Adic Semidualizing Modules

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Big Picture

Goals

1. Describe a notion that covers both "semidualizing modules" and "quasi-dualizing modules".
2. Show how to construct dualizing complexes over arbitrary complete local rings without using Cohen structure theorem.

Assumption

R is a commutative noetherian ring with identity.

Two Examples

Fact (Assume R is local and CM with canonical module D.)

1. *D is finitely generated over R,*
2. $\mathrm{Ext}^i_R(D, D) = 0$ *for all* $i \neq 0$, *and*
3. $R \xrightarrow{\chi} \mathrm{Hom}_R(D, D)$ *is an isomorphism.*

Fact (The module R has similar properties.)

1. *R is finitely generated over R,*
2. $\mathrm{Ext}^i_R(R, R) = 0$ *for all* $i \neq 0$, *and*
3. $R \xrightarrow{\chi} \mathrm{Hom}_R(R, R)$ *is an isomorphism.*

Remark

Each of these examples is used for an important duality.

Semidualizing Modules

Definition (Foxby '72)

An R-module C is semidualizing if

1. C is finitely generated over R,
2. $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \neq 0$, and
3. $R \xrightarrow{\chi} \mathrm{Hom}_R(C, C)$ is an isomorphism.

Applications

1. Understanding compositions of local ring homomorphisms of finite G-dimension.
2. Understanding growth of Bass numbers of local rings.

Quasi-dualizing Modules

Fact (Assume $(R, \mathfrak{m}, k)$ is local with $E = E_R(k)$.)

1. *If A is an artinian R-module, then it is $\mathfrak{m}$-torsion, so:*
   - *A has a natural $\widehat{R}$-module structure $A \xrightarrow{\cong} \widehat{R} \otimes_R A$, so*
   - *we have a natural homothety map $\widehat{R} \xrightarrow{\chi} \operatorname{Hom}_R(A, A)$.*
2. *E is artinian over R,*
3. $\operatorname{Ext}^i_R(E, E) = 0$ *for all* $i \neq 0$*, and*
4. $\widehat{R} \xrightarrow{\chi} \operatorname{Hom}_R(E, E)$ *is an isomorphism.*

Definition (Kubik '14)

An R-module A over a local ring is quasi-dualizing if

1. A is artinian over R,
2. $\operatorname{Ext}^i_R(A, A) = 0$ for all $i \neq 0$, and
3. $\widehat{R} \xrightarrow{\chi} \operatorname{Hom}_R(A, A)$ is an isomorphism.

Similarities and Differences

Definition

- An R-module C is semidualizing if
  1. C is finitely generated (i.e., noetherian) over R,
  2. $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \neq 0$, and
  3. $R \xrightarrow{\chi} \mathrm{Hom}_R(C, C)$ is an isomorphism.
- An R-module A over a local ring is quasi-dualizing if
  1. A is artinian over R,
  2. $\mathrm{Ext}^i_R(A, A) = 0$ for all $i \neq 0$, and
  3. $\widehat{R} \xrightarrow{\chi} \mathrm{Hom}_R(A, A)$ is an isomorphism.

Remark

- Similarities: a finiteness condition, Ext-vanishing, and a good endomorphism algebra.
- Differences: the finiteness condition, and the endomorphism algebra.

Common Ground (Noetherian vs. Artinian)

Assumption

$\mathfrak{a} \subsetneq R$ with finite generating sequence $\underline{x}$. Set $K = K^R(\underline{x})$.

Fact (Melkerson '05)

Given an R-module M, the following conditions are equivalent:

1. $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ *is finitely generated for all* i,
2. $\operatorname{Tor}_i^R(R/\mathfrak{a}, M)$ *is finitely generated for all* i, *and*
3. $\operatorname{H}_i(K \otimes_R M)$ *is finitely generated for all* i.

Definition (SW-Wicklein)

An R-module M is $\mathfrak{a}$-adically finite if

1. M is $\mathfrak{a}$-torsion, and
2. $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all i.

Common Ground (Noetherian vs. Artinian), cont.

Definition (SW-Wicklein)

An R-module M is $\mathfrak{a}$-adically finite if

1. M is $\mathfrak{a}$-torsion, and
2. $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all i.

Examples (Common Ground)

Let M be an R-module.

1. M is 0-adically finite if and only if it is finitely generated (i.e., noetherian) over R.
2. if $(R, \mathfrak{m})$ is local, then M is $\mathfrak{m}$-adically finite if and only if it is artinian over R.

Remark

Nice properties of $\mathfrak{a}$-adically finite modules include a version of Nakayama's Lemma.

Common Ground (Homothety Maps)

Fact (Let M be an $\mathfrak{a}$-torsion R-module.)

1. *Then M has a natural $\widehat{R}^{\mathfrak{a}}$-module structure $M \xrightarrow{\cong} \widehat{R}^{\mathfrak{a}} \otimes_R M$.*
2. *This yields a natural homothety map $\widehat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(M, M)$.*

Examples (Let M be an R-module.)

1. $\widehat{R}^{0} \cong R \xrightarrow{\chi} \operatorname{Hom}_R(M, M)$ is the same homothety map discussed for semidualizing modules.
2. If $(R, \mathfrak{m}, k)$ is local, then $\widehat{R}^{\mathfrak{m}} = \widehat{R} \xrightarrow{\chi} \operatorname{Hom}_R(M, M)$ is the same homothety map discussed for quasidualizing modules.

Goal 1: A notion that covers both semidualizing modules and quasi-dualizing modules.

Definition (SW-Wicklein)

An R-module M is $\mathfrak{a}$-adically semidualizing if

1. M is $\mathfrak{a}$-adically finite,
2. $\mathrm{Ext}^i_R(M, M) = 0$ for $i \neq 0$, and
3. $\widehat{R}^{\mathfrak{a}} \to \mathrm{Hom}_R(M, M)$ is an isomorphism.

Examples

1. an R-module is 0-adically semidualizing if and only if it is semidualizing.
2. if $(R, \mathfrak{m})$ is local, an R-module is $\mathfrak{m}$-adically semidualizing if and only if it is quasi-dualizing.

Goal 2: Construct dualizing complexes over complete local rings without using Cohen structure theorem.

Theorem (SW-Wicklein)

Let M be an R-module with (truncated) flat resolution F.

1. *M is $\mathfrak{a}$-adically finite over R if and only if $\mathrm{H}_i(\widehat{F}^{\mathfrak{a}})$ is finitely generated over $\widehat{R}^{\mathfrak{a}}$ for all i, i.e., $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M) \simeq \widehat{F}^{\mathfrak{a}} \in \mathcal{D}^f_b(\widehat{R}^{\mathfrak{a}})$.*
2. *M is $\mathfrak{a}$-adically semidualizing over R if and only if $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$ is a semidualizing complex over $\widehat{R}^{\mathfrak{a}}$.*
3. *M is $\mathfrak{a}$-adically semidualizing and $\mathrm{id}_R(M) < \infty$ if and only if $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$ is a dualizing complex over $\widehat{R}^{\mathfrak{a}}$.*
4. *If $(R, \mathfrak{m}, k)$ is local and $E = E_R(k)$, then $\mathbf{L}\widehat{\Lambda}^{\mathfrak{m}}(E)$ is a dualizing complex over $\widehat{R}$.*

Remark

All of this can be done for complexes.