A$ such that $xN = 0$, then $x \operatorname{Ext}^i_A(N,M) = 0$ for all i and for all M.

defn201114a DEFINITION IV.E.3.7. Assume R is Cohen-Macaulay and $\delta = \operatorname{depth}(R)$. Then the type of R (also known as the Cohen-Macaulay type) is

$$\operatorname{type}(R) = \mu^\delta_R(R) = \dim_k \operatorname{Ext}^\delta_R(k,R) = \dim_k \operatorname{Hom}_{\overline{R}}(k\overline{R}),$$

where $\overline{R} = R/\langle \mathbf{x} \rangle$ and $\mathbf{x}$ is a maximal R-regular sequence. More generally, for every $i \in \mathbb{Z}$ the i^{th} Bass number of R is

$$\mu^i_R(R) = \dim_k \operatorname{Ext}^i_R(k,R).$$

The Bass series of R (named after Hyman Bass) is

$$I_R(t) = \sum_{i=\delta}^{\infty} \mu^i_R(R) t^i.$$

ex201124b EXAMPLE IV.E.3.8. (a) By Example IV.E.3.3 we have

$$\operatorname{type}\left(\frac{k[X,Y]}{\langle X^2, Y^2 \rangle}\right) = 1.$$

(b) Also by Example IV.E.3.3 we have

$$\operatorname{type}\left(\frac{k[X,Y]}{\langle X^2, XY, Y^2 \rangle}\right) = 2.$$

(c) By Example IV.E.3.5 we have

$$\operatorname{type}\left(\frac{k[a,b,\alpha,\beta]}{\langle a\alpha, ab, b\beta \rangle}\right) = 2.$$

(d) More generally we have

$$\operatorname{type}\left(\frac{k[\mathbf{X},\mathbf{Y}]}{I(\Sigma G)}\right) = \text{number of minimal vertex covers of } G.$$

For instance, we obtain the conclusion of part (c) if we consider the graphs

$$G = \quad a \text{ ——— } b \qquad\qquad \Sigma G = \quad \alpha \text{ ——— } a \text{ ——— } b \text{ ——— } \beta,$$

since the two minimal vertex covers of G are $\{a\}$ and $\{b\}$.

PROOF. Set $R = k[\mathbf{X},\mathbf{Y}]/I(\Sigma G)$ and let $\overline{X_1 - Y_1}, \ldots, \overline{X_d - Y_d} \in R$. Then we have

$$\overline{R} = R/\left\langle \overline{X_1 - Y_1}, \ldots, \overline{X_d - Y_d} \right\rangle \cong \frac{k[\mathbf{X}]}{\langle X_1^2, \ldots, X_d^2 \rangle + I(G)}$$

and compute the irredundant irreducible decomposition

$$\begin{aligned}\left\langle X_1^2, \ldots, X_d^2 \right\rangle + I(G) &= \left\langle X_1^2, \ldots, X_d^2 \right\rangle + \bigcap_{\substack{V' \text{ a minimal} \\ \text{vertex cover}}} \langle V' \rangle \\ &= \bigcap_{\substack{V' \text{ a minimal} \\ \text{vertex cover}}} \left(\langle V' \rangle + \left\langle X_1^2, \ldots, X_d^2 \right\rangle \right). \end{aligned}$$

□

prop201124c

PROPOSITION IV.E.3.9. *Assume R is Cohen-Macaulay and let $\mathbf{y}$ be any R-regular sequence. Then*

$$\operatorname{type}(R/\langle \mathbf{y} \rangle) = \operatorname{type}(R).$$

PROOF. We may extend $\mathbf{y}$ to a maximal R-regular sequence $\mathbf{y}, \mathbf{z}$. Then by definition we have

$$\operatorname{type}(R) = \operatorname{type}(R/\langle \mathbf{y}, \mathbf{z} \rangle) = \operatorname{type}(R/\langle \mathbf{y} \rangle),$$

since $\mathbf{z}$ is a maximal $R/\langle \mathbf{y} \rangle$-regular sequence. □

prop201124d

PROPOSITION IV.E.3.10. *Assume R is local is Cohen-Macaulay. Then*

$$\operatorname{type}(\hat{R}) \overset{(a)}{=} \operatorname{type}(R) \overset{(b)}{=} \operatorname{type}(R[\![X_1, \ldots, X_e]\!]).$$

PROOF. (a) We present two proofs of this. First, since $\operatorname{Ext}^i_R(k, \hat{R}) \cong \operatorname{Ext}^i_R(k, R)$ for all $i \in \mathbb{Z}$ we have

$$\operatorname{type} \hat{R} = \dim_k \operatorname{Ext}^\delta_R(k, \hat{R}) = \dim_k \operatorname{Ext}^\delta_R(k, R) = \operatorname{type} R$$

as desired. Second, if $\mathbf{x}$ is a maximal R-regular sequence, then it is also an $\hat{R}$-regular sequence and

$$\hat{R}/\langle \mathbf{x} \rangle \cong \widehat{R/\langle \mathbf{x} \rangle} \cong R/\langle \mathbf{x} \rangle.$$

Thus

$$\operatorname{type} \hat{R} = \operatorname{type} \hat{R}/\langle \mathbf{x} \rangle = \operatorname{type} R/\langle \mathbf{x} \rangle = \operatorname{type} R.$$

(b) It is a fact that the sequence $\mathbf{X} = X_1, \ldots, X_e$ is $R[\![X_1, \ldots, X_e]\!]$-regular. Proposition IV.E.3.9 then implies

$$\operatorname{type}(R[\![X_1, \ldots, X_e]\!]) = \operatorname{type}(\underbrace{R[\![X_1, \ldots, X_e]\!]/\langle X_1, \ldots, X_e \rangle}_{\cong R}) = \operatorname{type} R.$$

□

prop201124e

PROPOSITION IV.E.3.11. *Assume R is standard graded and Cohen-Macaulay. Then*

$$\operatorname{type} R_{\mathfrak{m}} = \operatorname{type} R = \operatorname{type} R[X_1, \ldots, X_e].$$

PROOF. Note that $\operatorname{Ext}^i_{R_\mathfrak{m}}(k, R_\mathfrak{m}) \cong \operatorname{Ext}^i_R(k, R)$ and $X_1, \ldots, X_e$ is $R[X_1, \ldots, X_e]$-regular and then argue as in the proof of Proposition IV.E.3.10. □

cor201124f

COROLLARY IV.E.3.12. *If R is a geometric complete intersection, or a formal complete intersection, or a natural complete intersection, then* $\operatorname{type} R = 1$.

PROOF. We prove this in number of cases. In the first case, if R is a field, then $\operatorname{type} R = 1$ since $\operatorname{Hom}_k(k,k) \cong R^1$. In the second case, if R is a regular local ring, then a regular system of parameters $\mathbf{x}$ is an R-regular sequence such that $R/\langle \mathbf{x} \rangle$ is a field and therefore by the first case we have

$$\operatorname{type} R = \operatorname{type} R/\langle \mathbf{x} \rangle = 1.$$

In the third case, if R is a natural complete intersection, then $R \cong S/\langle \mathbf{y} \rangle$ where S is a regular local ring and $\mathbf{y}$ is an S-regular sequence. Therefore

$$\operatorname{type} R = \operatorname{type} S = 1,$$

where the first equality holds by Proposition IV.E.3.9 and the second equality holds by the second case. In the fourth case, if R is a formal complete intersection, then $\widehat{R}$ is a natural complete intersection and we have the following by Proposition IV.E.3.10 and the third case:

$$\operatorname{type} R = \operatorname{type} \widehat{R} = 1.$$

Finally, in the fifth case, if R is a geometric complete intersection, then by Proposition IV.E.3.11 and the third case we have

$$\operatorname{type} R = \operatorname{type} R_{\mathfrak{m}} = 1,$$

since $R_{\mathfrak{m}}$ is a natural complete intersection. □

disc201201a

DISCUSSION IV.E.3.13. What about localization? We do not expect $\operatorname{type} R_{\mathfrak{p}} = \operatorname{type} R$, but one might hope that $\operatorname{type} R_{\mathfrak{p}} \leq \operatorname{type} R$. One can try to show that

$$\dim_{k(\mathfrak{p})} \operatorname{Ext}^{\delta'}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), R_{\mathfrak{p}}) \leq \dim_k \operatorname{Ext}^{\delta}_R(k, R),$$

where δ and δ' are the depths of R and $R_{\mathfrak{p}}$, respectively, but this is difficult to do.

thm201201b

THEOREM IV.E.3.14. *Let $A \xrightarrow{\varphi} B$ be a flat, local homomorphism. Assume B is Cohen-Macaulay (i.e., A and $F = B/\mathfrak{m}_A B$ are Cohen-Macaulay). Then* $\operatorname{type} B = \operatorname{type} A \cdot \operatorname{type} F$ *and* $I_B(t) = I_A(t) \cdot I_F(t)$.

thm201201c

THEOREM IV.E.3.15. *Let $(T, \mathfrak{m}_T, k)$ be local and Cohen-Macaulay. Then for every ideal $\mathfrak{q} \in \operatorname{Spec}(T)$ we have*

$$\operatorname{type} T_{\mathfrak{q}} \leq \operatorname{type} T.$$

PROOF. We will first prove a special case. Suppose T is complete. Then the Cohen Structure Theorem implies $T = S/I$ for some regular local ring S. By the previous section we know $T_{\mathfrak{q}} = S_Q/I_Q$ is Cohen-Macaulay, and $p = \operatorname{pd}_S T = \operatorname{pd}_{S_Q} T_{\mathfrak{q}}$, and

$$\operatorname{type} T_{\mathfrak{q}} = \beta_p^{S_Q}(T_{\mathfrak{q}}) = \beta_p^{S_Q}(T_Q) \leq \beta_p^S(T) = \operatorname{type} T.$$

Alternatively, one also has

$$\begin{aligned}
\operatorname{type} T_{\mathfrak{q}} &= \beta_0^{S_Q}\left(\operatorname{Ext}^p_{S_Q}(T_Q, S_Q)\right) \\
&= \beta_0^{S_Q}\left(\operatorname{Ext}^p_{S_Q}(T_Q, S_Q)\right) \\
&= \beta_0^{S_Q}\left(\operatorname{Ext}^p_S(T, S)_Q\right) \\
&\leq \beta_0^{S}\left(\operatorname{Ext}^p_S(T, S)\right) \\
&= \operatorname{type} T.
\end{aligned}$$

(See the unnumbered Note below.) Now for the general case. Let T be local with prime ideal $\mathfrak{q} \in \operatorname{Spec}(T)$. Then there exists a prime ideal $Q \in \operatorname{Spec}(\hat{T})$ such that $\varphi(\mathfrak{q}) = Q$, where $\varphi : T \to \hat{T}$ is the typical map. We then have

$$\operatorname{type} T \overset{(1)}{=} \operatorname{type} \hat{T} \geq \operatorname{type} \hat{T}_Q \overset{(2)}{=} \operatorname{type} T_Q \cdot \operatorname{type}(F(Q)),$$

where $F(Q)$ is the closed fibre of φ_Q, i.e., $F(Q) = \hat{T}_Q/\mathfrak{q}\hat{T}_Q$. Equality (1) is a property of completions we established in a previous section. The inequality follows from the special case above. Equality (2) follows from Theorem IV.E.3.14, since $T_{\mathfrak{q}} \xrightarrow{\varphi_Q} \hat{T}_Q$ is flat and local. Since the completion $\hat{T}_Q$ is Cohen-Macaulay, so are $T_{\mathfrak{q}}$ and $F(Q)$, implying that $\operatorname{type} F(Q) \geq 1$. By the previous display the desired result then follows. □

NOTE. In general every generating sequence for a module M gives rise to a generating sequence for M_Q over S_Q, so $\beta_0^{S_Q}(M_Q) \leq \beta_0^S(M)$. Moreover, if L is a free resolution of M over S, then L_Q is a free resolution of M_Q over S_Q, and therefore $\beta_i^{S_Q}(M_Q) \leq \beta_i^S(M)$ and $\operatorname{pd}_{S_Q} M_Q \leq \operatorname{pd}_S M$.

ex201201d

EXAMPLE IV.E.3.16. Let K_d be the complete graph on d vertices. Then the ring $R = k[\mathbf{X}]/I(K_d)$ is Cohen-Macaulay of dimension one and $\operatorname{type} R = d - 1$. A maximal R-regular sequence is $X_d - (X_1 + \cdots + X_{d-1})$.

ex201201e

EXAMPLE IV.E.3.17. Let C_d be the d-cycle. Then the ring $R = k[\mathbf{X}]/I(C_d)$ is Cohen-Macaulay if and only if $d = 3$ or $d = 5$.

(a) If $d = 3$, then $C_3 = K_3$ and $\operatorname{type} R = \operatorname{type}(K[\mathbf{X}]/I(K_3)) = 3 - 1 = 2$.
(b) If $d = 5$, then $\operatorname{type} R = 1$. This takes work to show and we omit it here.

Exercises

Let k be a field. The point of the following four exercises is to give perspective on Theorem IV.E.2.9, which contains a homological characterization of the Cohen-Macaulay property.

exr210722ad

EXERCISE IV.E.3.18. Set $S = k[X]$ and $R = S/\langle X^a \rangle$ for a fixed integer $a \geq 2$.

(a) Is R Cohen-Macaulay or not?
(b) Verify directly, without invoking Theorem IV.E.2.9, that $\operatorname{Ext}^i_S(R, S) = 0$ for all $i \neq 1$ and $\operatorname{Ext}^1_S(R, S) \cong R$.

exr210722ae

EXERCISE IV.E.3.19. Set $S = k[X, Y]$ and $R = S/\langle X^a, Y^b \rangle$ for fixed integers $a, b \geq 2$.

(a) Is R Cohen-Macaulay or not?
(b) Verify directly, without invoking Theorem IV.E.2.9, that $\operatorname{Ext}^i_S(R, S) = 0$ for all $i \neq 2$ and $\operatorname{Ext}^2_S(R, S) \cong R$.

exr210722af

EXERCISE IV.E.3.20. Set $S = k[X, Y]$ and $R = S/\langle X^a, XY, Y^b \rangle$ for fixed integers $a, b \geq 2$.

(a) Is R Cohen-Macaulay or not?
(b) Verify directly, without invoking Theorem IV.E.2.9, that $\operatorname{Ext}^i_S(R, S) = 0$ for all $i \neq 2$ and $\operatorname{Ext}^2_S(R, S) \neq 0$.
(c) Prove or disprove: $\operatorname{Ext}^2_S(R, S) \cong R$.

exr210722ag

EXERCISE IV.E.3.21. Set $S = k[X, Y]$ and $R = S/\langle X^2, XY \rangle$.

(a) Is R Cohen-Macaulay or not?
(b) Verify directly, without invoking Theorem IV.E.2.9, that $\operatorname{Ext}^i_S(R, S) = 0$ for all $i \neq 1, 2$ and $\operatorname{Ext}^1_S(R, S), \operatorname{Ext}^2_S(R, S) \neq 0$.

CHAPTER IV.F

Gorenstein Rings

pter093021an

As usual, let S be local or standard graded, let $I \lneq S$ be a proper ideal (and homogeneous in the standard graded case), and set $R = S/I$. [SEAN: this page looks funny to me. Do you want this moved into the `\chapter` environment instead?

ction063021ae

IV.F.1. Foundational Properties

defn201201f

DEFINITION IV.F.1.1. A ring A is Gorenstein if it is Cohen-Macaulay of type 1, where A is local or standard graded.

ex201201g

EXAMPLE IV.F.1.2. (a) $k[X,Y]/\left\langle X^2, Y^2\right\rangle$ is Gorenstein.
(b) The ring $k[X,Y]/\left\langle X^2, XY, Y^2\right\rangle$ is not Gorenstein, because its type is 2.
(c) $k[a,b,\alpha,\beta]/\langle a\alpha, ab, b\beta\rangle$ is likewise not Gorenstein, since its type is 2.
(d) The ring $k[\mathbf{X},\mathbf{Y}]/I(\Sigma G)$ is Gorenstein if and only if the number of minimal vertex covers of G is 1.
(e) The ring $k[\mathbf{X}]/I(K_d)$ is Gorenstein if and only if $d-1=1$ if and only if $d=2$.
(f) The ring $k[\mathbf{X}]/I(C_d)$ is Gorenstein if and only if $d=5$.

prop201201h

PROPOSITION IV.F.1.3. *In the local setting, we have the following implications.*

$$\begin{tikzcd} \text{FIELD} \arrow[r, Rightarrow] & \text{RLR} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{NATURAL CI} \arrow[r, Rightarrow] & \text{FORMAL CI} \arrow[r, Rightarrow, "IV.E.3.12"'] & \text{GORENSTEIN} \arrow[r, Rightarrow, "def'n"'] & \text{CM} \\ & \text{UFD} \end{tikzcd}$$

ex201203a

EXAMPLE IV.F.1.4. (a) A ring being Cohen-Macaulay does not imply that the ring is Gorenstein, by Example IV.F.1.2.
(b) A ring being Gorenstein does not imply that the ring is a formal complete intersection. Consider $k[X_1,\ldots,X_5]/I(C_5)$, which is Gorenstein by Example IV.F.1.2. But $I(C_5)$ is minimally generated by five monomials and $\dim(R)=2$. If R were a formal complete intersection , then $\dim(R)$ is the number of variables minus the number of generators, so we would have $\dim(R) = 5-5 = 0$. So R is not a formal complete intersection.

prop201203b

PROPOSITION IV.F.1.5. *Let A be local or standard graded and let $\mathbf{x} = x_1,\ldots,x_n \in \mathfrak{m}_A$ be an A-regular sequence (which is homogeneous in standard graded case). Set $\overline{A} = A/\langle \mathbf{x}\rangle$. Then $\overline{A}$ is Gorenstein if and only if A is Gorenstein.*

PROOF. We have that $\overline{A}$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and that $\operatorname{type}(\overline{A}) = \operatorname{type}(A)$. $\square$

prop201203c

PROPOSITION IV.F.1.6. *Let A be local. Then A is Gorenstein if and only if $\widehat{A}$ is Gorenstein if and only if $A[\![\mathbf{X}]\!]$ is Gorenstein for some (or all) $\mathbf{X} = X_1,\ldots,X_d$.*

PROOF. We have that A is Cohen-Macaulay if and only if $\widehat{A}$ is Cohen-Macaulay if and only if $A[\![\mathbf{X}]\!]$ is Cohen Macaulay. Also, we have that $\operatorname{type}(A) = \operatorname{type}(\widehat{A}) = \operatorname{type}(A[\![\mathbf{X}]\!])$. $\square$

prop201203d

PROPOSITION IV.F.1.7. *Let A be standard graded. Then A is Gorenstein if and only if $A_{\mathfrak{m}_A}$ is Gorenstein if and only if $A[\mathbf{X}]$ is Gorenstein for some (or all) $\mathbf{X} = X_1,\ldots,X_d$.*

prop201203e

PROPOSITION IV.F.1.8. *If A is Gorenstein and local, and $\mathfrak{p} \in \operatorname{Spec}(A)$, then $A_{\mathfrak{p}}$ is Gorenstein.*

PROOF. If A is Cohen-Macaulay, then $A_{\mathfrak{p}}$ is Cohen-Macaulay. Also,

$$1 = \operatorname{type}(A) \geq \operatorname{type}(A_{\mathfrak{p}}) \geq 1,$$

so $\operatorname{type}(A_{\mathfrak{p}}) = 1$. $\square$

ex201203f EXAMPLE IV.F.1.9. The converse of Proposition IV.F.1.8 fails, using Exercise IV.C.2.20. There is a ring A which is not Cohen-Macaulay but $A_{\mathfrak{p}}$ is a field.

IV.F.2. Homological Properties

tion063021af

disc201203g DISCUSSION IV.F.2.1. A is an A-module and it is free. In particular, A is projective and flat. The question that follows is to ask if A is always injective as an A-module? The answer is that this is almost never true.

fact201213h FACT IV.F.2.2. *If A is injective as an A-module, then A is artinian. Moreover, if A has a finitely generated non-zero injective module, then A is artinian.*

disc201203i DISCUSSION IV.F.2.3. This is frustrating, because in linear algebra, a very important tool is the vector space dual: $V \mapsto V^* = \operatorname{Hom}_k(V,k)$. This is nice because it is exact, i.e., k is injective as a k-module. How can we make $\operatorname{Hom}_A(-,A)$ useful if A is not injective as an A-module? Most modules are not projective, but if we approximate a module by projective modules, this leads to projective resolutions and projective dimension. We are going to do an injective version of this.

fact201203j FACT IV.F.2.4. *For every A-module M, there exists an A-module monomorphism $M \overset{i}{\hookrightarrow} I$ such that I is injective. This implies that $M \cong \operatorname{Im}(i) \subseteq I$, so M is (isomorphic to) a submodule of an injective module.*

SKETCH OF PROOF. Proof by wishful thinking. Step 1 is to prove the result for $A = \mathbb{Z}$. Step 2 is to use the result of $\mathbb{Z}$ to get the result for A in general.
Step 1: A $\mathbb{Z}$-module N is injective if and only if it is divisible, i.e., for every $x \in N$ and for every $n \in \mathbb{Z}\setminus\{0\}$, there exists a $y \in N$ such that $ny = x$. We then get injectivity by looking at the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "n"] \arrow[d, "x"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z} \arrow[dl, dashed, "\exists y"] \\ & N & \end{tikzcd}$$

and using Baer's criterion. In general, injective implies divisible and the converse holds if A is a PID. For example, $\mathbb{Q}$ is divisible as a $\mathbb{Z}$-module. Furthermore, every quotient of a divisible module is divisible. In particular, $\mathbb{Q}/\mathbb{Z}$ is divisible, so it is injective over $\mathbb{Z}$. How can we show that every $\mathbb{Z}$-module is a submodule of an injective $\mathbb{Z}$-module? Use the Pontryagin dual

$$(-)^V = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}).$$

Then we take a dual twice to get the following maps:

$$M^V \overset{\tau}{\twoheadleftarrow} \mathbb{Z}^{(A)}$$

$$M \overset{\delta}{\hookrightarrow} M^{VV} \hookrightarrow (\mathbb{Z}^{(A)})V.$$

The δ map is defined by $\delta(m) = [f \mapsto f(m)]$, and so is injective. The final module is injective because

$$(\mathbb{Z}^{(A)})V = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}(A), \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})\widehat{\ } \cong (\mathbb{Q}/\mathbb{Z})\widehat{\ }.$$

The our conclusion is that $M \hookrightarrow I = (\mathbb{Z}^{(A)})^V$.
Step 2: There is a natural ring homomorphism $\psi : \mathbb{Z} \to A$. Therefore every A-module is a $\mathbb{Z}$-module by restriction of scalars. Let M be an A-module. Then M is a $\mathbb{Z}$-module, so there is a $\mathbb{Z}$-linear monomorphism $M \overset{\alpha}{\hookrightarrow} J$ such that J is injective over $\mathbb{Z}$. The goal is to show that M is a submodule of an injective A-module. So consider taking Hom of α and using Hom cancellation on the left to get

$$\begin{tikzcd} \operatorname{Hom}_A(A,M) & \subseteq & \operatorname{Hom}_{\mathbb{Z}}(A,M) \arrow[r, hook] & \operatorname{Hom}_{\mathbb{Z}}(A,J), \\ M \arrow[u, "\cong"] \arrow[urrr, hook] & & & \end{tikzcd}$$

where the constructed map needs to respect the A-linear structure. We can see that $\operatorname{Hom}_{\mathbb{Z}}(A,J)$ is injective over A using Hom-tensor adjointness, because

$$\operatorname{Hom}_A(-, \operatorname{Hom}_{\mathbb{Z}}(A,J)) \cong \operatorname{Hom}_{\mathbb{Z}}(A \otimes_A -, J) \cong \operatorname{Hom}_{\mathbb{Z}}(-, J),$$

which is exact. More generally, if $B \to C$ is a ring homomorphism and L is injective as a B-module, then $\mathrm{Hom}_B(C, L)$ is injective as a C-module. As another note, we have seen a map similar to $\mathrm{Hom}_{\mathbb{Z}}(A, -)$ with $\mathbb{Z} \to A$ before, in the map $\mathrm{Hom}_S(R, S)$ with $S \to R$. We used the latter map to detect Cohen-Macaulayness using $\mathrm{Ext}^i_S(R, S)$. $\square$

thm201209a THEOREM IV.F.2.5. *For every A-module M, there exists an exact sequence*

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

such that each I^j is injective over A.

PROOF. One can construct the following diagram from left-to-right where each diagonal sequence is exact.

$$\begin{tikzcd}
 & & & & 0 \arrow[dr] & & 0 & & & & 0 \\
 & 0 \arrow[dr] & & & & M' \arrow[ur] \arrow[dr] \arrow[d, phantom, "\circlearrowright"] & & & & M''' \arrow[ur] & \\
0 \arrow[rr] & & M \arrow[rr] \arrow[dr, "="'] & & I^0 \arrow[rr] \arrow[ur] & & I^1 \arrow[rr] \arrow[dr] & & I^2 \arrow[rr] \arrow[ur] & & \cdots \\
 & & & M \arrow[ur] \arrow[dr] \arrow[u, phantom, "\circlearrowright"] & & & & M'' \arrow[ur] \arrow[dr] \arrow[u, phantom, "\circlearrowright"] & & & \\
 & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

It follows that the horizontal sequence is also exact. $\square$

defn201209b DEFINITION IV.F.2.6. An augmented injective resolution of M is an exact sequence

$$^{+}I = \qquad 0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

such that each I^j is injective. The corresponding (truncated) injective resolution is the A-complex

$$I = \qquad 0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots,$$

where I is exact at every position except at I^0 where the homology is isomorphic to M. If M has an injective resolution I such that $I^j = 0$ for all sufficiently large j, then M has finite injective dimension and we write $\mathrm{id}_A M < \infty$. The injective dimension of M is the length of the shortest such injective resolution, i.e.,

$$\mathrm{id}_A M = \inf \left\{ \ell \;\middle|\; \exists \text{ an injective resolution of } M \text{ s.t. } I^j = 0,\ \forall j > \ell \right\}.$$

We say that $\mathrm{id}_A(M)$ measures how far M is from being injective.

ex201209c EXAMPLE IV.F.2.7. An injective resolution of $\mathbb{Z}$ over $\mathbb{Z}$ is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0. \\ & & & & \| & & \| & & \\ & & & & I^0 & & I^1 & & \end{array}$$

Therefore $\mathrm{id}_{\mathbb{Z}} \mathbb{Z} \leq 1$ and in fact $\mathrm{id}_{\mathbb{Z}} \mathbb{Z} = 1$, because $\mathbb{Z}$ is not divisible and therefore not injective, so $\mathrm{id}_{\mathbb{Z}} > 0$. (Note that $\mathrm{id}_A M = 0$ if and only if M is injective over A.)

thm201209d THEOREM IV.F.2.8. (a) *One can compute $\mathrm{Ext}^i_A(N, M)$ using an injective resolution of M:*

$$\mathrm{Ext}^i_A(N, M) = H^i(\mathrm{Hom}_A(N, I)).$$

thm201209d.b (b) *The following are equivalent.*

m201209d.b.i (i) *M is injective over A.*

n201209d.b.ii (ii) *$\mathrm{Ext}^i_A(N, M) = 0$ for all $i \geq 1$ and for all A-modules N.*

201209d.b.iii (iii) *$\mathrm{Ext}^1_A(N, M) = 0$ for all finitely generated A-modules N.*

n201209d.b.iv (iv) [Baer] $\mathrm{Ext}^1_A(A/\mathfrak{a}, M) = 0$ for all ideals $\mathfrak{a} \leq A$.

m201209d.b.v (v) $\mathrm{id}_A M = 0$

thm201209d.c (c) *Let $n \geq 0$ be given. The following are equivalent.*

m201209d.c.i (i) $\mathrm{id}_A M \le n$
201209d.c.ii (ii) $\mathrm{Ext}^i_A(N, M) = 0$ *for all* $i > n$ *and for all* A*-modules* N.
201209d.c.iii (iii) $\mathrm{Ext}^{n+1}_A(N, M) = 0$ *for all finitely generated* A*-modules* N.
201209d.c.iv (iv) $\mathrm{Ext}^{n+1}_A(A/\mathfrak{a}, M) = 0$ *for all ideals* $\mathfrak{a} \le A$.
thm201209d.d (d) [Bass] If A is local or standard graded and $M \neq 0$ is a finitely generated (graded) A-module with finite injective dimension, then $\mathrm{id}_A M = \mathrm{depth}(A)$.
thm201209d.e (e) If A is local or standard graded and there exists a finitely generated (graded) A-module $M \neq 0$ with finite injective dimension, the A is Cohen-Macaulay.

thm201209e THEOREM IV.F.2.9 (Auslander-Buchsbaum, Serre). *Assume A is local. The following are equivalent.*
thm201209e.i (i) *A is a regular local ring.*
hm201209e.ii (ii) *Every A-module has finite injective dimension over A.*
m201209e.iii (iii) *Every finitely generated A-module has finite injective dimension over A.*
hm201209e.iv (iv) *The residue field has finite injective dimension over A.*

ex201209f EXAMPLE IV.F.2.10. (a) Set $R = k[X]/\langle X^2\rangle$. Then R is injective over R, but $\mathrm{id}_R k = \infty$. To show that R is injective, we need to show that $\mathrm{Ext}^1_R(N, R) = 0$ for all finitely generated N. Notice that any such N is a finite dimensional vector space over $k \subseteq R$. We prove $\mathrm{Ext}^1_R(N, R) = 0$ by induction on $\ell = \dim_k N$. As a base case assume $\ell = 1$, i.e., $N \cong k$. Then a projective resolution of k is

$$P = \qquad \cdots \xrightarrow{X\cdot} R \xrightarrow{X\cdot} R \xrightarrow{X\cdot} R \xrightarrow{X\cdot} 0$$

and we obtain

$$\mathrm{Hom}_R(P, R) \cong \qquad 0 \longrightarrow R \xrightarrow{X\cdot} R \xrightarrow{X\cdot} R \xrightarrow{X\cdot} \cdots .$$

Thus we compute

$$\mathrm{Ext}^1_R(k, R) = H\left(R \xrightarrow{X\cdot} R \xrightarrow{X\cdot} R \right) = 0,$$

because of the exactness of the sequence at the relevant position. For the inductive step assume $\ell \ge 2$. By Nakayama's Lemma (Fact IV.A.2.4(d)), $N/\mathfrak{m}N \neq 0$ is a finite dimensional vector space over $k = A/\mathfrak{m}$, i.e., $N/\mathfrak{m}N \cong k^t$ for some $t \ge 1$, so we can surject $k^t \twoheadrightarrow k$. Therefore there exists an exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow k \longrightarrow 0,$$

which "splits". Thus we have $N \cong N' \oplus k$ as a k-vector space, so $\dim_k N' = \dim_k N - 1$ and therefore $\mathrm{Ext}^1_R(N', R) = 0$ by the inductive hypothesis. Considering the long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_R(k, R) \longrightarrow \mathrm{Hom}_R(N, R) \longrightarrow \mathrm{Hom}_R(N', R) \\ \longrightarrow \underbrace{\cancel{\mathrm{Ext}^1_R(k, R)}}_{=0} \longrightarrow \underbrace{\cancel{\mathrm{Ext}^1_R(N, R)}}_{\therefore=0} \longrightarrow \underbrace{\cancel{\mathrm{Ext}^1_R(N', R)}}_{=0} \end{array}$$

we conclude as desired. What about the infinite projective dimension? We present two proofs of the fact that $\mathrm{id}_R k = \infty$. First, if $\mathrm{id}_R k < \infty$, then Auslander-Buchsbaum, Serre implies R is a regular local ring, but $\mathrm{edim}\, R = 1 > 0 = \dim R$, a contradiction. Second, we can show that $\mathrm{Ext}^i_R(k, k) \neq 0$ for all i. We know $\mathrm{Ext}^i_R(k, k) = H^i(\mathrm{Hom}_R(P, k))$ and

$$\mathrm{Hom}_R(P, k) = \qquad 0 \longrightarrow k \xrightarrow[=0]{X\cdot} k \xrightarrow[=0]{X\cdot} k \xrightarrow[=0]{X\cdot} \cdots ,$$

where the multiplication map is the zero map because $k = R/\langle X\rangle$. Thus we have $\mathrm{Ext}^i_R(k, k) \cong k \neq 0$ for all i, as desired.

(b) Set $R = k[X, Y]/\langle X^2, XY, Y^2\rangle$. Then R is not injective over R and $\mathrm{id}_R R = \infty$. To justify this it suffices to show that $\mathrm{Ext}^i_R(k, R) \neq 0$ for all $i \ge 0$. We have the short exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow R \longrightarrow k \longrightarrow 0$$

which can be rewritten as

$$0 \longrightarrow k^2 \longrightarrow R \longrightarrow k \longrightarrow 0,$$

because $\mathfrak{m} \cong k^2$ as an R-module. We also have a minimal free resolution of k:

$$\cdots \longrightarrow R^{16} \longrightarrow R^8 \xrightarrow[\left(\begin{smallmatrix}\partial_2 & 0\\ 0 & \partial_2\end{smallmatrix}\right)]{\partial_3} R^4 \xrightarrow[\left(\begin{smallmatrix}\partial_1 & 0\\ 0 & \partial_1\end{smallmatrix}\right)]{\partial_2} R^2 \xrightarrow[(\,X\ Y\,)]{\partial_1} R \longrightarrow 0.$$

The long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_R(k,R) \arrow[r] & \operatorname{Hom}_R(R,R) \arrow[r] & \operatorname{Hom}_R(k^2,R) \arrow[dll, rounded corners] \\ & \operatorname{Ext}^1_R(k,R) \arrow[r] & \underbrace{\cancel{\operatorname{Ext}^1_R(R,R)}}_{=0} & \end{tikzcd}$$

can then be rewritten as

$$0 \longrightarrow k^2 \longrightarrow \underset{\substack{\wr\| \\ k^3}}{R} \longrightarrow k^4 \longrightarrow \operatorname{Ext}^1_R(k,R) \longrightarrow 0,$$

since $\operatorname{Hom}_R(k,R) \cong k^2$ and $\operatorname{Hom}_R(k^2,R) \cong k^4$ (and by Hom-cancellation). Thus the powers in a resolution of $\operatorname{Ext}^1_R(k,R)$ are 2, 3, and 4, so we compute

$$\dim_k \operatorname{Ext}^1_R(k,R) = 4 - 3 + 2 = 3$$

and it follows that $\operatorname{Ext}^1_R(k,R) \cong k^3 \neq 0$. Continuing the long exact sequence from above we have

$$\begin{tikzcd} & \overbrace{\cancel{\operatorname{Ext}^1_R(R,R)}}^{=0} \arrow[r] & \operatorname{Ext}^1_R(k^2,R) \arrow[dll, rounded corners] \\ \operatorname{Ext}^2_R(k,R) \arrow[r] & \underbrace{\cancel{\operatorname{Ext}^2_R(R,R)}}_{=0}. & \end{tikzcd}$$

Thus the snaking homomorphism om the previous display is an isomorphism and we have

$$\operatorname{Ext}^2_R(k,R) \cong \operatorname{Ext}^1_R(k^2,R) \cong \operatorname{Ext}^1_R(k,R)^2 \cong (k^3)^2 \cong k^6 \neq 0.$$

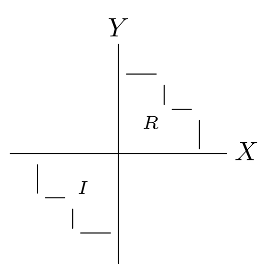
One can compute similarly

$$\begin{aligned} \operatorname{Ext}^i_R(k,R) &\cong \operatorname{Ext}^{i-1}_R(k^2,R) \cong \operatorname{Ext}^{i-1}_R(k,R)^2 \\ \operatorname{Ext}^3_R(k,R) &\cong (k^6)^2 \\ \operatorname{Ext}^4_R(k,R) &\cong (k^12)^2, \end{aligned}$$

i.e.,

$$\dim_k \operatorname{Ext}^n_R(k,R) = 3 \cdot 2^{n-1} \neq 0$$

for all $n \geq 1$. So R is not injective over R. Nonetheless, one can find a nice injective R-module using a staircase diagram.



I has two generators (i.e., $\beta_0^R(I) = 2 = \operatorname{type} R$) corresponding to the two downward-pointing corners in the interior of Quadrant III, its vector space dimension is three, and it is an injective R-module. In general, for any monomial ideal that determines an artinian ring R, one can always find an injective module I whose minimal number of generators is equal to the type of R.

thm201209g

THEOREM IV.F.2.11. *Let A be local or standard graded. Then A is Gorenstein if and only if* $\operatorname{id}_A(A) < \infty$.

How can we get $\mathrm{Hom}_A(-, A)$ to be nice? One way is to get A to be injective. Another way is if $\mathrm{id}_A(A) < \infty$. These conditions are both very restrictive.

ex201209h EXAMPLE IV.F.2.12. (a) Let $R = k[X]/\langle X^2 \rangle$, so $\mathrm{id}_R(R) = 0$. R is Gorenstein (and is Cohen-Macaulay type 1).
(b) Let $R = k[X,Y]/\langle X^2, XY, Y^2 \rangle$, so $\mathrm{id}_R(R) = \infty$. R is not Gorenstein (and is Cohen-Macaulay type 2).

IV.F.3. Dualizing Modules

tion063021ag disc201209i DISCUSSION IV.F.3.1. $\mathrm{Hom}_A(-, A)$ is nice when $\mathrm{id}_A(A) < \infty$, i.e., when A is Gorenstein. Can we get a nice duality when A is not Gorenstein? Not generally, but we can if we try to get a nice $\mathrm{Hom}_A(-, D)$.

disc201209j DISCUSSION IV.F.3.2. What properties make $\mathrm{Hom}_A(-, D)$ particularly nice?

(1) We want some form of exactness, either D to be injective or $\mathrm{id}_A(D) < \infty$.
(2) We want $\mathrm{Hom}_A(\text{f.g.}, D)$ to be finitely generated. The best way to guarantee that is to let D be finitely generated. Note these two properties means that D is Cohen-Macaulay.
(3) We want $\mathrm{Hom}_A(-, D)$ to not lose any information.

defn201209k DEFINITION IV.F.3.3. An A-module D is a dualizing A-module if it satisfies the following conditions.

(1) $\mathrm{id}_A(D) < \infty$.
(2) D is finitely generated.
(3) The natural map

$$x : A \xrightarrow{\cong} \mathrm{Hom}_A(D, D)$$

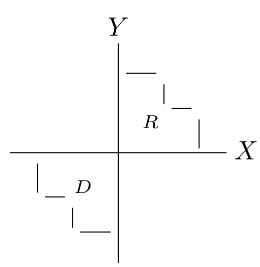
$$r \longmapsto [D \xrightarrow{r} D]$$

is an isomorphism, and $\mathrm{Ext}^i_A(D, D) = 0$ for all $i \geq 1$. We call x the homothety map.

ex20120912a EXAMPLE IV.F.3.4. (a) A is Gorenstein if and only if A is a dualizing module.
(2) A is finitely generated over A.
(3) We have $A \xrightarrow{\cong} \mathrm{Hom}_A(A, A)$ and $\mathrm{Ext}^i_A(A, A) = 0$ because A is projective.
(1) $\mathrm{id}_A(A) < \infty$ if and only if A is Gorenstein.

ex2012091.b (b) A^n is a dualizing module if and only if $n = 1$ and A is Gorenstein. This is because $\mathrm{Hom}_A(A^n, A^n) \cong A^{n^2} \overset{?}{\cong} A$, which occurs only when $n^2 = 1$.

ex2012091.c (c) Let $R = k[X,Y]/\langle X^2, XY, Y^2 \rangle$, which has the following staircase diagram.

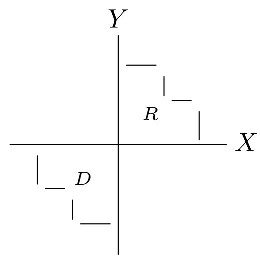


Then D is a dualizing R-module.

ex2012091.d (d) If we have any artinian monomial ideal, we can follow a similar process to part (c) to find a dualizing module. This dualizing module is also often called a canonical module (Grothendieck).

thm201209m THEOREM IV.F.3.5. *Assume $R = S/I$. If R is Cohen-Macaulay, then $D = \mathrm{Ext}^p_S(R, S)$ is a dualizing R-module with $p = \mathrm{pd}_S(R)$.*

ex201209n EXAMPLE IV.F.3.6. Consider the following staircase diagram for R.



We can count type$(R) = 2$ by counting the two corners in Quadrant I of the diagram, and we can count $\beta_0^R(D) = 2$ by counting the two corners in Quadrant III of the diagram.

thm201209o THEOREM IV.F.3.7 (Grothendieck, Sharp, Foxby, Reiten). (a) *Suppose A is local. Then there exists a dualizing A-module if and only if A is Cohen-Macaulay and $A \cong B/J$, where B is Gorenstein.*

(b) *Suppose A is standard graded. Then there exists a dualizing A-module if and only if A is Cohen-Macaulay.*

SKETCHY SKETCH OF PROOF. (a) For the reverse direction, let $A \cong B/J$ satisfying B Gorenstein and A Cohen-Macaulay. Then $\operatorname{Ext}_B^i(A, B) = 0$ for all $i \neq p$, so $D = \operatorname{Ext}_B^p(A, B)$ is a dualizing A-module, where $p = \operatorname{depth}(B) - \operatorname{depth}(A)$, because

$$\begin{aligned} \operatorname{Hom}_A(\operatorname{Hom}_B(A, B), \operatorname{Hom}_B(A, B)) &\cong \operatorname{Hom}_B(A \otimes_A (\operatorname{Hom}_B(A, B), B) \\ &\cong \operatorname{Hom}_B(\operatorname{Hom}_B(A, B), B) \approx A. \end{aligned}$$

For the forward direction, assume D is a dualizing A-module, so D is (non-zero) finitely generated and $\operatorname{id}_A(D) < \infty$. Therefore, A is Cohen-Macaulay. How do we find B? We use a construction of Nagata, called the idealization or trivial extension. We use

$$B = A \ltimes D = A \oplus D$$

as an A-module using the multiplication map

$$(a, d)(\alpha, \delta) = (a\alpha, a\delta + \alpha d).$$

We need to check that $A \ltimes D$ is a non-zero commutative ring with identity, so

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "1_A"'] & A \ltimes D = B \arrow[d, two heads, "\tau"] \arrow[dl, phantom, "\circlearrowleft"] \\ & A \end{tikzcd}$$

and the natural maps are ring homomorphisms. Then $A \cong B/\operatorname{Ker}(\tau)$, but $\ker(\tau) \cong D$.

$\square$

cor201209p COROLLARY IV.F.3.8. *$R = S/I$ has a dualizing module if and only if R is Cohen-Macaulay.*

cor201209q COROLLARY IV.F.3.9. *If A is local and Cohen-Macualay, then $\widehat{A}$ has a dualizing module.*

IV.F.4. Dualizing Complexes (Grothendieck and ~~Hartshorne~~)

ction063021ah

disc201209r DISCUSSION IV.F.4.1. How do we get a nice duality without assuming Cohen-Macaulayness? We use the derived category.

defn201209s DEFINITION IV.F.4.2. An A-complex D is a <u>dualizing A-complex</u> if it satisfies the following conditions.

(1) D is homologically finite, i.e.,

$$\bigoplus_{i \in \mathbb{Z}} H_i(D)$$

is a finitely generated A-module, i.e., $H_i(D)$ is finitely generated over A for all i and $H_i(D) = 0$ for all $|i| \gg 0$.

(2) $\operatorname{id}_A(D) < \infty$, i.e., there is a semiinjective resolution $D \xrightarrow{\simeq} I$ such that $I_j = 0$ for all $|j| \gg 0$, i.e., D has a bounded semiinjective resolution.

(3) The natural map

$$A \xrightarrow[\cong]{x} R\operatorname{Hom}_A(D, D)$$

is an isomorphism in the derived category, i.e.,

$$A \xrightarrow[\simeq]{x} \operatorname{Hom}_A(I, I)$$

is a quasiisomorphism.

ex201209t EXAMPLE IV.F.4.3. Every dualizing module is a dualizing complex. Morevoer, a dualizing module is a dualizing complex satisfying $H_i = 0$ for all $i \neq 0$.

thm201209u THEOREM IV.F.4.4. *A has a dualizing complex if and only if $A \cong B/J$ such that B is Gorenstein.*

ex201209v EXAMPLE IV.F.4.5. Let $R = k[X, Y]/\left\langle X^2, XY\right\rangle$. Then R is not Cohen-Macaulay, but there exists a dualizing R-complex.

cor201209w COROLLARY IV.F.4.6. *$R = S/I$ has a dualizing complex $D = R\operatorname{Hom}_S(R, S)$.*

cor201209x COROLLARY IV.F.4.7. *If A is local, then $\widehat{A}$ has a dualizing complex.*

Outro.

(1) There is a DG version of dualizing complexes.
(2) Some take-away points from this class:
  - Some rings are nicer than others.
  - Niceness is in the eye of the beholder.
  - Combinatorial constructions allow us to see the niceness sometimes.

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