Path Ideals for Weighted Graphs

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and Chelsey Paulsen (J. Algebra Appl., **12** (2013), no. 5)

Edge Ideals

Assumption

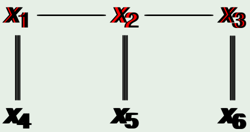
k is a field, $S = k[x_1, \dots, x_d]$, and $G = (V, E)$ is a (finite simple) graph with $V = \{x_1, \dots, x_d\}$.

Definition (Villareal '90)

The edge ideal $I(G) \subseteq S$ of G is $I(G) = (x_i x_j \in E)S$.

Example

$G =$



$$I(G) = (x_1x_2, x_2x_3, x_1x_4, x_2x_5, x_3x_6)S.$$

Irreducible Decompositions of Edge Ideals

Definition

A vertex cover of G is a subset $W \subseteq V$ such that for every $x_i x_j \in E$, either $x_i \in W$ or $x_j \in W$.

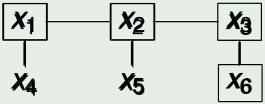
Fact

We have (irredundant) irreducible decompositions

$$I(G) = \bigcap_{W} (W)S = \bigcap_{W \text{ min}} (W)S.$$

Example

$G =$



$$I(G) = (x_1, x_2, x_3)S \cap (x_1, x_2, x_6)S \cap (x_1, x_3, x_5)S$$
$$\cap (x_2, x_3, x_4)S \cap (x_2, x_4, x_6)S$$

Cohen-Macaulayness of Trees

Theorem (Villareal 1990)

If T is a tree, then $S/I(T)$ is Cohen-Macaulay if and only if $I(T)$ is unmixed, if and only if T is a suspension of a tree. (Hence, Cohen-Macaulayness of $S/I(T)$ is characteristic-independent.)

Example

$$T = \quad \begin{array}{ccccc} x_1 & \text{———} & x_2 & \text{———} & x_3 \\ | & & | & & | \\ x_4 & & x_5 & & x_6 \end{array}$$

$S/I(T)$ is Cohen-Macaulay.

Weighted Edge Ideals

Definition

A weight function on G is a function $\omega\colon E \to \mathbb{N}$.
A weighted graph G_ω is a graph G, with a weight function ω.

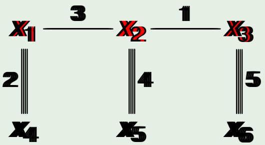
Definition (Paulsen-SW ’13)

The weighted edge ideal $I(G_\omega) \subseteq S$ of a weighted graph G_ω is

$$I(G_\omega) = (x_i^{\omega(e)} x_j^{\omega(e)} \mid e = x_i x_j \in E)S.$$

Example

$G_\omega =$



$$I(G_\omega) = (x_1^3 x_2^3, x_2 x_3, x_1^2 x_4^2, x_2^4 x_5^4, x_3^5 x_6^5)S.$$

Irreducible Decompositions of Weighted Edge Ideals

Definition

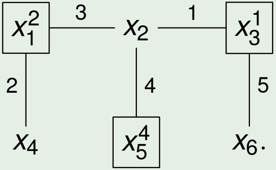
A weighted vertex cover W^σ of G_ω is a vertex cover $W \subseteq V$ with a function $\sigma\colon W \to \mathbb{N}$ such that for every $e = x_i x_j \in E$, one has

1. $x_i \in W$ and $\sigma(x_i) \leq \omega(e)$, or
2. $x_j \in W$ and $\sigma(x_j) \leq \omega(e)$.

Set $(W^\sigma)S = (x_i^{\sigma(x_i)} \mid x_i \in W)S$.

Example

$G_\omega =$



This weighted vertex cover is minimal: no vertices can be unboxed, and no weights (exponents) can be increased.

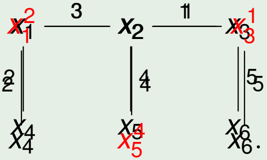
Theorem (Paulsen-SW '13)

We have (irredundant) irreducible decompositions

$$I(G_\omega) = \bigcap_{W^\sigma} (W^\sigma)S = \bigcap_{W^\sigma \ min} (W^\sigma)S$$

Example

$G_\omega =$



$$\begin{aligned} I(G_\omega) = & (x_1^2, x_2, x_3^5)S \cap (x_1^2, x_2^4, x_3)S \cap (x_1^2, x_2, x_6^5)S \\ & \cap (x_1^2, x_3, x_5^4)S \cap (x_2, x_3^5, x_4^2)S \cap (x_2^3, x_3, x_4^2)S \\ & \cap (x_2, x_4^2, x_6^5)S \cap (x_1^3, x_2^4, x_3, x_4^2)S \cap (x_1^3, x_3, x_4^2, x_5^4)S \end{aligned}$$

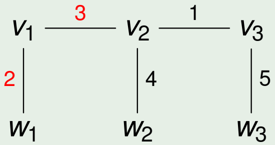
Cohen-Macaulay Weighted Trees

Theorem (Paulsen-SW ’13)

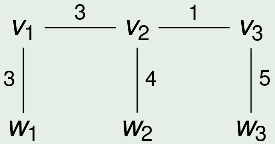
If T_ω is a weighted tree, then $S/I(T_\omega)$ is Cohen-Macaulay if and only if $I(T_\omega)$ is unmixed, if and only if T is a suspension of a tree Γ such that for each edge v_iv_j in Γ one has $\omega(v_iv_j) \leq \min\{\omega(v_iw_i), \omega(v_jw_j)\}$.

Example

non-CM



CM



Weighted Path Ideals

Definition (Kubik-SW)

Fix an integer $r \geq 1$. The **weighted r-path ideal** $I_r(G_\omega) \subseteq S$ of a weighted graph G_ω is the ideal of S generated by all monomials

$$x_{i_0}^{\omega(x_{i_0}x_{i_1})} x_{i_1}^{\max(\omega(x_{i_0}x_{i_1}),\omega(x_{i_1}x_{i_2}))} \cdots x_{i_{r-1}}^{\max(\omega(x_{i_{r-2}}x_{i_{r-1}}),\omega(x_{i_{r-1}}x_{i_r}))} x_{i_r}^{\omega(x_{i_{r-1}}x_{i_r})}$$

such that $x_{i_0}x_{i_1}\cdots x_{i_{r-1}}x_{i_r}$ is a path in G.

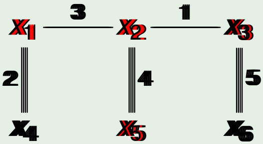
Example

1. If $r = 1$, then this is $I(G_\omega)$.
2. If $\omega = 1$ and G is a tree, then this is Conca's $I_r(G)$, generated by all the r-paths in G.

Weighted Path Ideals

Example

$G_\omega =$



$$I_2(G_\omega) = (x_1^3 x_2^3 x_3, x_1^3 x_2^4 x_5^4, x_2 x_3^5 x_6^5, x_1^3 x_2^3 x_4^2, x_2^4 x_3 x_5^4)S.$$

Theorem (Kubik-SW)

We have (irredundant) irreducible decompositions

$$I_r(G_\omega) = \bigcap_{W^\sigma} (W^\sigma)S = \bigcap_{W^\sigma \ min} (W^\sigma)S$$

Cohen-Macaulay Weighted Trees

Theorem (Kubik-SW)

Let G_ω be a weighted tree with no r-pathless leaves. TFAE:

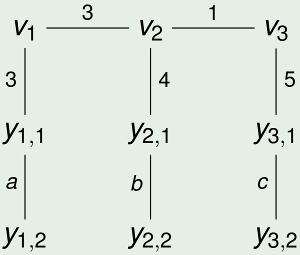
(i) $S/I_r(G_\omega)$ *is Cohen-Macaulay;*

(ii) $I_r(G_\omega)$ *is unmixed; and*

(iii) G_ω *is an r-path suspension of a weighted tree Γ_μ s.t. for all $v_i v_j \in E(\Gamma_\mu)$ one has* $\omega(v_i v_j) \leq \min\{\omega(v_i y_{i,1}), \omega(v_j y_{j,1})\}$.

Example

$I_2(G_\omega)$



Cohen-Macaulay Weighted Cliques

Theorem (Paulsen-SW '13)

Let K^d_ω be a weighted d-clique. Then $S/I(K^d_\omega)$ is Cohen-Macaulay of dimension 1.

Theorem (Kubik-SW)

Let K^3_ω be a weighted 3-clique. Then $S/I_2(K^3_\omega)$ is Cohen-Macaulay if and only if K^3_ω has edge weights of the form $a = a \leq b$.

Theorem (Kubik-SW)

Let K^d_ω be a weighted d-clique. Then $S/I_2(K^d_\omega)$ is Cohen-Macaulay if and only if $S/I_2(K^3_\lambda)$ is Cohen-Macaulay for each induced weighted sub-clique K^3_λ.