Free Resolutions

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Contents

Introduction

This course will be broken into three parts and we outline them here.

The reader will be familiar with the notion of using small amounts of summary data to gain insight into sometimes exceedingly complicated mathematical objects. For the statistician, one uses measures of center and spread, for instance, to understand immense data sets. In graduate-level abstract algebra, students are exposed to the Sylow Theorems, whereby significant structural information is gleaned from knowing only the order of a finite group. We know that one way to determine two groups G and H are not isomorphic is to show their respective orders are different, however, we also know that merely knowing $|G| = |H|$ does not imply $G \cong H$. So tools like the Sylow Theorems are powerful, yet are quite limited. In this course we are interested in similar tools that allow us to understand rings and modules.

In general, throughout the course we will use the following notation. We let k be a field and let $R = k[X_1, \ldots, X_d]$ be the polynomial ring in d variables with coefficients in k. We will write $I \leq R$ to denote an ideal I of the ring R. We want to understand the quotient ring $S = R/I$ and one of the aforementioned tools for doing so is free resolutions, the existence of which will be explored in Part I, along with Hilbert's Syzygy Theorem, presented below.

THEOREM 1 (Hilbert's Syzygy Theorem). *Let k be a field and $R = k[X_1, \ldots, X_d]$ the polynomial ring in d variables.*

(a) *If $I \leq R$ is $I = \langle f_1, \ldots, f_{\beta_1} \rangle$ where f_i is a polynomial in R for $i = 1, \ldots, \beta_1$, then there exists an exact sequence*

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & R^{\beta_d} & \xrightarrow{\partial_d} & \cdots & \xrightarrow{\partial_3} & R^{\beta_2} & \xrightarrow{\partial_2} & R^{\beta_1} & \xrightarrow[\begin{pmatrix} f_1 & \cdots & f_{\beta_1} \end{pmatrix}]{\partial_1} & R & \xrightarrow{\tau} & R/I \longrightarrow 0. \\ d+1 & & d & & & & 2 & & 1 & & 0 & & \end{array}$$

This is an <u>augmented free resolution</u> of R/I over R. The <u>free resolution</u> omits the module R/I. The maps ∂_i are the <u>differentials</u> in the resolution and the (homological) degree of each module in the resolution is given beneath it. It is common to write simply ∂ when the degree is understood.

(b) *If f_i is homogeneous for $i = 1, \ldots, d$, then this resolution can be built minimally and the β_j's are independent of the choice of minimal free resolution. The integer*

$$\beta_j = \beta_j^R(R/I)$$

is the <u>j^{th} Betti number</u> of R/I over R. This notion is originally from algebraic topology where it was named after Enrico Betti by Poincaré and modernized by Emmy Noether.

It should be noted that part (a) of the theorem guarantees the sequence will vanish beyond (homological) degree d, but it is not necessarily the case that $R^{\beta_i} \neq 0$ for $i = 1, \ldots, d$. An application of the theorem is as follows, and it resembles our finite group example above. If $J \leq R$ is another ideal generated by polynomials in R and $\beta_j^R(R/I) \neq \beta_j^R(R/J)$ for some j, then $R/I \not\cong R/J$. However, if $\beta_j^R(R/I) = \beta_j^R(R/J)$ for all j, then R/I may or may not be isomorphic to R/J.

Part II of the course will be spent exploring examples of free resolutions, including the Koszul complex named after J. L. Koszul, which we partially present here.

EXAMPLE 1 (Koszul Complex). Let $R = k[X_1, \ldots, X_d]$ and let $\{i_1, \ldots, i_{\beta_1}\}$ be a subset of $\{1, \ldots, d\}$. Then let $I = \left\langle X_{i_1}, \ldots, X_{i_{\beta_1}} \right\rangle \leq R$ be an ideal of R and we have the following free resolution.

$$0 \longrightarrow R^{\binom{\beta_1}{\beta_1}} \xrightarrow{\partial_{\beta_1}} \cdots \xrightarrow{\partial_4} R^{\binom{\beta_1}{3}} \xrightarrow{\partial_3} R^{\binom{\beta_1}{2}} \xrightarrow{\partial_2} R^{\binom{\beta_1}{1}} \xrightarrow[\begin{pmatrix} X_{i_1} & \cdots & X_{i_{\beta_1}} \end{pmatrix}]{\partial_1} R \longrightarrow 0$$

So the Betti numbers

$$\beta_j^R(R/I) = \binom{\beta_1}{j}$$

are independent of the list $\{i_1, \ldots, i_{\beta_1}\}$. This demonstrates that if I and J are ideals of R with equivalent Betti numbers $\beta_j^R(R/I) = \beta_j^R(R/J)$, then we need not have $R/J \cong R/I$.

In Part III of this course we will explore the theory of differential graded algebra (DGA) resolutions, as well as examples and applications of such resolutions. The Koszul complex is one such DGA resolution, as it admits a unital ring structure in the following way.

EXAMPLE 2. Consider the ideal $I = \langle X_1, X_2, X_3 \rangle \leq R = k[X_1, \ldots, X_d]$. Then the Koszul complex as seen in the previous example is

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & R & \xrightarrow{\partial_3} & R^3 & \xrightarrow{\partial_2} & R^3 & \xrightarrow[\begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}]{\partial_1} & R & \longrightarrow & 0 \\ & & e_{123} & & e_{12} & & e_1 & & 1 & & \\ & & & & e_{13} & & e_2 & & & & \\ & & & & e_{23} & & e_3 & & & & \end{array}$$

where we have denoted the basis elements below each R-module. We see that $\partial_1(e_i) = X_i$ for $i = 1, 2, 3$ and we also have the following.

$$\begin{aligned} \partial_2(e_{12}) &= X_1 e_2 - X_2 e_1 & \qquad \partial_3(e_{123}) &= X_1 e_{23} - X_2 e_{13} + X_3 e_{12} \\ \partial_2(e_{13}) &= X_1 e_3 - X_3 e_1 & & \\ \partial_2(e_{23}) &= X_2 e_3 - X_3 e_2 & & \end{aligned}$$

In general we have

$$\partial_m(e_{i_1 \cdots i_m}) = \sum_{j=1}^{m} (-1)^{j-1} X_{i_j} e_{i_1 \cdots \hat{i}_j \cdots i_m}$$

where $i_1 \cdots \hat{i}_j \cdots i_m$ denotes the ordered list $i_1 \cdots i_m$ with i_j omitted. This determines an R-linear map by respecting linear combinations of basis vectors with coefficients in R (this is the UMP for free modules). The multiplication goes as follows.

$$\begin{aligned} e_1 e_2 &= e_{12} & \qquad e_1 e_{12} &= 0 \\ e_1 e_1 &= 0 & e_2 e_{13} &= -e_{123} \\ e_2 e_1 &= -e_1 e_2 = -e_{12} & & \end{aligned}$$

This so-called "wedge product" is unital, associative and graded commutative, i.e.,

$$e_A e_B = (-1)^{|A| \cdot |B|} e_B e_A$$

where $A, B \subseteq \{1, 2, \ldots, m\}$, and $|A|$ and $|B|$ denote the homological degrees of e_A and e_B, respectively. The differentials and multiplication also satisfy the Leibniz rule:

$$\partial_{|A|+|B|}(e_A e_B) = \partial_{|A|}(e_A) e_B + (-1)^{|A|} e_A \partial_{|B|}(e_B).$$

For instance, we have

$$\partial_2(e_1 e_2) = \partial_2(e_{12}) = X_1 e_2 - X_2 e_1 \stackrel{\dagger}{=} \partial_1(e_1) e_2 + (-1)^1 e_1 \partial_1(e_2)$$

where † holds since degree-zero elements commute.

Part I

Homological Algebra

Linear Algebra

Throughout the chapter, we will assume R is a commutative ring with identity, unless stated otherwise.

Definition I.A.1. Let M be an R-module.

(a) A sequence $e_1, \ldots, e_n \in M$ is a finite basis for M if it generates M as an R-module and it is linearly independent over R, i.e., for every $m \in M$ there exist unique $r_1, \ldots, r_n \in R$ such that $m = \sum_{i=1}^{n} r_i e_i$.

(b) M is a finite rank free R-module if it has a finite basis.

Example I.A.2. (a) We define R^n to be the R-module whose elements are column vectors of size n with entries in R, i.e.,

$$R^n = \left\{ \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \;\middle|\; r_1, \ldots, r_n \in R \right\}.$$

This is a finite rank free R-module with standard basis $e_1, \ldots, e_n$ where $e_i = (\delta_{ij})_j$ and δ_{ij} is the Kronecker delta.

(b) If $0 \neq I \lneq R$, then R/I is not free, because it fails linear independence over R in the following way. If $0 \neq r \in I$ (which exists since $I \neq 0$), then for every $s \in R \setminus I$ we have $0 \neq \overline{s} = s + I \in R/I$ and $rs \in I$, which implies $\overline{rs} = 0 \in R/I$. Therefore $r \cdot \overline{s} = \overline{rs} = 0$ and we have thus exhibited a linear combination which sums to zero, but has a non-zero coefficient.

Fact I.A.3. (a) (Universal Mapping Property) *Let F be a free R-module with basis $e_1, \ldots, e_n \in F$. For every R-module M and any collection of elements $m_1, \ldots, m_n \in M$, there exists a unique R-module homomorphism $\phi\colon F \to M$ such that $\phi(e_i) = m_i$ for $i = 1, \ldots, n$.*

$$\begin{tikzcd}
\{e_1, \ldots, e_n\} \arrow[r, "\subseteq"] \arrow[dr] & F \arrow[d, dashed, "\exists!"] & \sum_i r_i e_i \arrow[d, mapsto] \\
e_i \arrow[r, mapsto] & m_i \quad M & \sum_i r_i m_i
\end{tikzcd}$$

(b) *If F and G are finite rank free R-modules with bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$, respectively, then $F \cong G$ as R-modules.*

(c) *If F is a finite rank free R-module with basis $e_1, \ldots, e_n$, then $F \cong R^n$ as R-modules.*

Proof. Part (a) is standard. We prove part (b) and then part (c) will follow as a corollary. Since F and G are free, by the universal mapping property we have R-module homomorphisms $\phi\colon F \to G$ and $\psi\colon G \to F$ such that $\phi(e_i) = f_i$ and $\psi(f_i) = e_i$ for $i = 1, \ldots, n$.

$$\begin{tikzcd}
\{e_1, \ldots, e_n\} \arrow[r, "\subseteq"] \arrow[dr] & F \arrow[d, dashed, "\phi"] \\
& G
\end{tikzcd}
\qquad
\begin{tikzcd}
\{f_1, \ldots, f_n\} \arrow[r, "\subseteq"] \arrow[dr] & G \arrow[d, dashed, "\psi"] \\
& F
\end{tikzcd}$$

Since the composition $\psi \circ \phi$ is a homomorphism from F to F and id_F is likewise a homomorphism from F to F, by the uniqueness given in the universal mapping property we have $\psi \circ \phi = \mathrm{id}_F$. By the same reasoning we know $\phi \circ \psi = \mathrm{id}_G$ and we conclude both ϕ and ψ are isomorphisms, i.e., $F \cong G$.

$$\begin{tikzcd}
\{e_1, \ldots, e_n\} \arrow[r, "\subseteq"] \arrow[dr] & F \arrow[d, shift right, "\mathrm{id}_F"'] \arrow[d, shift left, "\psi\circ\phi"] \\
& F
\end{tikzcd}
\qquad
\begin{tikzcd}
\{f_1, \ldots, f_n\} \arrow[r, "\subseteq"] \arrow[dr] & G \arrow[d, shift right, "\mathrm{id}_G"'] \arrow[d, shift left, "\phi\circ\psi"] \\
& G
\end{tikzcd}$$

□

Notation I.A.4. Let $\phi\colon R^n \to R^m$ be R-linear. We represent ϕ by a matrix A where the j^{th} column of A consists of the coefficients needed to represent $\phi(e_j)$. That is, if $e_1, \dots, e_n \in R^n$ and $f_1, \dots, f_m \in R^m$ form the standard bases, then we let $A = (a_{ij})$ where

$$\phi(e_j) = \sum_{i=1}^{m} a_{ij} f_i$$

for $j = 1, \dots, n$. This partially justifies the following:

$$\mathrm{Hom}_R(R^n, R^m) \cong \mathrm{Mat}_{m \times n}(R) \cong R^{mn}.$$

Fact I.A.5. *An R-module M is finitely generated over R if and only if there exists an R-module epimorphism (surjective homomorphism) $\tau\colon R^n \to M$ for some $n \in \mathbb{N}$, in which case M is generated by $\tau(e_1), \dots, \tau(e_n) \in M$.*

Theorem I.A.6. *The following are equivalent.*

(i) *Every ideal of R is finitely generated.*
(ii) *R satisfies the ascending chain condition for ideals, i.e., for every chain of ideals $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$, there exists an integer $N \in \mathbb{N}$ such that $I_N = I_{N+1} = I_{N+2} = \dots$.*
(iii) *R satisfies the maximum condition for ideals, i.e., every non-empty set of ideals of R contains a maximal element with respect to containment.*
(iv) *For every $n \in \mathbb{N}$, every submodule of R^n is finitely generated.*
(v) *For every $n \in \mathbb{N}$, R^n satisfies the ascending chain condition for submodules.*
(vi) *For every $n \in \mathbb{N}$, R^n satisfies the maximum condition for submodules.*

Definition I.A.7. R is noetherian if it satisfies the equivalent conditions of Theorem I.A.6.

Fact I.A.8.

(a) (Hilbert's Basis Theorem) *If R is noetherian, then $R[X]$ is noetherian.*
(b) *If R is noetherian, then for every $n \in \mathbb{N}$ and for every ideal $I \leq R[X_1, \dots, X_n]$, the quotient ring $R[X_1, \dots, X_n]/I$ is noetherian.*
(c) *If k is a field, then k is noetherian and by Hilbert's Basis Theorem, $k[X_1, \dots, X_n]$ is noetherian as well for every $n \in \mathbb{N}$.*

CHAPTER I.B

Exact Sequences

Definition I.B.1.

(a) A sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ of R-module homomorphisms is exact if $\mathrm{Im}\,(\alpha) = \mathrm{Ker}\,(\beta)$.

(b) A sequence

$$\cdots \xrightarrow{f_{i+2}} A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \xrightarrow{f_{i-1}} \cdots$$

is exact if $\mathrm{Im}\,(f_{i+1}) = \mathrm{Ker}\,(f_i)$ for all $i \in \mathbb{Z}$.

(c) A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Fact I.B.2.

(a) $0 \longrightarrow A \xrightarrow{\alpha} B$ *is exact if and only if* α *is injective.*

(b) $B \xrightarrow{\beta} C \longrightarrow 0$ *is exact if and only if* β *is surjective.*

(c) $0 \longrightarrow A \longrightarrow 0$ *is exact if and only if* $A = 0$.

(d) $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$ *is exact if and only if* α *is an isomorphism.*

Theorem I.B.3. *If R is noetherian and M is a finitely generated R-module, then there exists an exact sequence*

$$\cdots \xrightarrow{\partial_{i+1}} R^{\beta_i} \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \xrightarrow{\tau} M \longrightarrow 0.$$

Proof. Since M is assumed to be finitely generated, by Fact I.A.5 the exists an integer $\beta_0 \in \mathbb{N}$ and an epimorphism $\tau\colon R^{\beta_0} \to M$. Since $\mathrm{Ker}\,(\tau)$ is a submodule of R^{β_0} and R is noetherian, by Theorem I.A.6 and Fact I.A.5 there exists an integer $\beta_1 \in \mathbb{N}$ and epimorphism $\tau_1\colon R^{\beta_2} \to \mathrm{Ker}\,(\tau)$. This procedure continues and yields the following commutative diagram of short exact diagonal sequences.

$$\begin{tikzcd}
 & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & & & \\
 & & \mathrm{Ker}\,(\tau_2) \arrow[ur] \arrow[dr, "\iota_2", "\subseteq"'] \arrow[d, phantom, "\curvearrowright"] & & & & \mathrm{Ker}\,(\tau) \arrow[ur] \arrow[dr, "\iota_0", "\subseteq"'] \arrow[d, phantom, "\curvearrowright"] & & & & 0 & \\
\cdots \arrow[r] & R^{\beta_3} \arrow[ur, "\tau_3"] \arrow[rr, "\partial_3=\iota_2\tau_3"'] & {} & R^{\beta_2} \arrow[rr, "\partial_2=\iota_1\tau_2"] \arrow[dr, "\tau_2"'] & {} & R^{\beta_1} \arrow[ur, "\tau_1"] \arrow[rr, "\partial_1=\iota_0\tau_1"'] & {} & R^{\beta_0} \arrow[rr, "\tau"] \arrow[dr, "\tau"'] & {} & M \arrow[ur] \arrow[rr] & & 0 \\
 & & & & \mathrm{Ker}\,(\tau_1) \arrow[ur, "\subseteq", "\iota_1"'] \arrow[dr] \arrow[u, phantom, "\curvearrowright"] & & & & M \arrow[ur, "\mathrm{id}_M"'] \arrow[dr] \arrow[u, phantom, "\curvearrowright"] & & & \\
 & & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

A diagram chase shows the horizontal sequence is exact. □

Definition I.B.4. The exact sequence in Theorem I.B.3 is an augmented free resolution of M.

Remark I.B.5. In general, these are difficult to compute. Thus, the following examples are particularly nice. A main point of this course is to construct other examples explicitly.

Example I.B.6. We give three examples of free resolutions.

(a) From the fundamental theorem of finitely generated abelian groups, if G is a finitely generated abelian group, there exist positive integers $d_1, \ldots, d_n, r \in \mathbb{N}$ such that

$$G \cong \frac{\mathbb{Z}}{(d_1)} \oplus \cdots \oplus \frac{\mathbb{Z}}{(d_n)} \oplus \mathbb{Z}^r.$$

The following is an augmented free resolution of G (as a $\mathbb{Z}$-module).

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^n \arrow[rr, "{\begin{pmatrix} d_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & d_n \\ & \mathbf{0} & \end{pmatrix}}"'] & & \mathbb{Z}^{n+r} \arrow[r] & G \arrow[r] & 0 \end{tikzcd}$$

(b) If R is an integral domain and $0 \neq r \in R \setminus R^\times$, then the following is an augmented free resolution of $R/(r)$.

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[rr, "r\cdot"] \arrow[dr] & & R \arrow[r] & \frac{R}{(r)} \arrow[r] & 0 \\ & & (r) \arrow[ur] \arrow[dr] & & & \\ & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

(c) These resolutions need not be finite. Consider the ring $R = k[X,Y]/(XY)$ and the R-module $M = R/(\overline{X})$. Then we have the following augmented free resolution of M.

$$\begin{tikzcd} & & 0 \arrow[dr] & & 0 & & & & \\ & & & (\overline{X}) \arrow[ur] \arrow[dr] & & & & & \\ \cdots \arrow[r, "\overline{X}\cdot"] & R \arrow[r, "\overline{Y}\cdot"] & R \arrow[ur] \arrow[rr, "\overline{X}\cdot"'] & & R \arrow[rr, "\overline{Y}\cdot"] \arrow[dr] & & R \arrow[r] & \frac{R}{(\overline{X})} \arrow[r] & 0 \\ & & & & & (\overline{Y}) \arrow[ur] \arrow[dr] & & & \\ & & & & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

One can find a similar resolution when $R = k[X]/(X^2)$ and $M = R/\left(\overline{X}\right)$.

Theorem I.B.7 (Hilbert's Syzygy Theorem). *Let $R = k[X_1, \ldots, X_d]$ where k is a field and let M be a finitely generated R-module. Then there exists a free resolution*

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow M \longrightarrow 0$$

with $\beta_i \geq 0$ for $i = 0, \ldots, d$. (One can replace k with a principal ideal domain and the conclusion holds if one also replaces d with $d+1$ in the above sequence.)

Remark I.B.8. (a) If $d = 0$, then R is a field and $M \cong R^{\beta_0}$, so we have the augmented free resolution

$$0 \longrightarrow R^{\beta_0} \longrightarrow M \longrightarrow 0.$$

(b) If $d = 1$, then R is a principal ideal domain and we can construct the following augmented free resolution.

$$\begin{tikzcd}
& 0 \arrow[dr] & & & & & \\
0 \arrow[r] & & R^{\beta_1} \arrow[rr] \arrow[dr, "\cong"'] & & R^{\beta_0} \arrow[r, "\tau"] & M \arrow[r] & 0 \\
& & & \operatorname{Ker}(\tau) \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Here we use the assumptions that $\operatorname{Ker}(\tau)$ is a submodule of R^{β_0} and R is a principal ideal domain to conclude that $\operatorname{Ker}(\tau) \cong R^{\beta_1}$ for some $\beta_1 \leq \beta_0$.

The following results were given as exercises. For each exercise we consider the following sequence of R-modules and R-module homomorphisms.

$$A = \qquad \cdots \xrightarrow{\partial^A_{i+1}} A_i \xrightarrow{\partial^A_i} \cdots$$

Assume that each R-module A_i is free with finite basis B_i.

Exercise I.B.9. Fix an integer i. If $\partial^A_{i-1}\left(\partial^A_i(b)\right) = 0$ for all $b \in B_i$, then we have $\partial^A_{i-1}\partial^A_i = 0$.

Proof. Set $\{b_1, \ldots, b_r\} = B_i$. By assumption, the following diagram commutes for each of the vertical maps.

$$\begin{tikzcd}
\{b_1, \ldots, b_r\} \arrow[r, "\subseteq"] \arrow[dr, "0"'] & A_i \arrow[d, shift right, "0"'] \arrow[d, shift left, "\partial^A_{i-1}\circ\partial^A_i"] \\
& A_{i-2}
\end{tikzcd}$$

Therefore by the uniqueness given in the Universal Mapping Property (Fact I.A.3) we have $\partial^A_{i-1}\circ\partial^A_i = 0$. $\square$

Exercise I.B.10. Fix integers i and j, and let $f\colon B_i \times B_j \to A_{i+j}$ be a function. Then there is a unique well-defined R-bilinear map $\mu_{i,j}\colon A_i \times A_j \to A_{i+j}$ such that $\mu_{i,j}(b, b') = f(b, b')$ for all $b \in B_i$ and $b' \in B_j$.

Proof. We define $\mu_{i,j}$ the only way we can, because of the requirement of bilinearity.

$$\mu_{i,j}\left(\sum_{b\in B_i} r_b b\,,\ \sum_{b'\in B_j} s_{b'} b'\right) := \sum_{b\in B_i}\sum_{b'\in B_j} r_b s_{b'} \cdot f(b, b')$$

Well-definedness follows readily from the linear independence of B_i and of B_j. If we suppose that ρ is another R-bilinear map satisfying $\rho(b, b') = f(b, b')$ for all $b \in B_i$ and all $b' \in B_j$, then we have

$$\rho\left(\sum_{b\in B_i} r_b b\,,\ \sum_{b'\in B_j} s_{b'} b'\right) = \sum_{b\in B_i}\sum_{b'\in B_j} r_b s_{b'} \rho(b, b') = \sum_{b\in B_i}\sum_{b'\in B_j} r_b s_{b'} f(b, b') = \mu_{i,j}\left(\sum_{b\in B_i} r_b b\,,\ \sum_{b'\in B_j} s_{b'} b'\right),$$

so $\mu_{i,j}$ is unique. $\square$

Exercise I.B.11. Fix integers i and j, and let $\mu_{i,j}\colon A_i \times A_j \to A_{i+j}$ be an R-bilinear map. For all $a \in A_i$ and $a' \in A_j$, set $aa' = \mu_{i,j}(a, a')$.

(a) If $i = 0$ and there exists and element $1 \in A_0$ such that $1b' = b'$ for all $b' \in B_j$, then $1a' = a'$ for all $a' \in A_j$.
(b) If $bb' = (-1)^{ij} b'b$ for all $b \in B_i$ and $b' \in B_j$, then $aa' = (-1)^{ij} a'a$ for all $a \in A_i$ and $a' \in A_j$.
(c) If $b(b' + b'') = bb' + bb''$ for all $b \in B_i$ and $b', b'' \in B_j$ (with the standard order of operations), then $a(a' + a'') = aa' + aa''$ for all $a \in A_i$ and $a', a'' \in A_j$.

Exercise I.B.12. For all integers i and j, let $\mu_{i,j}\colon A_i \times A_j \to A_{i+j}$ be an R-bilinear map. For all $a \in A_i$ and $a' \in A_j$, set $aa' = \mu_{i,j}(a, a')$. Fix integers i, j, and k. If we have $b(b'b'') = (bb')b''$ for all $b \in B_i$, $b' \in B_j$, and $b'' \in B_k$, then $a(a'a'') = (aa')a''$ for all $a \in A_i$, $a' \in A_j$, and $a'' \in A_k$.

CHAPTER I.C

Graded Resolutions

In this chapter we are interested in being able to keep track of finer information about certain resolutions.

ASSUMPTION I.C.1. In this chapter, assume k is a field and that $R = k[X_1, \ldots, X_d]$ is the polynomial ring in d variables with the standard grading, i.e., $\deg X_i = 1$ for all i.

DEFINITION I.C.2. A <u>homogeneous</u> (or <u>graded</u>) ideal in R is an ideal generated by homogeneous polynomials (not necessarily of the same degree).

EXAMPLE I.C.3. Let $R = k[X, Y]$ and consider the ideal $I = \langle X^2, XY^2 \rangle$. Since X^2 and XY^2 are each homogeneous, I is a homogeneous ideal. Note that $X^2 - XY^2$ is not homogeneous, yet we have $I = \langle X^2 - XY^2, XY^2 \rangle$, so the existence of a non-homogeneous generator in the representation of an ideal does not imply the ideal is not graded.

THEOREM I.C.4 (Hilbert's Syzygy Theorem (graded version)). *If $I = \langle f_1, \ldots, f_{\beta_1} \rangle$ such that each f_i is homogeneous, then there exists an (augmented) free resolution*

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_1} \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

such that each differential in the resolution is represented by a matrix of homogeneous polynomials.

EXAMPLE I.C.5. Let $R = k[X, Y]$ and consider the ideals $I_1 = \langle X, Y \rangle$, $I_2 = \langle X^a, Y^b \rangle$, and $J = \langle X^a, XY, Y^b \rangle$. In the case of I_2 we assume $a, b \geq 1$ and in the case of J we assume $a, b \geq 2$. Then we claim the following are respective free resolutions of R/I_1, R/I_2, and R/J.

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -Y \\ X \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} X & Y \end{pmatrix}} R \longrightarrow \frac{R}{I_1} \longrightarrow 0$$

$$0 \longrightarrow R \xrightarrow[\partial_2^I]{\begin{pmatrix} -Y^b \\ X^a \end{pmatrix}} R^2 \xrightarrow[\partial_1^I]{\begin{pmatrix} X^a & Y^b \end{pmatrix}} R \longrightarrow \frac{R}{I_2} \longrightarrow 0$$

$$0 \longrightarrow R^2 \xrightarrow[\partial_2^J]{\begin{pmatrix} -Y & 0 \\ X^{a-1} & -Y^{b-1} \\ 0 & X \end{pmatrix}} R^3 \xrightarrow[\partial_1^J]{\begin{pmatrix} X^a & XY & Y^b \end{pmatrix}} R \longrightarrow \frac{R}{J} \longrightarrow 0$$

Since the first diagram is just a special case of the second, we need only justify the exactness of the resolutions of R/I_2 and R/J. The exactness at the (homological) degree -1 and 0 positions are by construction. The exactness at the degree 2 position in the second resolution follows from the fact that R is an integral domain and ∂_2^I amounts to the standard scalar multiplication of $(-Y^b \ X^a)^T$ by elements $r \in R$. Also for the second resolution, argue as we do for the third resolution to show that one also has exactness at the degree 1 position.

To show the third resolution is exact at the degree 1 position, we will show $\operatorname{Ker}(\partial_1^J) = \operatorname{Im}(\partial_2^J)$, i.e.,

$$\operatorname{Ker}(\partial_1^J) = \left\langle \begin{pmatrix} -Y \\ X^{a-1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -Y^{b-1} \\ X \end{pmatrix} \right\rangle.$$

The proof of the reverse containment is short.

$$\begin{pmatrix} X^a & XY & Y^b \end{pmatrix} \cdot \begin{pmatrix} -Y \\ X^{a-1} \\ 0 \end{pmatrix} = 0 \qquad\qquad \begin{pmatrix} X^a & XY & Y^b \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -Y^{b-1} \\ X \end{pmatrix} = 0$$

For the forward containment, let $(f\ g\ h)^T \in \operatorname{Ker}\left(\partial_1^J\right)$ and note this implies

$$X^a f + XYg + Y^b h = 0. \tag{I.C.5.1}$$

Since $X|X^a f$ and $X|XYg$, it follows that $X|Y^b h$ and therefore $X|h$. Let $h' \in R$ such that $h = Xh'$. By similar reasoning, we let $f' \in R$ such that $f = Yf'$. Hence (I.C.5.1) becomes

$$0 = X^a Y f' + XYg + Y^b X h' = XY(X^{a-1} f' + g + Y^{b-1} h').$$

Since we are working in an integral domain, this implies $X^{a-1}f' + g + Y^{b-1}h' = 0$ and therefore $g = -X^{a-1}f' - Y^{b-1}h'$. Hence we conclude our argument as follows.

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} Yf' \\ -X^{a-1}f' - Y^{b-1}h' \\ Xh' \end{pmatrix} = f' \begin{pmatrix} Y \\ -X^{a-1} \\ 0 \end{pmatrix} + h' \begin{pmatrix} 0 \\ -Y^{b-1} \\ X \end{pmatrix}$$

To see that the resolution of R/J is exact in the degree 2 position (i.e., that ∂_2^J is injective), let $(c\ d)^T \in \operatorname{Ker}\left(\partial_2^J\right)$ and observe that

$$0 = c \begin{pmatrix} -Y \\ X^{a-1} \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ -Y^{b-1} \\ X \end{pmatrix}$$

implies $dX = 0$ and $cY = 0$, so $c = 0 = d$.

As an aside, we point out that if one deletes one row of the matrix representing ∂_2^J and takes the determinant of the resulting matrix, then one obtains the entries of $(X^a\ XY\ Y^b)$, up to a sign. This is a special case of the Hilbert-Burch Theorem, which we discuss in Part II.

Remark I.C.6. Notice the resolutions in Example I.C.5 are all of the form

$$0 \longrightarrow R^{b_2} \longrightarrow R^{b_1} \longrightarrow R^{b_0} \longrightarrow 0$$

and their exponents all satisfy $b_0 - b_1 + b_2 = 0$. This leads us to the following exercise and theorem.

Exercise I.C.7. Let K be a field and consider the following exact sequence of K-vector spaces.

$$0 \longrightarrow K^{\beta_d} \longrightarrow \cdots \longrightarrow K^{\beta_0} \longrightarrow 0$$

Then

$$\sum_{i=0}^{d} (-1)^i \beta_i = 0.$$

Theorem I.C.8. *Let $I \leq R$ be a non-zero ideal and let*

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_0} \longrightarrow R/I \longrightarrow 0$$

be an exact sequence. Then

$$\sum_{i=0}^{d} (-1)^i \beta_i = 0.$$

Proof. Let $0 = \mathfrak{p} \lneq R$, which is a prime ideal and let $K = R_{\mathfrak{p}} = k(X_1, \ldots, X_d)$ be the field of fractions of R (i.e., localize at $\mathfrak{p}$). Then for any $0 \neq s \in I$ and any $\overline{r}/t \in (R/I)_{\mathfrak{p}}$, we have

$$\frac{\overline{r}}{t} = \frac{\overline{sr}}{st} = \frac{0}{st} = 0,$$

so $(R/I)_{\mathfrak{p}} = 0$. We can localize the given resolution to obtain the following resolution.

$$\begin{tikzcd} 0 \arrow[r] & R_{\mathfrak{p}}^{\beta_d} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & R_{\mathfrak{p}}^{\beta_0} \arrow[r] \arrow[d, equal] & (R/I)_{\mathfrak{p}} \arrow[r] \arrow[d, equal] & 0 \\ & K^{\beta_d} & & K^{\beta_0} & 0 & \end{tikzcd}$$

Thus the desired conclusion follows from Exercise I.C.7. $\square$

NOTATION I.C.9. Let $n \in \mathbb{N}$ and set

$$R_n = \{\text{homogeneous polynomials in } R \text{ of degree } n\} \cup \{0\}.$$

REMARK I.C.10. $R_n \subset R$ is a k-subspace, but is not an ideal unless $d = 0$.

NOTATION I.C.11. Let $m \in \mathbb{Z}$. We say $R(-m)$ is a "shifted" or "twisted" copy of R. It has $R(-m) = R$ as an R-module, but if $f \in R$ is homogeneous, then

$$\deg_{R(-m)}(f) = \deg_R(f) + m,$$

i.e.,

$$R(-m)_n = R_{n-m}.$$

For instance, $1 \in k = R_0 = R(-m)_m$. It follows that $R(-m)$ is a free R-module with basis $\{1\}$ such that $\deg_{R(-m)}(1) = m$. More generally we have that

$$F = \bigoplus_{i=1}^{r} R(-m_i)$$

is a graded free R-module of rank r for $m_1, \ldots, m_r \in \mathbb{Z}$ and

$$F_n = \left(\bigoplus_{i=1}^{r} R(-m_i)\right)_n = \bigoplus_{i=1}^{r} R(-m_i)_n = \bigoplus_{i=1}^{r} R_{n-m_i}.$$

For instance, if $e_1, \ldots, e_r \in F$ is the standard basis, then $\deg_F(e_i) = m_i$. The homogeneous elements of F of degree n are of the form

$$\sum_{i=1}^{r} s_i e_i$$

where each s_i is homogeneous in R with $\deg_R s_i = n - m_i$, because we need $\deg_F(s_i e_i) = \deg_R s_i + m_i$.

EXAMPLE I.C.12. In the R-module

$$\begin{matrix} R(-a) \\ \oplus \\ R(-b) \end{matrix}$$

we have the element

$$\begin{pmatrix} -Y^b \\ X^a \end{pmatrix} \in \begin{pmatrix} R(-a) \\ \oplus \\ R(-b) \end{pmatrix}_{a+b}$$

because this element can be written as

$$-Y^b e_1 + X^a e_2,$$

where $-Y^b$ and e_2 each have degree b, and X^a and e_1 have degree a.

DEFINITION I.C.13. Let F and G be free graded R-modules of finite rank. A homomorphism $\phi\colon F \to G$ is graded (or homogeneous) if $\phi(F_n) \subseteq G_n$ for all $n \in \mathbb{Z}$.

FACT I.C.14. *A homomorphism $\phi\colon F \to G$ between graded free modules of finite rank is graded if and only if $\phi(e_i) \in G_{m_i}$ for all $i = 1, \ldots, r$, where $F = \oplus_{i=1}^{n} R(-m_i)$.*

EXAMPLE I.C.15. Let $R = k[X, Y]$ and let $I = \langle X^a, Y^b \rangle \leq R$ be an ideal where $a, b \geq 2$. Then we have the following (augmented) free resolution of R/I.

$$\begin{tikzcd}
0 \arrow[r] & R(-a-b) \arrow[r, "\begin{pmatrix} -Y^b \\ X^a \end{pmatrix}"] & \begin{matrix} R(-a) \\ \oplus \\ R(-b) \end{matrix} \arrow[r, "\begin{pmatrix} X^a & Y^b \end{pmatrix}"] & R \arrow[r] & R/I \arrow[r] & 0
\end{tikzcd}$$

$$\epsilon \longmapsto \begin{pmatrix} -Y^b \\ X^a \end{pmatrix}$$

$$e_1 \longmapsto X^a$$

$$e_2 \longmapsto Y^b$$

This is graded because, for instance, the elements $\epsilon \in R(-a-b)$ and $-Y^b \in R(-a)$ and $X^a \in R(-b)$ all have degree $a+b$.

FACT I.C.16. *With notation as in Fact I.C.14, if ϕ is graded, then*

$$\operatorname{Im}(\phi) = \langle \phi(e_1), \ldots, \phi(e_r) \rangle$$

is generated by finitely many homogeneous elements. One can also show that $\operatorname{Ker}(\phi)$ *is generated by finitely many homogeneous elements of F.*

EXAMPLE I.C.17. The graded homomorphism

$$\phi\colon \begin{matrix} R(-a) \\ \oplus \\ R(-b) \end{matrix} \xrightarrow{\begin{pmatrix} X^a & Y^b \end{pmatrix}} R$$

has kernel generated by the vector $(-Y^b\ X^a)^T$, a homogeneous element of degree $a+b$.

We now give a sketch of the proof of Hilbert's Syzygy Theorem (graded version).

PROOF. By assumption $I = \langle f_1, \ldots, f_{\beta_1} \rangle$ and we let $\deg_R f_i = m_i$. We begin computing the resolution in the usual manner, surjecting onto R/I from R in the natural way and then onto $\operatorname{Ker}(\tau) = I$ from a free module.

$$\bigoplus_{i=1}^{\beta_1} R(-m_i) \xrightarrow[\partial_1]{\begin{pmatrix} f_1 & \cdots & f_{\beta_1} \end{pmatrix}} R \xrightarrow{\tau} R/I \longrightarrow 0$$

By construction $\operatorname{Im}(\partial_1) = I = \operatorname{Ker}(\tau)$ and we consider that by Fact I.C.16, we know $\operatorname{Ker}(\partial_1)$ is free and generated by finitely many homogeneous elements of $\bigoplus_{i=1}^{\beta_1} R(-m_i)$. So there exists a non-negative integer β_2 and homogeneous column vectors $f_{1,i}, \ldots, f_{1,\beta_2} \in \bigoplus_{i=1}^{\beta_1} R(-m_i)$ such that $\operatorname{Ker}(\partial_1) = \langle f_{1,i}, \ldots, f_{1,\beta_2} \rangle$. For each $i = 1, \ldots, \beta_2$ let $m_{1,i}$ denote the degree of $f_{1,i}$ and we may surject onto $\operatorname{Ker}(\partial_1)$ from the free module $\bigoplus_{i=1}^{\beta_2} R(-m_{1,i})$. Call this map τ_1. If $\iota_1\colon \operatorname{Ker}(\partial_1) \to \bigoplus_{i=1}^{\beta_1} R(-m_i)$ is the natural injection, then we define $\partial_2 = \iota_1 \circ \tau_1$ to produce the following commutative diagram .

$$\begin{tikzcd}
\bigoplus_{i=1}^{\beta_2} R(-m_{1,i}) \arrow[rr, "\begin{pmatrix} f_{1,1} & \cdots & f_{1,\beta_2} \end{pmatrix}", "\partial_2"'] \arrow[dr, "\tau_1"'] & & \bigoplus_{i=1}^{\beta_1} R(-m_i) \arrow[r, "\begin{pmatrix} f_1 & \cdots & f_{\beta_1} \end{pmatrix}", "\partial_1"'] & R \arrow[r, "\tau"] & R/I \arrow[r] & 0 \\
& \operatorname{Ker}(\partial_1) \arrow[ur, "\iota_1"'] \arrow[dr] & & & & \\
0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

Note that ∂_2 is given by a $\beta_1 \times \beta_2$ matrix. Note also that for each fixed i, each entry of $f_{1,i}$ is also homogeneous by the notation given in I.C.11. One can continue this procedure to produce the desired diagram. $\square$

CHAPTER I.D

Chain Complexes

Throughout this chapter, assume only that R is a commutative ring with identity.

FACT I.D.1. *Given R-module homomorphisms* $L \xrightarrow{f} M \xrightarrow{g} N$ *, we have* $\operatorname{Im}(f) \subseteq \operatorname{Ker}(g)$ *if and only if* $g \circ f = 0$.

DEFINITION I.D.2. A chain complex over R is a sequence of R-module homomorphisms

$$A = \qquad \cdots \xrightarrow{\partial^A_{i+2}} A_{i+1} \xrightarrow{\partial^A_{i+1}} A_i \xrightarrow{\partial^A_i} A_{i-1} \xrightarrow{\partial^A_{i-1}} \cdots$$

such that $\partial^A_i \circ \partial^A_{i+1} = 0$ for all $i \in \mathbb{Z}$. If A is an R-complex, then elements $a \in A_n$ have homological degree $|a| = n$.

NOTE I.D.3. We give a few remarks about the relationship between exact sequences and chain complexes.

(a) An exact sequence of R-module homomorphisms is an R-complex.

(b) R-complexes are not necessarily exact. For example, the R-complex $0 \longrightarrow M \longrightarrow 0$ is exact if and only if $M = 0$.

(c) Given an augmented free resolution

$$P^+ = \qquad \cdots \xrightarrow{\partial^P_3} P_2 \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \xrightarrow{\tau} M \longrightarrow 0,$$

the sequence

$$P^+ = \qquad \cdots \xrightarrow{\partial^P_3} P_2 \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \longrightarrow 0$$

is not exact in general, but is an R-complex called a (truncated) free resolution of M.

DEFINITION I.D.4. Let A be an R-complex as in Definition I.D.2. For all $n \in \mathbb{Z}$, denote by $Z_n(A) = Z_n$ the set of cycles of homological degree n and denote by $B_n(A) = B_n$ the set of boundaries of homological degree n, i.e.,

$$Z_n(A) = Z_n = \operatorname{Ker}\left(\partial^A_n\right) \subseteq A_n$$
$$B_n(A) = B_n = \operatorname{Im}\left(\partial^A_{n+1}\right) \subseteq Z_n,$$

where the containments on the right are as submodules. The n^{th} homology module of A is

$$H_n(A) = \frac{Z_n(A)}{B_n(A)}.$$

NOTE I.D.5. Let A be an R-complex.

(a) A is exact if and only if $Z_n = B_n$ for all $n \in \mathbb{Z}$ if and only if $H_n(A) = 0$ for all $n \in \mathbb{Z}$.

(b) Given an augmented free resolution P^+ as in Note I.D.3(c), we have

$$H_n(P) \cong \begin{cases} 0 & n \neq 0 \\ M & n = 0, \end{cases}$$

because of the following.

$$M = \operatorname{Im}(\tau) \cong \frac{P_0}{\operatorname{Ker}(\tau)} = \frac{P_0}{\operatorname{Im}\left(\partial^P_1\right)} = \frac{\operatorname{Ker}\left(\partial^P_0\right)}{\operatorname{Im}\left(\partial^P_1\right)} = H_0(P)$$

(c) Given an R-complex as in Definition I.D.2, if $\partial^A_{i+1} = 0$ and $\partial^A_i = 0$, then we have

$$H_i(A) = \frac{\operatorname{Ker}\left(\partial^A_i\right)}{\operatorname{Im}\left(\partial^A_{i+1}\right)} = \frac{A_i}{0} \cong A_i.$$

(d) Given an R-complex as in Definition I.D.2, if $\partial_{i+1}^A = 0$, then we have

$$H_i(A) = \frac{\operatorname{Ker}\left(\partial_i^A\right)}{0} \cong \operatorname{Ker}\left(\partial_i^A\right).$$

(e) Given an R-complex as in Definition I.D.2, if $\partial_i^A = 0$, then we have

$$H_i(A) = \frac{A_i}{\operatorname{Im}\left(\partial_{i+1}^A\right)} = \operatorname{Coker}\left(\partial_{i+1}^A\right).$$

Definition I.D.6. Let A and Y be R-complexes.

(a) The shift or suspension of A is an R-complex denoted ΣA where $(\Sigma A)_i = A_{i-1}$ and $\partial_i^{\Sigma A} = -\partial_{i-1}^A$.
(b) The direct sum of A and Y is the R-complex $A \oplus Y$ where $(A \oplus Y)_i = A_i \oplus Y_i$ and $\partial_i^{A\oplus Y}(a, y) = \left(\partial_i^A(a), \partial_i^Y(y)\right)$.

Remark I.D.7. The homology modules of ΣA are the homology modules of the original complex A:

$$H_i(\Sigma A) = H_{i-1}(A).$$

To see this, we observe

$$A = \quad \cdots \xrightarrow{\partial_{i+2}^A} A_{i+1} \xrightarrow{\partial_{i+1}^A} A_i \xrightarrow{\partial_i^A} A_{i-1} \xrightarrow{\partial_{i-1}^A} \cdots$$

$$\Sigma A = \quad \cdots \xrightarrow{-\partial_{i+1}^A} A_i \xrightarrow{-\partial_i^A} A_{i-1} \xrightarrow{-\partial_{i-1}^A} A_{i-2} \xrightarrow{-\partial_{i-2}^A} \cdots$$

and compute

$$H_i(\Sigma A) = \frac{\operatorname{Ker}\left(\partial_i^{\Sigma A}\right)}{\operatorname{Im}\left(\partial_{i+1}^{\Sigma A}\right)} = \frac{\operatorname{Ker}\left(-\partial_{i-1}^A\right)}{\operatorname{Im}\left(-\partial_i^A\right)} = \frac{\operatorname{Ker}\left(\partial_{i-1}^A\right)}{\operatorname{Im}\left(\partial_i^A\right)} = H_{i-1}(A).$$

The homology modules of $A \oplus Y$ are exactly what one might want them to be:

$$H_i(A \oplus Y) \cong H_i(A) \oplus H_i(Y).$$

This follows from the definition of $A \oplus Y$ and the first isomorphism theorem, which we show below.

$$\begin{matrix} A \\ \oplus \\ Y \end{matrix} \qquad \cdots \longrightarrow \begin{matrix} A_{i+1} \\ \oplus \\ Y_{i+1} \end{matrix} \xrightarrow{\begin{pmatrix} \partial_{i+1}^A & 0 \\ 0 & \partial_{i+1}^Y \end{pmatrix}} \begin{matrix} A_i \\ \oplus \\ Y_i \end{matrix} \xrightarrow{\begin{pmatrix} \partial_i^A & 0 \\ 0 & \partial_i^Y \end{pmatrix}} \begin{matrix} A_{i-1} \\ \oplus \\ Y_{i-1} \end{matrix} \longrightarrow \cdots$$

$$H_i(A \oplus Y) = \frac{\operatorname{Ker}\begin{pmatrix} \partial_i^A & 0 \\ 0 & \partial_i^Y \end{pmatrix}}{\operatorname{Im}\begin{pmatrix} \partial_{i+1}^A & 0 \\ 0 & \partial_{i+1}^Y \end{pmatrix}} = \frac{\begin{pmatrix} \operatorname{Ker}\left(\partial_i^A\right) \\ \oplus \\ \operatorname{Ker}\left(\partial_i^Y\right) \end{pmatrix}}{\begin{pmatrix} \operatorname{Im}\left(\partial_{i+1}^A\right) \\ \oplus \\ \operatorname{Im}\left(\partial_{i+1}^Y\right) \end{pmatrix}} \cong \begin{matrix} \left(\operatorname{Ker}\left(\partial_i^A\right) / \operatorname{Im}\left(\partial_{i+1}^A\right)\right) \\ \oplus \\ \left(\operatorname{Ker}\left(\partial_i^Y\right) / \operatorname{Im}\left(\partial_{i+1}^Y\right)\right) \end{matrix}$$

Definition I.D.8. A chain map between R-complexes A and Y is a commutative ladder diagram.

$$\begin{tikzcd} A \arrow[d, "\phi"'] & \cdots \arrow[r, "\partial_{i+2}^A"] & A_{i+1} \arrow[r, "\partial_{i+1}^A"] \arrow[d, "\phi_{i+1}"] & A_i \arrow[r, "\partial_i^A"] \arrow[d, "\phi_i"] & A_{i-1} \arrow[r, "\partial_{i-1}^A"] \arrow[d, "\phi_{i-1}"] & \cdots \\ Y & \cdots \arrow[r, "\partial_{i+2}^Y"'] & Y_{i+1} \arrow[r, "\partial_{i+1}^Y"'] & Y_i \arrow[r, "\partial_i^Y"'] & Y_{i-1} \arrow[r, "\partial_{i-1}^Y"'] & \cdots \end{tikzcd}$$

In other words, $\phi = \{\phi_i\}$ is a sequence of R-module homomorphisms $\phi_i \colon A_i \to Y_i$ such that the above diagram commutes, i.e., such that $\partial_i^Y \circ \phi_i = \phi_{i-1} \circ \partial_i^A$ for all $i \in \mathbb{Z}$. We say the ϕ_i's are "compatible with the differentials" of the complexes. (For those familiar with the language of categories, chain maps are the "morphisms in the category of R-complexes".) The chain map ϕ is an isomorphism if it has a two-sided inverse, i.e., if there exists a chain map $\psi \colon Y \to A$ such that $\psi_i \circ \phi_i = \mathrm{id}_{A_i}$ and $\phi_i \circ \psi_i = \mathrm{id}_{Y_i}$ for all $i \in \mathbb{Z}$.

EXAMPLE I.D.9. Let A and Y be R-complexes.

(a) The zero map $A \xrightarrow{0} Y$ is a chain map, since the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & A_i \arrow[r] \arrow[d, "0"'] & A_{i-1} \arrow[r] \arrow[d, "0"] & \cdots \\
\cdots \arrow[r] & Y_i \arrow[r] & Y_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

(b) For any $x \in R$, the "homothety" map $A \xrightarrow{x} A$ is a chain map, because the differentials ∂_i^A are R-linear, i.e., we have $\partial_i^A(xa) = x \cdot \partial_i^A(a)$ for all $a \in A_i$, so the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & A_i \arrow[r, "\partial_i^A"] \arrow[d, "x"'] & A_{i-1} \arrow[r] \arrow[d, "x"] & \cdots \\
\cdots \arrow[r] & A_i \arrow[r, "\partial_i^A"] & A_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

(c) Let the following be a free resolution of an R-module M.

$$P^+ = \qquad \cdots \xrightarrow{\partial_2^P} R^{\beta_1} \xrightarrow{\partial_1^P} R^{\beta_0} \xrightarrow{\tau} M \longrightarrow 0$$

Then the surjection τ determines the following chain map.

$$\begin{tikzcd}
P \arrow[d, "\tau"'] & \cdots \arrow[r, "\partial_2^P"] & R^{\beta_1} \arrow[r, "\partial_1^P"] \arrow[d] & R^{\beta_0} \arrow[r] \arrow[d, "\tau"] & 0 \arrow[d] \\
M & \cdots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

(d) If $\phi\colon A \to Y$ is a chain map, then ϕ is an isomorphism if and only if each ϕ_i is an isomorphism, i.e., if and only if each ϕ_i is 1-1 and onto. (One proves this with a relatively standard diagram chase.)

The next result says that chain maps induce maps on homology modules.

THEOREM I.D.10. *Let $\phi\colon A \to Y$ be a chain map.*

(a) *We have $\phi_i(Z_i(A)) \subseteq Z_i(Y)$ and $\phi_i(B_i(A)) \subseteq B_i(Y)$ for all $i \in \mathbb{Z}$.*

(b) *There exists a well-defined R-module homomorphism $H_i(\phi)\colon H_i(A) \to H_i(Y)$ given by*

$$H_i(\phi)(\overline{a}) = \overline{\phi(a)}.$$

PROOF. To prove part (a), first let $a \in Z_i(A)$. Then we have

$$\left(\phi_{i-1} \circ \partial_i^A\right)(a) = \phi_{i-1}(0) = 0$$

by definition of $Z_i(A)$. Therefore since ϕ is chain map we have

$$\left(\partial_i^Y \circ \phi_i\right)(a) = \left(\phi_{i-1} \circ \partial_i^A\right)(a) = 0$$

which implies $\phi_i(a) \in Z_i(Y)$.

Second, if we let $b \in B_i(A)$, then there exists an element $c \in A_{i+1}$ such that $\partial_{i+1}^A(c) = b$. Since ϕ is a chain map we have

$$\phi(b) = \left(\phi_i \circ \partial_{i+1}^A\right)(c) = \left(\partial_{i+1}^Y \circ \phi_{i+1}\right)(c) \in B_i(Y).$$

Now we prove (b). Let $Z_i(\phi)$ and $B_i(\phi)$ each be given by the same rule as ϕ_i with the appropriate restricted domain and codomain. By part (a) we have the following commutative diagram.

$$\begin{tikzcd}
B_i(A) \arrow[r, "\subseteq"] \arrow[d, "B_i(\phi)"'] & Z_i(A) \arrow[r, "\subseteq"] \arrow[d, "Z_i(\phi)"'] & A_i \arrow[d, "\phi_i"'] \\
B_i(Y) \arrow[r, "\subseteq"'] & Z_i(Y) \arrow[r, "\subseteq"'] & Y_i
\end{tikzcd}$$

We claim the following is also a commutative diagram, where τ_A and τ_Y are the natural surjections.

$$\begin{tikzcd}
B_i(A) \arrow[r, "\subseteq"] \arrow[d, "B_i(\phi)"'] & Z_i(A) \arrow[r, "\tau_A"] \arrow[d, "Z_i(\phi)"'] & H_i(A) \ni \overline{a} \arrow[d, dashed, "H_i(\phi)"'] \arrow[d, mapsto, shift left=6] \\
B_i(Y) \arrow[r, "\subseteq"'] & Z_i(Y) \arrow[r, "\tau_Y"'] & H_i(Y) \ni \overline{\phi(a)}
\end{tikzcd}$$

Note it suffices to show that $H_i(\phi)$ is well-defined and R-linear, since the commutivity of the diagram is by construction. We know $H_i(\phi)$ lands well by the first equality in part (a). To show $H_i(\phi)$ preserves equality (i.e., is independent of our choice of representative), let $a, a' \in Z_i(A)$ such that $\overline{a} = \overline{a'}$ in $H_i(A)$. This implies $a - a' \in B_i(A)$ and therefore it is now straightforward to show that $H_i(\phi)$ is R-linear. By the second equality in part (a) we have $\phi(a) - \phi(a') = \phi(a - a') \in B_i(Y)$, i.e., $\overline{\phi(a)} - \overline{\phi(a')} = 0 \in H_i(Y)$. $\square$

DEFINITION I.D.11. A quasiisomorphism is a chain map $\phi\colon A \to Y$ such that $H_i(\phi)\colon H_i(A) \to H_i(Y)$ is an isomorphism for all $i \in \mathbb{Z}$.

EXAMPLE I.D.12. Let A and Y be R-complexes.

(a) The zero map $A \xrightarrow{0} Y$ induces the zero map on homology since

$$H_i(0)(\overline{a}) = \overline{0(a)} = \overline{0} = 0.$$

(b) For a fixed $x \in R$, the homothety map $A \xrightarrow{x} A$ induces a homothety map on homology, since

$$H_i(x)(\overline{a}) = \overline{x \cdot a} = x \cdot \overline{a}.$$

One might also use the more cumbersome, yet more transparent notation $\mu^{A,x}$ to denote the homothety map on A by the element x. With this notation, the above display says that $H_i(\mu^{A,x}) = \mu^{H_i(A),x}$.

As a for instance, if $x \in R$ is a unit, then $\mu^{A,x}$ is an isomorphism and the preceding paragraph implies $H_i(\mu^{A,x})$ is an isomorphism for all $i \in \mathbb{Z}$. Hence $\mu^{A,x}$ is a quasiisomorphism. In general, we will see in Proposition I.D.13 that if ϕ is an isomorphism, then it is also a quasiisomorphism.

(c) Recall from Example I.D.9 that the augmented free resolution P^+ of an R-module M determines a chain map $\tau\colon P \to M$. We claim τ is a quasiisomorphism. To see that $H_i(\tau)$ is an isomorphism for every $i \in \mathbb{Z}$, first note that for all $i \neq 0$ one has $H_i(P) = 0 = H_i(M)$ since P is exact and $C_i = 0$ for all $i \neq 0$. Hence $H_i(\tau)$ is the identity map on the zero module and is therefore an isomorphism for all $i \neq 0$. When $i = 0$, we have the following commutative diagram.

$$\begin{tikzcd}
H_0(P) \arrow[r, "H_0(\tau)", "\therefore\cong"'] \arrow[d, "\alpha"', "\cong"] & H_0(M) \arrow[d, equal] \\
M \arrow[r, "\cong", "\gamma"'] & \frac{M}{0}
\end{tikzcd}$$

The map γ is the natural surjection and is an isomorphism by the first isomorphism theorem. Observe also that

$$H_0(P) = \frac{R^{\beta_0}}{\operatorname{Im}\left(\partial_1^P\right)} = \frac{R^{\beta_0}}{\operatorname{Ker}\left(\tau\right)}$$

so the surjection α induced by τ is an isomorphism also by the first isomorphism theorem. Hence $H_0(\tau)$ is a composition of isomorphisms and is therefore itself an isomorphism.

PROPOSITION I.D.13. *Let $\phi\colon A \to Y$ and $\psi\colon C \to A$ be chain maps.*

(a) *The composition $\phi \circ \psi\colon C \to Y$ is a chain map.*
(b) *$H_i(-)$ respects compositions, i.e., $H_i(\phi \circ \psi) = H_i(\phi) \circ H_i(\psi)$ for all $i \in \mathbb{Z}$.*
(c) *If ϕ is an isomorphism, then ϕ is a quasiisomorphism.*

PROOF. (a) This is proved using a standard diagram chase on the following section of a ladder diagram.

$$\begin{tikzcd} C_i \arrow[r, "\partial_i^C"] \arrow[d, "\psi_i"'] & C_{i-1} \arrow[d, "\psi_{i-1}"] \\ A_i \arrow[r, "\partial_i^A"] \arrow[d, "\phi_i"'] & A_{i-1} \arrow[d, "\phi_{i-1}"] \\ Y_i \arrow[r, "\partial_i^Y"] & Y_{i-1} \end{tikzcd}$$

(b) Note that we are trying to prove that the commutative diagram

$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[dr, "\phi\circ\psi"'] & A \arrow[d, "\phi"] \\ & Y \end{tikzcd}$$

given in (a) induces the following commutative diagram.

$$\begin{tikzcd} H_i(C) \arrow[r, "H_i(\psi)"] \arrow[dr, "H_i(\phi\circ\psi)"'] & H_i(A) \arrow[d, "H_i(\phi)"] \\ & H_i(Y) \end{tikzcd}$$

To show this, for any $c \in C_i$ we have

$$H_i(\phi \circ \psi)(\overline{c}) = \overline{(\phi \circ \psi)(c)} = \overline{\phi(\psi(c))} = H_i(\phi)\left(\overline{\psi(c)}\right) = H_i(\phi)\left(H_i(\psi)(\overline{c})\right) = \left(H_i(\phi) \circ H_i(\psi)\right)(\overline{c}).$$

(c) Assume ϕ is an isomorphism and let $\zeta\colon Y \to A$ be its two-sided inverse. Then the composition $\phi \circ \zeta$ is equal to the homothety map $\mu^{Y,1}$. Moreover, by Example I.D.12(b) and by part (b) we have

$$H_i(\phi) \circ H_i(\zeta) = H_i(\phi \circ \zeta) = H_i(\mu^{Y,1}) = \mu^{H_i(Y),1} = \mathrm{id}_{H_i(Y)}.$$

Similarly we have

$$H_i(\zeta) \circ H_i(\phi) = H_i(\zeta \circ \phi) = H_i(\mu^{A,1}) = \mu^{H_i(A),1} = \mathrm{id}_{H_i(A)}.$$

Hence $H_i(\phi)$ is an isomorphism with the two-sided inverse $H_i(\zeta)$, i.e.,

$$(H_i(\phi))^{-1} = H_i(\phi^{-1}).$$

□

EXAMPLE I.D.14. The converse of Proposition I.D.13 fails in general. The chain map $\tau\colon P \to M$ from Example I.D.12 is a quasiisomorphism, but is almost never an isomorphism. For instance, if $M \ncong R^{\beta_0}$, then it is not an isomorphism.

DEFINITION I.D.15. A short exact sequence of chain maps is a sequence

$$0 \longrightarrow A \xrightarrow{\phi} C \xrightarrow{\psi} D \longrightarrow 0$$

of chain maps such that each "level" is a short exact sequence

$$0 \longrightarrow A_i \xrightarrow{\phi_i} C_i \xrightarrow{\psi_i} D_i \longrightarrow 0.$$

For diagram chases it may also be convenient to display a short exact sequence of chain maps as follows.

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial^A_{i+2}"'] & \vdots \arrow[d, "\partial^C_{i+2}"'] & \vdots \arrow[d, "\partial^D_{i+2}"'] & \\
0 \arrow[r] & A_{i+1} \arrow[r, "\phi_{i+1}"] \arrow[d, "\partial^A_{i+1}"'] & C_{i+1} \arrow[r, "\psi_{i+1}"] \arrow[d, "\partial^C_{i+1}"'] & D_{i+1} \arrow[r] \arrow[d, "\partial^D_{i+1}"'] & 0 \\
0 \arrow[r] & A_i \arrow[r, "\phi_i"] \arrow[d, "\partial^A_i"'] & C_i \arrow[r, "\psi_i"] \arrow[d, "\partial^C_i"'] & D_i \arrow[r] \arrow[d, "\partial^D_i"'] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r, "\phi_{i-1}"] \arrow[d, "\partial^A_{i-1}"'] & C_{i-1} \arrow[r, "\psi_{i-1}"] \arrow[d, "\partial^C_{i-1}"'] & D_{i-1} \arrow[r] \arrow[d, "\partial^D_{i-1}"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Below we present Theorem I.D.16 along with two different proofs. The first requires the Snake Lemma, which we present as an unnumbered result before giving the theorem.

Lemma (Snake Lemma). *Given a commutative diagram of R-modules with exact rows*

$$\begin{tikzcd}
 & U \arrow[r, "f"] \arrow[d, "u"] & V \arrow[r, "g"] \arrow[d, "v"] & W \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & U' \arrow[r, "f'"'] & V' \arrow[r, "g'"'] & W' &
\end{tikzcd}$$

there exists an exact sequence

$$\begin{tikzcd}
\operatorname{Ker}(u) \arrow[r, "\tilde{f}"] & \operatorname{Ker}(v) \arrow[r, "\tilde{g}"] & \operatorname{Ker}(w) \arrow[r, "\sigma"] & \operatorname{Coker}(u) \arrow[r, "\overline{f'}"] & \operatorname{Coker}(v) \arrow[r, "\overline{g'}"] & \operatorname{Coker}(w)
\end{tikzcd}$$

$$x \longmapsto f(x) \qquad\qquad \overline{x'} \longmapsto \overline{f'(x')}$$

$$y \longmapsto g(y) \qquad\qquad \overline{y'} \longmapsto \overline{g'(y')}$$

where $\operatorname{Coker}(u) = U'/\operatorname{Im}(u)$*, and the other cokernels are defined similarly. The map σ is defined as follows. Let $z \in \operatorname{Ker}(w)$. Then $w(z) = 0$ and since g is surjective let $y \in V$ such that $g(y) = z$. By the commutivity of the diagram we have*

$$g'(v(y)) = w(g(y)) = w(z) = 0,$$

so $v(y) \in \operatorname{Ker}(g') = \operatorname{Im}(f')$. Let $x' \in U'$ such that $f'(x') = v(y)$. Then $\sigma(z)$ is defined as

$$\sigma(z) = \overline{x'}.$$

Now we present the theorem promised. The first proof uses the Snake Lemma and the second is a more "manual" proof.

Theorem I.D.16. *Given a short exact sequence of chain maps as in Definition I.D.15, there exists the following long exact sequence on homology.*

$$\begin{tikzcd}
\cdots \arrow[r, "\eth_{i+1}"] & H_i(A) \arrow[r, "H_i(\phi)"] & H_i(C) \arrow[r, "H_i(\psi)"] & H_i(D) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}, "\eth_i"'] \\
 & H_{i-1}(A) \arrow[r, "H_{i-1}(\phi)"] & \cdots &
\end{tikzcd}$$

We call $\eth_i$ a <u>*connecting homomorphism*</u>*.*

Proof. First let us construct $\eth_i$. It will be helpful to have the following section of ladder diagram in view for this part.

$$\begin{tikzcd} & & C_i \arrow[r, "\psi_i"] \arrow[d, "\partial_i^C"'] & D_i \arrow[r] \arrow[d, "\partial_i^D"] & 0 \\ 0 \arrow[r] & A_{i-1} \arrow[r, "\phi_{i-1}"'] & C_{i-1} \arrow[r, "\psi_{i-1}"'] & D_{i-1} \arrow[r] & 0 \end{tikzcd}$$

Let $\overline{d} \in H_i(D)$ and we want to define $\eth_i(\overline{d})$. Since $d \in Z_i(D)$, we know $\partial_i^D(d) = 0$ and since ψ_i is surjective let $c \in C_i$ such that $\psi_i(c) = d$. Since ψ is a chain map we have

$$\psi_{i-1}\left(\partial_i^C(c)\right) = \partial_i^D\left(\psi_i(c)\right) = \partial_i^D(d) = 0,$$

so $\partial_i^C(c) \in \operatorname{Ker}(\psi_{i-1}) = \operatorname{Im}(\phi_{i-1})$. Therefore let $a \in A_{i-1}$ such that $\phi_{i-1}(a) = \partial_i(c)$. We define

$$\eth_i\left(\overline{d}\right) = \overline{a} \in H_{i-1}(A).$$

We claim the following is a commutative diagram R-module homomorphisms with exact rows.

$$\begin{tikzcd} & \operatorname{Coker}\left(\partial_{i+2}^A\right) \arrow[r, "\overline{\phi_{i+1}}"] \arrow[d, "\widehat{\partial_{i+1}^A}"'] & \operatorname{Coker}\left(\partial_{i+2}^C\right) \arrow[r, "\overline{\psi_{i+1}}"] \arrow[d, "\widehat{\partial_{i+1}^C}"'] & \operatorname{Coker}\left(\partial_{i+2}^D\right) \arrow[r] \arrow[d, "\widehat{\partial_{i+1}^D}"'] & 0 \\ 0 \arrow[r] & \operatorname{Ker}\left(\partial_i^A\right) \arrow[r, "\widetilde{\phi_i}"'] & \operatorname{Ker}\left(\partial_i^C\right) \arrow[r, "\widetilde{\psi_i}"'] & \operatorname{Ker}\left(\partial_i^D\right) & \end{tikzcd} \tag{I.D.16.1}$$

Step 1: We show that

$$\widehat{\partial_{i+1}^A} \colon \operatorname{Coker}\left(\partial_{i+2}^A\right) \longmapsto \operatorname{Ker}\left(\partial_i^A\right)$$

$$\overline{a} \longmapsto \partial_{i+1}^A(a)$$

is a well-defined R-module homomorphism. Since $\partial_{i+1}^A(A_{i+1}) = B_i(A) \subseteq Z_i(A)$, we may restrict the codomain of ∂_{i+1}^A to get the well-defined R-module homomorphism $\zeta\colon A_{i+1} \to Z_i(A)$. Since $B_{i+1}(A) \subseteq Z_{i+1}(A)$ we also have $\zeta(B_{i+1}(A)) = \partial_{i+1}^A(B_{i+1}(A)) = 0$. Therefore we have the commutative diagram

$$\begin{tikzcd} A_{i+1} \arrow[r, two heads] \arrow[d, "\zeta"'] & A_{i+1}/B_{i+1}(A) \arrow[dl, dashed, "\exists!\widehat{\partial_{i+1}^A}"] \\ Z_i(A) & \end{tikzcd}$$

where $\widehat{\partial_{i+1}^A}(\overline{x}) = \partial_{i+1}^A(x)$. Moreover, we have

$$\operatorname{Im}\left(\widehat{\partial_{i+1}^A}\right) = \operatorname{Im}(\zeta) = \operatorname{Im}\left(\partial_{i+1}^A\right)$$

and

$$\operatorname{Ker}\left(\widehat{\partial_{i+1}^A}\right) = \frac{\operatorname{Ker}(\zeta)}{B_{i+1}(A)} = \frac{\operatorname{Ker}\left(\partial_{i+1}^A\right)}{\operatorname{Im}\left(\partial_{i+2}^A\right)} = H_{i+1}(A).$$

Identical arguments can be used to show the well-definedness of $\widehat{\partial_{i+1}^C}$ and $\widehat{\partial_{i+1}^D}$ as well. One also finds that $\operatorname{Im}\left(\widehat{\partial_{i+1}^C}\right) = \operatorname{Im}\left(\partial_{i+1}^C\right)$, $\operatorname{Im}\left(\widehat{\partial_{i+1}^D}\right) = \operatorname{Im}\left(\partial_{i+1}^D\right)$, $\operatorname{Ker}\left(\widehat{\partial_{i+1}^C}\right) = H_{i+1}(C)$, and $\operatorname{Ker}\left(\widehat{\partial_{i+1}^D}\right) = H_{i+1}(D)$ as well.

Step 2: We apply the Snake Lemma to

$$\begin{tikzcd} 0 \arrow[r] & A_{i+2} \arrow[r, "\phi_{i+2}"] \arrow[d, "\partial_{i+2}^A"'] & C_{i+2} \arrow[r, "\psi_{i+2}"] \arrow[d, "\partial_{i+2}^C"'] & D_{i+2} \arrow[r] \arrow[d, "\partial_{i+2}^D"'] & 0 \\ 0 \arrow[r] & A_{i+1} \arrow[r, "\phi_{i+1}"'] & C_{i+1} \arrow[r, "\psi_{i+1}"'] & D_{i+1} \arrow[r] & 0 \end{tikzcd}$$

to get the exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & \operatorname{Coker}\left(\partial^A_{i+2}\right) \arrow[r, "\overline{\phi_{i+1}}"] & \operatorname{Coker}\left(\partial^C_{i+2}\right) \arrow[r, "\overline{\psi_{i+1}}"] & \operatorname{Coker}\left(\partial^D_{i+2}\right) \arrow[r] & 0 \end{tikzcd}$$

$$\overline{a} \longmapsto \overline{\phi_{i+1}(a)} \qquad \overline{c} \longmapsto \overline{\psi_{i+1}(c)}$$

where the exactness on the right follows from the surjectivity of ψ_{i+1}.

Step 3: We apply the Snake Lemma to

$$\begin{tikzcd} 0 \arrow[r] & A_i \arrow[r, "\phi_i"] \arrow[d, "\partial^A_i"'] & C_i \arrow[r, "\psi_i"] \arrow[d, "\partial^C_i"'] & D_i \arrow[r] \arrow[d, "\partial^D_i"'] & 0 \\ 0 \arrow[r] & A_{i-1} \arrow[r, "\phi_{i-1}"'] & C_{i-1} \arrow[r, "\psi_{i-1}"'] & D_{i-1} \arrow[r] & 0 \end{tikzcd}$$

to get the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker}\left(\partial^A_i\right) \arrow[r, "\widetilde{\phi}_i"] & \operatorname{Ker}\left(\partial^C_i\right) \arrow[r, "\widetilde{\psi}_i"] & \operatorname{Ker}\left(\partial^D_i\right) \arrow[r] & \cdots \end{tikzcd}$$

$$a \longmapsto \phi_i(a) \qquad c \longmapsto \psi_i(c)$$

where the exactness on the left follows from the injectivity of ϕ_i.

Step 4: We show that (I.D.16.1) commutes. For any $\overline{c} \in \operatorname{Coker}\left(\partial^C_{i+2}\right)$ we have

$$\widehat{\partial^D_{i+1}}\left(\overline{\psi_{i+1}}(\overline{c})\right) = \widehat{\partial^D_{i+1}}\left(\overline{\psi_{i+1}(c)}\right) = \partial^D_{i+1}\left(\psi_{i+1}(c)\right)$$

and

$$\widetilde{\psi}_i\left(\widehat{\partial^C_{i+1}}(\overline{c})\right) = \widetilde{\psi}_i\left(\partial^C_{i+1}(c)\right) = \psi_i\left(\partial^C_{i+1}(c)\right),$$

which are equivalent since ψ is a chain map. One can similarly show that the left square commutes using the fact that ϕ is a chain map.

Step 5: From the conclusion of Step 1 we have

$$\operatorname{Coker}\left(\widehat{\partial^A_{i+1}}\right) = \frac{\operatorname{Ker}\left(\partial^A_i\right)}{\operatorname{Im}\left(\widehat{\partial^A_{i+1}}\right)} = \frac{\operatorname{Ker}\left(\partial^A_i\right)}{\operatorname{Im}\left(\partial^A_{i+1}\right)} = H_i(A),$$

as well as $\operatorname{Coker}\left(\widehat{\partial^C_{i+1}}\right) = H_i(C)$ and $\operatorname{Coker}\left(\widehat{\partial^D_{i+1}}\right) = H_i(D)$.

Step 6: Having established our claim that (I.D.16.1) is a commutative diagram, we apply the Snake Lemma once more to obtain the following exact sequence.

$$\begin{tikzcd}[column sep=small] \operatorname{Ker}\left(\widehat{\partial^A_{i+1}}\right) \arrow[r, "\widetilde{\overline{\phi_{i+1}}}", "H_{i+1}(\phi)"'] \arrow[d, equal] & \operatorname{Ker}\left(\widehat{\partial^C_{i+1}}\right) \arrow[r, "\widetilde{\overline{\psi_{i+1}}}", "H_{i+1}(\psi)"'] \arrow[d, equal] & \operatorname{Ker}\left(\widehat{\partial^D_{i+1}}\right) \arrow[r, "\sigma", "\eth_{i+1}"'] \arrow[d, equal] & \operatorname{Coker}\left(\widehat{\partial^A_{i+1}}\right) \arrow[r, "\overline{\widetilde{\phi_i}}", "H_i(\phi)"'] \arrow[d, equal] & \operatorname{Coker}\left(\widehat{\partial^C_{i+1}}\right) \arrow[r, "\overline{\widetilde{\psi_i}}", "H_i(\psi)"'] \arrow[d, equal] & \operatorname{Coker}\left(\widehat{\partial^D_{i+1}}\right) \arrow[d, equal] \\ H_{i+1}(A) & H_{i+1}(C) & H_{i+1}(D) & H_i(A) & H_i(C) & H_i(D) \end{tikzcd}$$

$$\overline{a} \longmapsto \overline{\phi_{i+1}(a)} \qquad\qquad \overline{c} \longmapsto \overline{\psi_i(c)}$$

It remains only to justify that the map σ given by the Snake Lemma is the same map $\eth_{i+1}$ that we constructed. To do so we perform a diagram chase on (I.D.16.1). For any $\overline{d} \in \operatorname{Ker}\left(\widehat{\partial^D_{i+2}}\right)$ we have $\partial^D_{i+2}(d) = \widehat{\partial^D_{i+2}}\left(\overline{d}\right) = 0$ and by the exactness of the top row we let $\overline{c} \in \operatorname{Coker}\left(\partial^C_{i+2}\right)$ such that $\overline{\psi_{i+1}(c)} = \overline{\psi_{i+1}}(\overline{c}) = \overline{d}$. Since the right square commutes we have

$$\psi_i(\partial^C_{i+1}(c)) = \widetilde{\psi}_i\left(\widehat{\partial^C_{i+1}}(\overline{c})\right) = \widehat{\partial^D_{i+1}}\left(\overline{\psi_{i+1}}(\overline{c})\right) = \widehat{\partial^D_{i+1}}\left(\overline{d}\right) = 0,$$

so $\widehat{\partial^C_{i+1}}(\overline{c}) \in \operatorname{Ker}\left(\widetilde{\psi_i}\right) = \operatorname{Im}\left(\widetilde{\phi_i}\right)$ and $\partial^C_{i+1}(c) \in \ker(\psi_i) = \operatorname{Im}(\phi_i)$. Let $a \in \operatorname{Ker}\left(\partial^A_i\right)$ such that $\phi_i(a) = \widetilde{\phi_i}(a) = \widehat{\partial^C_{i+1}}(\overline{c}) = \partial^C_{i+1}(c)$. Comparing the rules defining σ and $\eth_i$, we conclude that

$$\sigma\left(\overline{d}\right) = \overline{a} = \eth_i\left(\overline{d}\right).$$

□

Alternate proof. As with the previous proof, the first step is to construct $\eth$. Since this argument is the same as that in the previous proof, we begin with the second step.

Step 2: We show $\eth_i$ is well-defined. First we have

$$\begin{aligned}
\phi_{i-2}\left(\partial^A_{i-1}(a)\right) &= \partial^C_{i-1}\left(\phi_{i-1}(a)\right) && \phi \text{ a chain map}\\
&= \partial^C_{i-1}\left(\partial^C_i(c)\right) && \text{definition of } a\\
&= 0. && C \text{ an } R\text{-complex}
\end{aligned}$$

Since ϕ_{i-2} is injective, this implies $\partial^A_{i-1}(a) = 0$, i.e., $a \in \operatorname{Ker}\left(\partial^A_{i-1}\right)$, as desired.

Second we will show $\overline{a} \in H_{i-1}(A)$ is independent of any choices made in Step 1. Let $d, d' \in \operatorname{Ker}\left(\partial^D_i\right)$ such that $\overline{d} = \xi = \overline{d'}$, let $c, c' \in C_i$ such that $\psi_i(c) = d$ and $\psi_i(c') = d'$, and let $a, a' \in A_{i-1}$ such that $\phi_{i-1}(a) = \partial^C_i(c)$ and $\phi_{i-1}(a') = \partial^C_i(c')$. We need to show $\overline{a} = \overline{a'}$ in $H_{i-1}(A) = \operatorname{Ker}\left(\partial^A_{i-1}\right) / \operatorname{Im}\left(\partial^A_i\right)$, or in other words, we need to show $a - a' \in \operatorname{Im}\left(\partial^A_i\right)$.

By assumption $\overline{d} = \overline{d'} \in H_i(A) = \operatorname{Ker}\left(\partial^A_i\right) / \operatorname{Im}\left(\partial^A_{i+1}\right)$, so $d - d' \in \operatorname{Im}\left(\partial^A_{i+1}\right)$ and we let $\eta \in D_{i+1}$ such that $\partial^D_{i+1}(\eta) = d - d'$. Since ψ_{i+1} is surjective, we may let $\nu \in C_{i+1}$ such that $\psi_{i+1}(\nu) = \eta$ and we compute the following.

$$\psi_i(c - c' - \partial^C_{i+1}(\nu)) = \psi_i(c) - \psi_i(c') - (\psi_i \circ \partial^C_{i+1})(\nu) = d - d' - (d - d') = 0$$

In the above calculation we rely only on the definitions of our elements and the linearity of ψ_i. By this calculation we know $c - c' - \partial^C_{i+1}(\nu) \in \ker(\psi_i) = \operatorname{Im}(\phi_i)$ so let $\omega \in A_i$ such that $\phi_i(\omega) = c - c' - \partial^C_{i+1}(\nu)$. Since $a, a', \partial^A_i(\omega) \in A_{i-1}$, we compute as follows.

$$\begin{aligned}
\phi_{i-1}\left(\partial^A_i(\omega) - (a - a')\right) &= \left(\phi_{i-1} \circ \partial^A_i\right)(\omega) - \phi_{i-1}(a) + \phi_{i-1}(a') && \text{linearity}\\
&= \left(\partial^C_i \circ \phi_i\right)(\omega) - \partial^C_i(c) + \partial^C_i(c') && \phi \text{ a chain map}\\
&= \partial^C_i\left(c - c' - \partial^C_{i+1}(\nu)\right) - \partial^C_i(c) + \partial^C_i(c') && \text{definition of } \omega\\
&= \partial^C_i\left(c - c' - \partial^C_{i+1}(\nu) - c + c'\right) && \text{linearity}\\
&= -\left(\partial^C_i \circ \partial^C_{i+1}\right)(\nu) &&\\
&= 0 && C \text{ an } R\text{-complex}
\end{aligned}$$

Since ϕ_{i-1} is injective, this implies $\partial^A_i(\omega) - (a - a') = 0$ or equivalently

$$a - a' = \partial^A_i(\omega) \in \operatorname{Im}\left(\partial^A_i\right)$$

completing this step.

Step 3: Here we prove $\eth_i$ is an R-module homomorphism. Let $\xi, \xi' \in H_i(D)$ and $r \in R$. Also let $d, d' \in \operatorname{Ker}\left(\partial^D_i\right)$ such that $\overline{d} = \xi$ and $\overline{d'} = \xi'$, let $c, c' \in C_i$ such that $\psi_i(c) = d$ and $\psi_i(c') = d'$, and let $a, a' \in A_{i-1}$ such that $\phi_{i-1}(a) = \partial^C_i(c)$ and $\phi_{i-1}(a') = \partial^C_i(c')$.

Notice that $rd + d' \in \operatorname{Ker}\left(\partial^D_i\right)$ and hence it makes sense to write $\overline{rd + d'} = r\xi + \xi'$. Notice also that $rc + c' \in C_i$ so we have

$$\psi_i(rc + c') = \psi_i(rc) + \psi_i(c') = r \cdot \psi_i(c) + \psi_i(c') = rd + d'.$$

Finally note that $ra + a' \in A_{i-1}$ for which we have

$$\begin{aligned}
\phi_{i-1}(ra + a') &= \phi_{i-1}(ra) + \phi_{i-1}(a') = r \cdot \phi_{i-1}(a) + \phi_{i-1}(a')\\
&= r \cdot \partial^C_i(c) + \partial^C_i(c') = \partial^C_i(rc) + \partial^C_i(c') = \partial^C_i(rc + c').
\end{aligned}$$

Therefore we have an element satisfying the definition of $\eth_i$ described in Step 1 of the previous proof so we conclude this step in the following display.

$$\eth_i(r\xi + \xi') = \overline{ra + a'} = r \cdot \overline{a} + \overline{a} = r \cdot \eth_i(\xi) + \eth_i(\xi)$$

Step 4: We tackle the first of several questions of exactness. Here we show $\mathrm{Im}\,(H_i(\phi)) \subseteq \mathrm{Ker}\,(H_i(\psi))$. Let $\delta \in H_i(A)$ and let $\rho \in \mathrm{Ker}\,\left(\partial_i^A\right)$ such that $\overline{\rho} = \delta$. Therefore we have

$$H_i(\psi)\,(H_i(\phi)(\delta)) = H_i(\psi)\left(\overline{\phi_i(\rho)}\right) = \overline{(\psi_i \circ \phi_i)(\rho)} = \overline{0} = 0$$

where the third equality comes from the exactness of the original sequence of chain maps.

Step 5: We now show $\mathrm{Im}\,(H_i(\phi)) \supseteq \mathrm{Ker}\,(H_i(\psi))$. Let $\delta \in \mathrm{Ker}\,(H_i(\psi))$ and let $\rho \in \mathrm{Ker}\,\left(\partial_i^C\right)$ such that $\overline{\rho} = \delta$. This gives

$$0 = H_i(\psi)(\overline{\rho}) = \overline{\psi_i(\rho)} \in H_i(D) = \frac{\mathrm{Ker}\,\left(\partial_i^D\right)}{\mathrm{Im}\,\left(\partial_{i+1}^D\right)}.$$

Therefore $\psi_i(\rho) \in \mathrm{Im}\,\left(\partial_{i+1}^D\right)$ so we lift to some $\mu \in D_{i+1}$ such that $\partial_{i+1}^D(\mu) = \psi_i(\rho)$ and lift again to some $\sigma \in C_{i+1}$ such that $\psi_{i+1}(\sigma) = \mu$ (since ψ_{i+1} is surjective). Since $\rho, \partial_{i+1}^C(\sigma) \in C_i$, we consider the element $\rho - \partial_{i+1}^C(\sigma) \in C_i$. Using linearity and the fact that ψ is a chain map we compute

$$\psi_i(\rho - \partial_{i+1}^C(\sigma)) = \psi_i(\rho) - (\psi_i \circ \partial_{i+1}^C)(\sigma) = \psi_i(\rho) - (\partial_{i+1}^D \circ \psi_{i+1})(\sigma) = \psi_i(\rho) - \partial_{i+1}^D(\mu) = 0.$$

Hence $\rho - \partial_{i+1}^C(\sigma) \in \ker(\psi_i) = \mathrm{Im}\,(\phi_i)$ and we let $\tau \in A_i$ such that $\phi_i(\tau) = \rho - \partial_{i+1}^C(\sigma)$. We claim $\tau \in \mathrm{Ker}\,\left(\partial_i^A\right)$ and point out it suffices to show $\left(\phi_{i-1} \circ \partial_i^A\right)(\tau) = 0$ since ϕ_{i-1} is injective. We compute

$$\left(\phi_{i-1} \circ \partial_i^A\right)(\tau) = \partial_i^C(\phi_i(\tau)) = \partial_i^C(\rho - \partial_{i+1}^C(\sigma)) = \partial_i^C(\rho) - \left(\partial_i^C \circ \partial_{i+1}^C\right)(\sigma) = 0$$

where the last equality holds by definition of ρ and because C is a chain complex.

We consider $\rho, \partial_{i+1}^C(\sigma) \in \mathrm{Ker}\,\left(\partial_i^C\right)$ and $\tau \in \mathrm{Ker}\,\left(\partial_i^A\right)$, which represent the cosets $\overline{\rho}, \overline{\partial_{i+1}^C(\sigma)} \in H_i(C)$ and $\overline{\tau} \in H_i(A)$. Therefore it makes sense to compute

$$H_i(\phi)(\overline{\tau}) = \overline{\phi_i(\tau)} = \overline{\rho - \partial_{i+1}^C(\sigma)} = \overline{\rho} - \overline{\partial_{i+1}^C(\sigma)} = \overline{\rho} - \overline{0} = \overline{\rho} = \delta.$$

Hence $\delta \in \mathrm{Im}\,(H_i(\phi))$, completing this step.

Step 6: Continuing our proof of exactness, we show here that $\mathrm{Im}\,(H_i(\psi)) \subseteq \mathrm{Ker}\,(\eth_i)$. Let $\zeta \in H_i(C)$ and let $c \in \mathrm{Ker}\,\left(\partial_i^C\right)$ such that $\overline{c} = \zeta$. We want to show that $(\eth_i \circ H_i(\psi))\,(\overline{c}) = 0$. Define $d = \psi_i(c)$ and we have

$$H_i(\psi)(\overline{c}) = \overline{\psi_i(c)} = \overline{d}.$$

Computing $\eth_i(H_i(\psi)(\overline{c})) = \eth_i(\overline{d})$ requires some $a \in \mathrm{Ker}\,\left(\partial_{i-1}^A\right)$ such that $\phi_{i-1}(a) = \partial_i^C(c)$. Since $c \in \mathrm{Ker}\,\left(\partial_i^C\right)$ by assumption, $\partial_i^C(c) = 0 = \phi_{i-1}(0)$, so setting $a = 0$ we get

$$\eth_i(\overline{d}) = \overline{a} = \overline{0} = 0.$$

Step 7: We now show $\mathrm{Im}\,(H_i(\psi)) \supseteq \mathrm{Ker}\,(\eth_i)$. Let $\xi \in \mathrm{Ker}\,(\eth_i) \subseteq H_i(D)$ and let $d \in \mathrm{Ker}\,\left(\partial_i^D\right)$ such that $\xi = \overline{d}$. Fix some $c \in C_i$ such that $\psi_i(c) = d$ and some $a \in A_{i-1}$ such that $\phi_{i-1}(a) = \partial_i^C(c) \in \ker(\psi_{i-1}) = \mathrm{Im}\,(\phi_{i-1})$. Our construction in Step 1 implies $\eth_i(\xi) = \overline{a}$ so we have

$$0 = \eth_i(\xi) = \overline{a} \in H_{i-1}(A) = \frac{\mathrm{Ker}\,\left(\partial_{i-1}^A\right)}{\mathrm{Im}\,\left(\partial_i^A\right)}.$$

Hence $a \in \mathrm{Im}\,\left(\partial_i^A\right)$ and we let $\omega \in A_i$ such that $\partial_i^A(\omega) = a$. Moreover, $\phi_i(\omega)$, $c \in C_i$ so we compute the following.

$$\begin{aligned}
\partial_i^C(c - \phi_i(\omega)) &= \partial_i^C(c) - \left(\partial_i^C \circ \phi_i\right)(\omega) && \text{linearity} \\
&= \partial_i^C(c) - \left(\phi_{i-1} \circ \partial_i^A\right)(\omega) && \phi \text{ a chain complex} \\
&= \partial_i^C(c) - \phi_{i-1}(a) && \text{definition of } \omega \\
&= \partial_i^C(c) - \partial_i^C(c) && \text{definition of } a \\
&= 0
\end{aligned}$$

Therefore $c - \phi_i(\omega) \in \mathrm{Ker}\,\left(\partial_i^C\right)$ and hence $\overline{c - \phi_i(\omega)} \in H_i(C)$. We may also compute

$$H_i(\psi)(\overline{c - \phi_i(\omega)}) = \overline{\psi_i(c - \phi_i(\omega))} = \overline{\psi_i(c) - (\psi_i \circ \phi_i)(\omega)} = \overline{\psi_i(c)} = \overline{d} = \xi$$

where the third equality holds by the exactness of the i^{th} row of the given diagram. Hence $\xi \in \mathrm{Im}\,(H_i(\psi))$, which completes this step.

Step 8: Here we show $\operatorname{Im}(\eth_i) \subseteq \operatorname{Ker}(H_{i-1}(\phi))$. Let $\xi \in H_i(D)$ and let $d \in \operatorname{Ker}\left(\partial_i^D\right)$ such that $\xi = \overline{d}$. We want to show that $H_{i-1}(\phi)\left(\eth_i(\overline{d})\right) = 0$. Since ψ_i is surjective, let $c \in C_i$ such that $\psi_i(c) = d$ and since $\partial_i^C(c) \in \operatorname{Ker}(\psi_{i-1}) = \operatorname{Im}(\phi_{i-1})$, let $a \in A_{i-1}$ such that $\phi_{i-1}(a) = \partial_i^C(c)$. We therefore have

$$H_{i-1}(\phi)(\eth_i(\overline{d})) = H_{i-1}(\phi)(\overline{a}) = \overline{\phi_{i-1}(a)} = \overline{\partial_i^C(c)} = 0$$

which completes this step.

Step 9: We finally show that $\operatorname{Im}(\eth_i) \supseteq \operatorname{Ker}(H_{i-1}(\phi))$. Let $\lambda \in \operatorname{Ker}(H_{i-1}(\phi))$ and fix some element $a \in \operatorname{Ker}\left(\partial_{i-1}^A\right)$ such that $\lambda = \overline{a} \in H_{i-1}(A)$. By assumption we have

$$0 = H_{i-1}(\phi)(\lambda) = H_{i-1}(\phi)(\overline{a}) = \overline{\phi_{i-1}(a)} \in H_{i-1}(C) = \frac{\operatorname{Ker}\left(\partial_{i-1}^C\right)}{\operatorname{Im}\left(\partial_i^C\right)}.$$

It follows that $\phi_{i-1}(a) \in \operatorname{Im}\left(\partial_i^C\right)$, so we may let $c \in C_i$ such that $\partial_i^C(c) = \phi_{i-1}(a)$. Denote $\psi_i(c) = d$ and notice by our construction in Step 1, this element is a good candidate on which to apply $\eth_i$. Observe that

$$\partial_i^D(d) = \partial_i^D(\psi_i(c)) = \psi_{i-1}(\partial_i^C(c)) = (\psi_{i-1} \circ \phi_{i-1})(a) = 0$$

so $d \in \operatorname{Ker}\left(\partial_i^D\right)$. Therefore $\overline{d} \in H_i(D)$ and

$$\eth_i(\overline{d}) = \overline{a} = \lambda.$$

This completes this proof of the theorem. □

Definition I.D.17. Let $\alpha\colon A \to Y$ be a chain map. The <u>mapping cone</u> of α is an R-complex $\operatorname{Cone}(\alpha)$ defined by

$$\operatorname{Cone}(\alpha)_i = \begin{matrix} A_{i-1} \\ \oplus \\ Y_i \end{matrix} \xrightarrow[\partial_i^{\operatorname{Cone}(\alpha)}]{\begin{pmatrix} -\partial_{i-1}^A & 0 \\ \alpha_{i-1} & \partial_i^Y \end{pmatrix}} \begin{matrix} A_{i-2} \\ \oplus \\ Y_{i-1} \end{matrix} = \operatorname{Cone}(\alpha)_{i-1}.$$

Example I.D.18. Recall again Examples I.D.9 and I.D.12.

(a) The mapping cone of the zero map $A \xrightarrow{0} Y$ is given by

$$\begin{matrix} A_{i-1} \\ \oplus \\ Y_i \end{matrix} \xrightarrow{\begin{pmatrix} -\partial_{i-1}^A & 0 \\ 0 & \partial_i^Y \end{pmatrix}} \begin{matrix} A_{i-2} \\ \oplus \\ Y_{i-1} \end{matrix}.$$

Hence $\operatorname{Cone}(0) = \Sigma A \oplus Y$.

(b) The mapping cone of the homothety map $\mu^{A,x}\colon A \to A$ where $x \in R$ is given by

$$\begin{matrix} A_{i-1} \\ \oplus \\ A_i \end{matrix} \xrightarrow{\begin{pmatrix} -\partial_i^A & 0 \\ x & \partial_{i-1}^A \end{pmatrix}} \begin{matrix} A_{i-2} \\ \oplus \\ A_{i-1} \end{matrix}$$

where the entry x denotes the map $x \cdot \operatorname{id}_{A_{i-1}}$.

(c) Finally, the mapping cone of $\tau\colon P \to M$ is

$$\begin{matrix} & \operatorname{Cone}(\tau)_2 & & \operatorname{Cone}(\tau)_1 & & \operatorname{Cone}(\tau)_0 & & \operatorname{Cone}(\tau)_{-1} \\ & \| & & \| & & \| & & \| \\ \cdots \longrightarrow & \begin{matrix} R^{\beta_1} \\ \oplus \\ 0 \end{matrix} & \xrightarrow{\begin{pmatrix} -\partial_1^P & 0 \\ 0 & 0 \end{pmatrix}} & \begin{matrix} R^{\beta_0} \\ \oplus \\ 0 \end{matrix} & \xrightarrow{\begin{pmatrix} 0 & 0 \\ \tau & 0 \end{pmatrix}} & \begin{matrix} 0 \\ \oplus \\ M \end{matrix} & \longrightarrow & 0 \end{matrix}$$

and is isomorphic to

$$\cdots \longrightarrow R^{\beta_1} \xrightarrow{-\partial_1^P} R^{\beta_0} \xrightarrow{\tau} M \longrightarrow 0$$

where M is of homological degree 0. Note also that

$$\Sigma^{-1}\operatorname{Cone}(\tau) = \qquad \cdots \longrightarrow R^{\beta_1} \xrightarrow{\partial_1^P} R^{\beta_0} \xrightarrow{-\tau} M \longrightarrow 0 \qquad \cong P^+$$

where M is of homological degree -1.

(d) If $\alpha\colon A \to Y$ is a chain map, then $\operatorname{Cone}(\alpha)_{i+1}$ is the direct sum of the two R-modules indicated below.

$$\begin{tikzcd}
A \arrow[d, "\alpha"'] & \cdots \arrow[r] & A_{i+1} \arrow[r] \arrow[d] & A_i \arrow[r] \arrow[d] & A_{i-1} \arrow[r] \arrow[d] & \cdots \\
Y & \cdots \arrow[r] & Y_{i+1} \arrow[r] & Y_i \arrow[r] & Y_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

Proposition I.D.19. *If $\alpha\colon A \to Y$ is a chain map, then $\operatorname{Cone}(\alpha)$ is an R-complex.*

Proof. It suffices to show $\partial_{i-1}^{\operatorname{Cone}(\alpha)} \circ \partial_i^{\operatorname{Cone}(\alpha)} = 0$. To this end we compute

$$\begin{pmatrix} -\partial_{i-2}^A & 0 \\ \alpha_{i-2} & \partial_{i-1}^Y \end{pmatrix} \begin{pmatrix} -\partial_{i-1}^A & 0 \\ \alpha_{i-1} & \partial_i^Y \end{pmatrix} = \begin{pmatrix} \partial_{i-2}^A \circ \partial_{i-1}^A & 0 \\ \partial_{i-1}^Y \circ \alpha_{i-1} - \alpha_{i-2} \circ \partial_{i-1}^A & \partial_{i-1}^Y \circ \partial_i^Y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that the $2,2$ and $2,1$ and $1,1$-entries of the third matrix here are zero since Y is an R-complex, α is a chain map, and A is an R-complex, respectively. $\square$

Theorem I.D.20. *Let $\alpha\colon A \to Y$ be a chain map.*

(a) *There exists a short exact sequence of chain maps*

$$0 \longrightarrow Y \xrightarrow{\binom{1}{0}} \operatorname{Cone}(\alpha) \xrightarrow{(0\ 1)} \Sigma A \longrightarrow 0.$$

(b) *The connecting homomorphism $\eth_i$ given by Theorem I.D.16 is actually the map induced on homology modules given by Theorem I.D.10, i.e.,*

$$\begin{tikzcd}
\eth_i \arrow[d, equal] & : & H_i(\Sigma A) \arrow[r] \arrow[d, equal] & H_{i-1}(Y). \\
H_{i-1}(\alpha) & & H_{i-1}(A) &
\end{tikzcd}$$

Proof. (a) Consider the following diagram with split exact rows:

$$\begin{tikzcd}
0 \arrow[r] & Y_i \arrow[r, "\binom{1}{0}"] \arrow[d, "\partial_i^Y"'] & \begin{matrix} A_{i-1} \\ \oplus \\ Y_i \end{matrix} \arrow[r, "(0\ 1)"] \arrow[d, "\begin{pmatrix} -\partial_{i-1}^A & 0 \\ \alpha_{i-1} & \partial_i^Y \end{pmatrix}"'] & A_{i-1} \arrow[r] \arrow[d, "-\partial_{i-1}^A"'] & 0 \\
0 \arrow[r] & Y_{i-1} \arrow[r, "\binom{1}{0}"'] & \begin{matrix} A_{i-2} \\ \oplus \\ Y_{i-1} \end{matrix} \arrow[r, "(0\ 1)"'] & A_{i-2} \arrow[r] & 0.
\end{tikzcd}$$

A diagram chase shows that this diagram commutes, so it is an exact sequence of chain maps.

(b) Let $\overline{a} \in H_{i-1}(A)$. Then we can chase this element from $H_i(\Sigma A)$ to $H_{i-1}(Y)$, following the diagram from the proof of (a).

$$\begin{tikzcd}
 & \begin{pmatrix} a \\ 0 \end{pmatrix} \arrow[r, mapsto] \arrow[d, mapsto] & a \arrow[d, mapsto] \\
\alpha_{i-1}(a) \arrow[r, mapsto] & \begin{pmatrix} 0 \\ \alpha_{i-1}(a) \end{pmatrix} & 0.
\end{tikzcd}$$

Therefore, we have $\eth_i(\overline{a}) = \overline{\alpha_{i-1}(a)} = H_{i-1}(\alpha)(\overline{a})$.

$\square$

Corollary I.D.21. *Let* $\phi\colon A \to C$ *be a chain map. Then* ϕ *is a quasiisomorphism if and only if* $\mathrm{Cone}(\varphi)$ *is exact.*

Proof. Consider the following long exact sequence from Theorems I.D.16 and I.D.20:

$$\cdots \longrightarrow H_i(\mathrm{Cone}(\phi)) \longrightarrow H_{i-1}(A) \xrightarrow{H_{i-1}(\phi)} H_{i-1}(C) \longrightarrow H_{i-1}(\mathrm{Cone}(\phi)) \longrightarrow \cdots.$$

($\Leftarrow$) Suppose $\mathrm{Cone}(\phi)$ is exact. This implies that $H_i(\mathrm{Cone}(\phi)) = 0$ for all i, so the above long exact sequence looks like

$$0 \longrightarrow H_{i-1}(A) \xrightarrow[H_{i-1}(\phi)]{\cong} H_{i-1}(C) \longrightarrow 0.$$

The isomorphism here is from Fact I.B.2(d). Therefore ϕ is a quasiisomorphism by definition.

($\Rightarrow$) Suppose ϕ is a quasiisomorphism. Then a different piece of the above long exact sequence looks like

$$\cdots \longrightarrow H_i(A) \xrightarrow[H_i(\phi)]{\cong} H_i(C) \longrightarrow H_i(\mathrm{Cone}(\phi)) \longrightarrow H_{i-1}(A) \xrightarrow[H_{i-1}(\phi)]{\cong} H_{i-1}(C) \longrightarrow \cdots.$$

This implies the unlabeled middle two maps are both 0, and it follows that $H_i(\mathrm{Cone}(\phi)) = 0$ for all i. Therefore, $\mathrm{Cone}(\phi)$ is exact. $\square$

CHAPTER I.E

Application: Long Exact Sequences in Ext and Tor

Fact I.E.1. *Let N be an R-module and let*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be exact. Then Hom_R *is left exact and* $\otimes$ *is right exact.*

In other words, the following sequences are exact:

(a) $\mathrm{Hom}_R(N,-) = \qquad 0 \longrightarrow \mathrm{Hom}_R(N,M') \xrightarrow[\mathrm{Hom}_R(N,\alpha)]{\alpha_*} \mathrm{Hom}_R(N,M) \xrightarrow[\mathrm{Hom}_R(N,\beta)]{\beta_*} \mathrm{Hom}_R(N,M'') \longrightarrow ?$

(b) $\mathrm{Hom}_R(-,N) = \qquad 0 \longrightarrow \mathrm{Hom}_R(M'',N) \xrightarrow[\mathrm{Hom}_R(\alpha,N)]{\alpha^*} \mathrm{Hom}_R(M,N) \xrightarrow[\mathrm{Hom}_R(\beta,N)]{\beta^*} \mathrm{Hom}_R(M',N) \longrightarrow ?$

(c) $-\otimes_R N = \qquad ? \longrightarrow M'\otimes_R N \xrightarrow{\alpha\otimes N} M\otimes_R N \xrightarrow{\beta\otimes N} M''\otimes_R N \longrightarrow 0$

Question I.E.2. What goes in the <u>?'s</u>?

Definition I.E.3. Let P be a free resolution of M. Then
(a) $\mathrm{Ext}^i_R(M,N) = H_{-i}(\mathrm{Hom}_R(P,N))$, and
(b) $\mathrm{Tor}^R_i(M,N) = H_i(P\otimes_R N)$.

Proposition I.E.4. *Let A be an R-complex and N and R-module. Then the following are R-complexes:*

(a) $\mathrm{Hom}_R(A,N) = \qquad \cdots \longrightarrow \mathrm{Hom}_R(A_{i-1},N) \xrightarrow{(\partial_i^A)^*} \mathrm{Hom}_R(A_i,N) \xrightarrow{(\partial_{i+1}^A)^*} \cdots$

(b) $A_i\otimes_R N = \qquad \cdots \longrightarrow A_i\otimes_R N \xrightarrow{\partial_i^A\otimes N} A_{i-1}\otimes_R N \xrightarrow{\partial_{i-1}^A\otimes N} \cdots$

Proof. (a) Let $\gamma\in\mathrm{Hom}_R(A_{i-1},N)$, and consider the diagram:

$$\begin{array}{lccccc} \mathrm{Hom}_R(A,N) = & \cdots \longrightarrow \mathrm{Hom}_R(A_{i-1},N) & \xrightarrow{(\partial_i^A)^*} & \mathrm{Hom}_R(A_i,N) & \xrightarrow{(\partial_{i+1}^A)^*} & \cdots \\ & \gamma & \longmapsto & \gamma\circ\partial_i^A & \longmapsto & \gamma\circ\partial_i^A\circ_{i+1}^A = 0 \end{array}$$

(b) Let $a\otimes n\in A_i\otimes_R N$, and consider the diagram:

$$\begin{array}{lccccc} A_i\otimes_R N = & \cdots \longrightarrow A_i\otimes_R N & \xrightarrow{\partial_i^A\otimes N} & A_{i-1}\otimes_R N & \xrightarrow{\partial_{i-1}^A\otimes N} & \cdots \\ & a\otimes n & \longmapsto & \partial_i^A(a)\otimes n & \longmapsto & \partial_{i-1}^A\partial_i^A(a)\otimes n = 0\otimes n = 0 \end{array}$$

By definition, this shows that both of the above are R-complexes.

□

Fact I.E.5. Ext^i_R *and* Tor^R_i *are independent of choice of P.*

Theorem I.E.6. *Let N be an R-module and let*

$$0 \longrightarrow M' \longrightarrow M \xrightarrow{\alpha} M'' \xrightarrow{\beta} 0$$

be exact. Then there exist the following long exact sequences:

(a)

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(N, M') \arrow[r, "\alpha_*"] & \mathrm{Hom}_R(N, M) \arrow[r, "\beta_*"] & \mathrm{Hom}_R(N, M'') \arrow[dll, rounded corners] \\
 & \mathrm{Ext}^1_R(N, M') \arrow[r] & \mathrm{Ext}^1_R(N, M) \arrow[r] & \mathrm{Ext}^1_R(N, M'') \arrow[dll, rounded corners] \\
 & \mathrm{Ext}^2_R(N, M') \arrow[r] & \cdots &
\end{tikzcd}
$$

(b)

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M'', N) \arrow[r, "\alpha^*"] & \mathrm{Hom}_R(M, N) \arrow[r, "\beta^*"] & \mathrm{Hom}_R(M', N) \arrow[dll, rounded corners] \\
 & \mathrm{Ext}^1_R(M'', N) \arrow[r] & \mathrm{Ext}^1_R(M, N) \arrow[r] & \mathrm{Ext}^1_R(M', N) \arrow[dll, rounded corners] \\
 & \mathrm{Ext}^2_R(M'', N) \arrow[r] & \cdots &
\end{tikzcd}
$$

(c)

$$
\begin{tikzcd}
 & & \cdots \arrow[r] & \mathrm{Tor}^R_2(M'', N) \arrow[dll, rounded corners] & \\
 & \mathrm{Tor}^R_1(M', N) \arrow[r] & \mathrm{Tor}^R_1(M, N) \arrow[r] & \mathrm{Tor}^R_1(M'', N) \arrow[dll, rounded corners] & \\
 & M' \otimes_R N \arrow[r, "\alpha \otimes N"] & M \otimes_R N \arrow[r, "\beta \otimes N"] & M'' \otimes_R N \arrow[r] & 0
\end{tikzcd}
$$

Proof. We will prove part (a). Let F be a free resolution of N, and construct the following commutative diagram with exact rows:

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(F, M') \arrow[r, "\alpha_*"] & \mathrm{Hom}_R(F, M) \arrow[r, "\beta_*"] & \mathrm{Hom}_R(F, M'') \arrow[r] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_R(F_0, M') \arrow[r, "\alpha_*"] \arrow[d, "(\partial_1^F)^*"] & \mathrm{Hom}_R(F_0, M) \arrow[r, "\beta_*"] \arrow[d, "(\partial_1^F)^*"] & \mathrm{Hom}_R(F_0, M'') \arrow[r] \arrow[d, "(\partial_1^F)^*"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(F_1, M') \arrow[r, "\alpha_*"] \arrow[d, "(\partial_2^F)^*"] & \mathrm{Hom}_R(F_1, M) \arrow[r, "\beta_*"] \arrow[d, "(\partial_2^F)^*"] & \mathrm{Hom}_R(F_1, M'') \arrow[r] \arrow[d, "(\partial_2^F)^*"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Here, we have β_* defined so that for $\delta \in \mathrm{Hom}_R(F_i, M)$, we have $\beta_*(\delta) = \beta \circ \delta$. Since the F_i's are projective modules, this implies exactness at $\mathrm{Hom}_R(F_i, M'')$ in each row of the above diagram. Then by Theorem I.D.16, we can form the following long exact sequence from each of the above short exact sequences of chain maps:

$$
\begin{tikzcd}
& & 0 \arrow[dll, rounded corners] \\
H_0(\mathrm{Hom}_R(F, M')) \arrow[r, "H_0(\alpha_*)"] & H_0(\mathrm{Hom}_R(F, M)) \arrow[r, "H_0(\beta_*)"] & H_0(\mathrm{Hom}_R(F, M'')) \arrow[dll, rounded corners] \\
H_{-1}(\mathrm{Hom}_R(F, M')) \arrow[r] & H_{-1}(\mathrm{Hom}_R(F, M)) \arrow[r] & H_{-1}(\mathrm{Hom}_R(F, M'')) \arrow[dll, rounded corners] \\
H_{-2}(\mathrm{Hom}_R(F, M')) \arrow[r] & \cdots .
\end{tikzcd}
$$

Each of the homology groups of degree $-i$ for $i \geq 1$ are exactly the corresponding Ext_R^i from the statement of the result. Therefore, we only have to show the first row corresponds to the stated theorem. First, we check that $\mathrm{Hom}_R(N, M') \cong H_0(\mathrm{Hom}_R(F, M'))$. Notice that

$$
F^+ = \qquad \cdots \xrightarrow{\partial_2^F} F_1 \xrightarrow{\partial_1^F} F_0 \xrightarrow{\tau} N \longrightarrow 0
$$

is exact, so the following sequence

$$
(F^+)^* = \qquad 0 \longrightarrow \mathrm{Hom}_R(N, M') \xrightarrow{\tau^*} \mathrm{Hom}_R(F_0, M') \xrightarrow{(\partial_1^F)^*} \mathrm{Hom}_R(F_1, M')
$$

is exact. We can use the injectivity of τ^* and the exactness at $\mathrm{Hom}_R(F_0, M')$ to build the following sequence of isomorphisms and equalities:

$$
\mathrm{Hom}_R(N, M') \cong \mathrm{Im}\,(\tau^*) = \mathrm{Ker}\left((\partial_1^F)^*\right) = H_0(\mathrm{Hom}_R(F, M')).
$$

The argument is similar for the sequences containing M and M''. Next, one can check the following diagram is commutative.

$$
\begin{tikzcd}
\mathrm{Hom}_R(N, M') \arrow[r, "\cong"] \arrow[d, "\alpha_*"] & H_0(\mathrm{Hom}_R(F, M')) \arrow[d, "H_0(\alpha_*)"] \\
\mathrm{Hom}_R(N, M) \arrow[r, "\cong"] & H_0(\mathrm{Hom}_R(F, M))
\end{tikzcd}
$$

One obtains an analogous diagram for β, and these show how the long exact sequence given in the proof above matches up with the one in the statement of the result. This proves part (a). In the interest of time, we omit the remainder of the proof. $\square$

We end this chapter with some computations of long exact sequences.

EXAMPLE I.E.7. Let $R = k[X, Y]$ and $N = R/\langle X, Y\rangle$. Consider the following short exact sequence

$$
0 \longrightarrow R \xrightarrow{X} R \longrightarrow R/(X) \longrightarrow 0.
$$

To compute the associated long exact sequence for $\mathrm{Ext}_R(N, -)$, we use the following projective resolutions of N (augmented and truncated), then we dualize.

$$
\begin{tikzcd}
P^+ = & 0 \arrow[r] & R \arrow[r, "\begin{pmatrix} -Y \\ X \end{pmatrix}"] & R^2 \arrow[r, "\begin{pmatrix} X & Y \end{pmatrix}"] & R \arrow[r] & N \arrow[r] & 0 \\
P = & 0 \arrow[r] & R \arrow[r, "\begin{pmatrix} -Y \\ X \end{pmatrix}"] & R^2 \arrow[r, "\begin{pmatrix} X & Y \end{pmatrix}"] & R \arrow[r] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
P^* = \mathrm{Hom}_R(P,R) = \arrow[d, "\cong"] & 0 \arrow[r] & \mathrm{Hom}_R(R,R) \arrow[r, "\partial_1^*"] \arrow[d, "\cong"] & \mathrm{Hom}_R(R^2,R) \arrow[r, "\partial_2^*"] \arrow[d, "\cong"] & \mathrm{Hom}_R(R,R) \arrow[r] \arrow[d, "\cong"] & 0 \\
\Sigma^{-2}P = & 0 \arrow[r] & R \arrow[r, "\begin{pmatrix} -Y \\ X \end{pmatrix}"'] & R^2 \arrow[r, "\begin{pmatrix} X & Y \end{pmatrix}"'] & R \arrow[r] & 0
\end{tikzcd}
$$

The isomorphism above is straightforward to verify. We will discus this "self-duality" isomorphism in more detail later in the course. Furthermore, from the way that $\Sigma^{-2}P$ is defined, we have

$$
\mathrm{Ext}^i_R(N,R) \cong H_{-i}(P^*) \cong H_{-i}(\Sigma^{-2}P) = H_{2-i}(P) \cong \begin{cases} R/(X,Y) = N & \text{if } i = 2 \\ 0 & \text{if } i \neq 2 \end{cases}
$$

Next, we need to find $\mathrm{Ext}^i_R(N, R/(X))$. To do so, we consider the isomorphic sequences

$$
\begin{tikzcd}
\mathrm{Hom}_R(P,R/(X)) = \arrow[d, "\cong"'] & 0 \arrow[r] & \mathrm{Hom}_R(R,R/(X)) \arrow[r, "\partial_1^*"] & \mathrm{Hom}_R(R^2,R/(X)) \arrow[r, "\partial_2^*"] & \mathrm{Hom}_R(R,R/(X)) \arrow[r] & 0 \\
A = \arrow[d, "\cong"'] & 0 \arrow[r] & R/(X) \arrow[r, "\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ Y \end{smallmatrix}\right)"] & (R/(X))^2 \arrow[r, "\left(\begin{smallmatrix} -Y & X \end{smallmatrix}\right) = \left(\begin{smallmatrix} -Y & 0 \end{smallmatrix}\right)"] & R/(X) \arrow[r] & 0 \\
C = & 0 \arrow[r] & k[Y] \arrow[r, "\left(\begin{smallmatrix} 0 \\ Y \end{smallmatrix}\right)"] & (k[Y])^2 \arrow[r, "\left(\begin{smallmatrix} -Y & 0 \end{smallmatrix}\right)"] & k[Y] \arrow[r] & 0
\end{tikzcd}
$$

To compute $H_i(C)$, first consider the kernel of the map $\begin{pmatrix} 0 \\ Y \end{pmatrix} : k[Y] \to (k[Y])^2$. This map is injective since Y is a non-zero-divisor on $k[Y]$. Therefore the kernel is 0, so for homological degree 0 we get

$$
\mathrm{Ext}^0_R(N, R/(X)) = H_0(\mathrm{Hom}_R(P, R/(X))) = 0.
$$

Next consider the image of the map $\begin{pmatrix} -Y & 0 \end{pmatrix} : (k[Y])^2 \to k[Y]$. The image of this map is $\langle Y \rangle$, so for homological degree -2 we get

$$
\mathrm{Ext}^2_R(N, R/(X)) = H_{-2}(\mathrm{Hom}_R(P, R/(X))) \cong k[Y]/\langle Y \rangle \cong N.
$$

Finally consider the image of the first map and the kernel of the second map. We have $\mathrm{Im}\left(\begin{pmatrix} 0 \\ Y \end{pmatrix}\right) = \left\langle \begin{pmatrix} 0 \\ Y \end{pmatrix} \right\rangle$. Let $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathrm{Ker}\left(\begin{pmatrix} -Y & 0 \end{pmatrix}\right)$, so $-Yf = 0$. Since Y is a non-zero-divisor of $(k[Y])^2$, then $f = 0$. Therefore,

$$
\mathrm{Ker}\left(\begin{pmatrix} -Y & 0 \end{pmatrix}\right) = \left\{ \begin{pmatrix} 0 \\ g \end{pmatrix} \mid g \in k[Y] \right\} = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \text{ so}
$$

$$
\mathrm{Ext}^1_R(N, R/(X)) = \frac{\mathrm{Ker}\left(\begin{pmatrix} -Y & 0 \end{pmatrix}\right)}{\mathrm{Im}\left(\begin{pmatrix} 0 \\ Y \end{pmatrix}\right)} = \frac{\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 0 \\ Y \end{pmatrix} \right\rangle} \cong \frac{k[Y]}{\langle Y \rangle} \cong \frac{k[X,Y]}{\langle X, Y \rangle} = N.
$$

Now we have the information needed to construct the long exact sequence in $\mathrm{Ext}_R(N, -)$ following the notation from Theorem I.E.6(a):

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}^0_R(N,R)=0 \arrow[r] & \operatorname{Ext}^0_R(N,R)=0 \arrow[r] & \operatorname{Ext}^0_R(N,R/(X)) \arrow[dlll, rounded corners] \\
& \operatorname{Ext}^1_R(N,R)=0 \arrow[r] & \operatorname{Ext}^1_R(N,R)=0 \arrow[r] & \operatorname{Ext}^1_R(N,R/(X)) \cong N \arrow[dlll, rounded corners] \\
& \operatorname{Ext}^2_R(N,R)\cong N \arrow[r, "0", "X"'] & \operatorname{Ext}^2_R(N,R)\cong N \arrow[r] & \operatorname{Ext}^2_R(N,R/(X)) \cong N \arrow[dlll, rounded corners] \\
& \operatorname{Ext}^3_R(N,R)=0 \arrow[r] & \cdots &
\end{tikzcd}
$$

Since most of the terms are 0, the only interesting part simplifies to the following exact sequence, where the labelled properties can be determined using the exactness.

$$0 \longrightarrow N \xrightarrow{\cong} N \xrightarrow{0} N \xrightarrow{\cong} N \longrightarrow 0.$$

Additionally, it is possible to show that

$$\operatorname{Ext}^i_R(N,N) \cong \begin{cases} N^2 & \text{if } i=1 \\ N & \text{if } i=0,2 \\ 0 & \text{otherwise} \end{cases}.$$

We leave it as an exercise here to verify this and to compute the long exact sequence in $\operatorname{Ext}_R(N,-)$ associated to the following short exact sequence:

$$0 \longrightarrow R/(X) \xrightarrow{Y} R/(X) \longrightarrow N \longrightarrow 0.$$

Part II

Examples of Free Resolutions

Resolutions of Mapping Cones

Our first goal of this section is, given an R-module homomorphism $f : M \to N$ and resolutions of M and N, to build a resolution of $N/f(M)$.

Lemma II.A.1 (Lifting Lemma). *Let $f : M \to N$ be an R-module homomorphism, let P^+ be a free resolution of M, and let Q^+ be a free resolution of N. Then there exist chain maps $F : P \to Q$ and $F^+ : P^+ \to Q^+$ such that $F^+_{-1} = f$ and such that the following diagram commutes:*

$$\begin{tikzcd} H_0(P) \arrow[r, "H_0(F)"] \arrow[d, "\overline{\tau}", "\cong"'] & H_0(Q) \arrow[d, "\overline{\pi}"', "\cong"] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

Graphically, the following diagram commutes:

$$\begin{tikzcd} P^+ = \arrow[d, "F^+"'] & \cdots \arrow[r, "\partial_2^P"] & P_1 \arrow[r, "\partial_1^P"] \arrow[d, dashed, "F_1"] & P_0 \arrow[r, "\tau"] \arrow[d, dashed, "F_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\ Q^+ = & \cdots \arrow[r, "\partial_2^Q"] & Q_1 \arrow[r, "\partial_1^Q"] & Q_0 \arrow[r, "\pi"] & N \arrow[r] & 0. \end{tikzcd}$$

Proof. Consider the commutative diagram II.A.3.1 on the following page, which is obtained via repeated application of Exercise 1.3. We can see the diagram below commutes, since the equality in the bottom right comes from the commutativity of diagram II.A.3.1. Furthermore, the diagram shows that $F^+_{-1} = f$.

$$\begin{tikzcd} \overline{p} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \tau(p) \arrow[ddd, mapsto] \\ & H_0(P) \arrow[r, "\overline{\tau}"] \arrow[d, "H_0(F)"] & M \arrow[d, "f"] & \\ & H_0(Q) \arrow[r, "\overline{\pi}"] & N & \\ \overline{F_0(p)} \arrow[rrr, mapsto] & & & \pi(F_0(p)) = f(\tau(p)) \end{tikzcd}$$

$\square$

Theorem II.A.2. *Keep the same notation from II.A.1. If f is one-to-one, then* $\mathrm{Cone}(F)$ *is a free resolution of $N/f(M)$.*

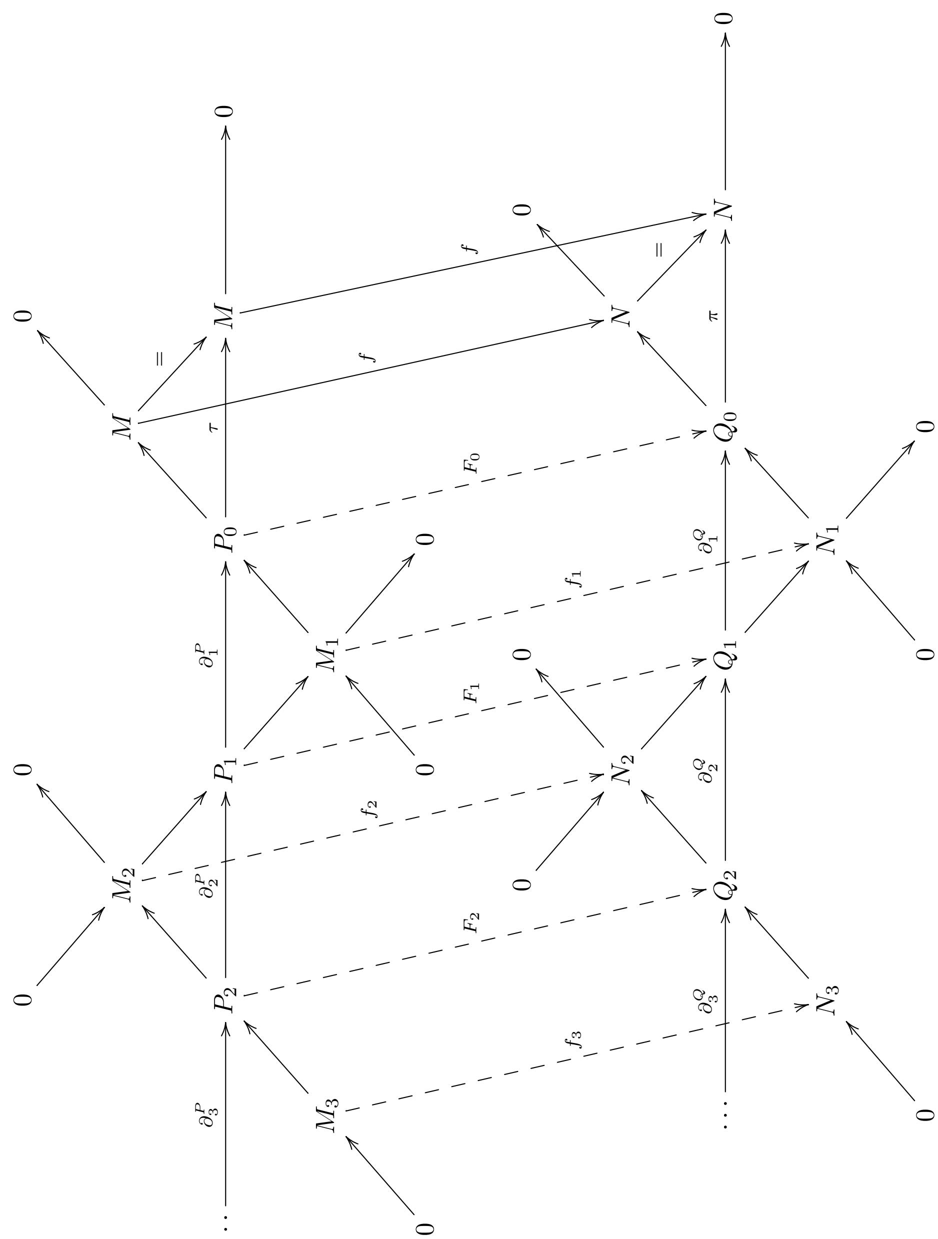
To prove this theorem, we require the following lemma.

Lemma II.A.3. *Let* $F = (\cdots \xrightarrow{\partial_2^F} F_1 \xrightarrow{\partial_1^F} F_0 \longrightarrow 0)$ *be an R-complex such that each F_i is free and $H_i(F) = 0$ for all $i \neq 0$. Then F is a free resolution of $H_0(F)$.*

Proof. Define $\tau \colon F_0 \to H_0(F)$ to be the natural projection:

$$F_0 \xrightarrow{\tau} \frac{F_0}{\mathrm{Im}\,(F_1 \to F_0)} = \frac{\mathrm{Ker}\,(F_0 \to 0)}{\mathrm{Im}\,(F_1 \to F_0)} = H_0(F).$$

Then $F^+ = (\cdots \xrightarrow{\partial_2^F} F_1 \xrightarrow{\partial_1^F} F_0 \xrightarrow{\tau} H_0(F) \longrightarrow 0)$ is exact, since τ is onto and $\mathrm{Ker}\,(\tau) = \mathrm{Im}\left(\partial_1^F\right)$ by construction. $\square$



(II.A.3.1)

Now we can prove the theorem and thus acheive our first goal for this section.

Proof of II.A.2. First, we know $\operatorname{Cone}(F)_i = \begin{matrix} P_{i-1} \\ \oplus \\ Q_i \end{matrix}$ is free because P_{i-1} and Q_i are free modules. Then, compute $H_i(\operatorname{Cone}(F))$ using the following short exact sequence:

$$0 \longrightarrow Q \longrightarrow \operatorname{Cone}(F) \longrightarrow \Sigma P \longrightarrow 0.$$

Then the corresponding long exact sequence of homology modules is

$$H_i(Q) \longrightarrow H_i(\mathrm{Cone}(F)) \longrightarrow H_{i-1}(P) \xrightarrow{\eth_{i-1}} H_{i-1}(Q).$$

Here, $\eth_{i-1} = H_{i-1}(F)$. If $i \geq 2$, then $H_i(Q) = 0 = H_{i-1}(P)$, so the section of this sequence becomes

$$0 \longrightarrow H_i(\mathrm{Cone}(F)) \longrightarrow 0,$$

so $H_i(\mathrm{Cone}(F)) = 0$ as well. If $i = 1$, then $\eth_0 = H_0(F)$, which is related to f in Lemma II.A.1, so if f is one-to-one, then $H_0(F)$ is one-to-one as well. Then we have

$$0 \longrightarrow H_1(\mathrm{Cone}(F)) \longrightarrow H_0(P) \xrightarrow{H_0(F)} H_0(Q).$$

A diagram chase shows that $H_1(\mathrm{Cone}(F)) = 0$. If $i = 0$, then we have

$$\begin{tikzcd}
0 \arrow[r] & H_0(P) \arrow[r, "H_0(F)"] \arrow[d, "\overline{\tau}", "\cong"'] & H_0(Q) \arrow[r] \arrow[d, "\overline{\pi}", "\cong"'] & H_0(\mathrm{Cone}(F)) \arrow[r] \arrow[d, dashed, "\exists!\rho"] & H_{-1}(P) = 0 \\
0 \arrow[r] & M \arrow[r, "f"] & N \arrow[r] & N/f(M) \arrow[r] & 0.
\end{tikzcd}$$

By the Snake Lemma, ρ is an isomorphism. Finally, we look at the structure of $\mathrm{Cone}(F)$.

$$\begin{tikzcd}
P = \arrow[d, "F"] & (\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "F_1"] & P_0 \arrow[r] \arrow[d, "F_0"] & 0 \arrow[r] & 0) \\
Q = & (\cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & 0) &
\end{tikzcd}$$

$$\mathrm{Cone}(F) = \qquad \cdots \longrightarrow \begin{matrix} P_0 \\ \oplus \\ Q_1 \end{matrix} \longrightarrow \begin{matrix} 0 \\ \oplus \\ Q_0 \end{matrix} \longrightarrow 0$$

By Lemma II.A.3, $\mathrm{Cone}(F)$ is a free resolution of $N/f(M)$.

□

Our next goal is to build a free resolution for R-modules of the form $R/\langle f_1, \ldots, f_n \rangle$.

Definition II.A.4. For any ideal $J \leq R$ and any element $r \in R$, the colon ideal is

$$(J : r) = \{s \in R \mid sr \in J\}.$$

Example II.A.5.

(a) In the ring $\mathbb{Z}$, for the ideal $36\mathbb{Z}$ and the element 15 we have the colon ideal $(36\mathbb{Z} : 15) = 12\mathbb{Z} \leq \mathbb{Z}$.

(b) More generally, let R be a unique factorization domain and let $f, g \in R$ be elements with respective prime factorizations $f = up_1^{e_1} \cdots p_n^{e_n}$ and $g = vp_1^{d_1} \cdots p_n^{d_n}$. Then we have the colon ideal $(J : g) = p_1^{c_1} \cdots p_n^{c_n} R$ where we set

$$c_i = (e_i - d_i)_+ = \begin{cases} e_i - d_i & e_i \geq d_i \\ 0 & e_i < d_i. \end{cases}$$

(c) Let $R = k[X_1, \ldots, X_d]$ be a polynomial ring and for any vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^d$ set

$$\mathbf{X}^{\mathbf{a}} = X_1^{a_1} \cdots X_d^{a_d}.$$

If $\mathbf{b_1}, \ldots, \mathbf{b_n} \in \mathbb{N}^d$, then for the ideal $J = \langle \mathbf{X}^{\mathbf{b_1}}, \ldots, \mathbf{X}^{\mathbf{b_n}} \rangle$ one can show

$$(J :\, \mathbf{X}^{\mathbf{a}}) = \left\langle \mathbf{X}^{(\mathbf{b_1}-\mathbf{a})_+}, \ldots, \mathbf{X}^{(\mathbf{b_n}-\mathbf{a})_+} \right\rangle.$$

For instance, in the ring $R = k[X, Y]$ we have

$$(\langle X^3, X^2Y^2, Y^4 \rangle :\, XY^2) = \langle X^2, X, Y^2 \rangle = \langle X, Y^2 \rangle.$$

PROPOSITION II.A.6. *Let $J \leq R$ be an ideal and let $r \in R$.*

(a) $J \subseteq (J :\, r) \leq R$
(b) $(J :\, r) = R$ *if and only if* $r \in J$.
(c) *The sequence*

$$0 \longrightarrow \frac{R}{(J :\, r)} \xrightarrow{r\cdot} \frac{R}{J} \xrightarrow{\psi} \frac{R}{(J + rR)} \longrightarrow 0$$

is exact where ψ is the natural surjection.

PROOF. (a) The inclusion follows from the definition of an ideal. Hence $(J :\, r)$ is non-empty. For any $s_1, s_2 \in (J :\, r)$ and any $t \in R$ we have

$$(ts_1 + s_2)r = ts_1r + s_2r \in J$$

since J is an ideal. This proves colon ideals are indeed ideals.

(b) We observe that $(J :\, r) = R$ if and only if $1 \in (J :\, r)$, if and only if $r = 1 \cdot r \in J$.

(c) Since $J \subseteq J + rR$, the map ψ above is a well-defined surjective R-module homomorphism. By the second isomorphism theorem

$$\frac{J + rR}{J} \cong \frac{rR}{J \cap rR}.$$

Since $rR/(J \cap rR)$ is cyclic with generator $\overline{r} = r + J \cap rR$, the module $(J + rR)/J$ is cyclic with generator $\overline{r} = r + J \cap rR$. Hence by the third isomorphism theorem we have

$$\frac{R/J}{\langle \overline{r} \rangle (R/J)} = \frac{R/J}{(J + rR)/J} \cong \frac{R}{J + rR}. \tag{II.A.6.1}$$

Therefore the sequence

$$R \xrightarrow{\overline{r}\cdot} \frac{R}{J} \xrightarrow{\psi} \frac{R}{J + rR} \longrightarrow 0$$

is exact since the display (II.A.6.1) above shows that

$$\operatorname{Ker}(\psi) = \langle \overline{r} \rangle (R/J) = \operatorname{Im}\left(R \xrightarrow{\overline{r}\cdot} R/J\right).$$

Next we observe that

$$\operatorname{Ker}\left(R \xrightarrow{\overline{r}\cdot} R/J\right) = \{s \in R \mid s\overline{r} = 0 \in R/J\} = \{s \in R \mid sr \in J\} = (J :\, r)$$

and consider the commutative diagram

$$\begin{tikzcd} R \arrow[r, "\overline{r}\cdot"] \arrow[d, "\pi"'] & R/J \\ R/(J :\, r) \arrow[ur, dashed, "\exists \overline{\pi}"'] & \end{tikzcd}$$

where π is the natural surjection and $\overline{\pi}$ is a well-defined R-module monomorphism. Moreover a short diagram chase shows that $\overline{\pi}$ is the homothety map $r\cdot$ and $\operatorname{Im}(r\cdot) = \operatorname{Im}(\overline{r})$. It follows that the desired short exact sequence exists. $\square$

THEOREM II.A.7. *Let $f_1, \ldots, f_j, r \in R$, let $J = \langle f_1, \ldots, f_n \rangle \leq R$ be an ideal, let P be a free resolution of $R/(J :\, r)$, and let Q be a free resolution of R/J. Then there is a chain map $\Phi^+ \colon P^+ \to Q^+$ such that* $\operatorname{Cone}(\Phi)$ *is a free resolution of $R/(J + rR)$.*

Proof. The existence of Φ^+ follows from Lemma II.A.1:

$$\begin{tikzcd}
P^+ = \arrow[d, dashed, "\exists\Phi^+"'] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, dashed, "\Phi_1^+"] & P_0 \arrow[r] \arrow[d, dashed, "\Phi_0^+"] & \dfrac{R}{(J:\, r)} \arrow[r] \arrow[d, hook, "r\cdot"] & 0 \\
Q^+ = & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & \dfrac{R}{J} \arrow[r] & 0.
\end{tikzcd}$$

By Theorem II.A.2 we know Cone(Φ) is a free resolution of $(R/J)/\operatorname{Im}(r\cdot)$ where $r\cdot$ is the vertical homothety map in the above ladder diagram. Moreover, by the proof of Proposition II.A.6 we have

$$\frac{R/J}{\operatorname{Im}(r\cdot)} = \frac{R/J}{\langle \overline{r} \rangle (R/J)} \cong \frac{R}{J + rR} = \frac{R}{\langle f_1, \ldots, f_n \rangle + \langle r \rangle} = \frac{R}{\langle f_1, \ldots, f_n, r \rangle}$$

which completes the proof. □

The Koszul Complex

EXAMPLE II.B.1. Set $R = k[X, Y]$. A free resolution of $R/\langle Y\rangle$ is

$$P^+ = \left(0 \longrightarrow R \xrightarrow{Y\cdot} R \xrightarrow{\tau} R/\langle Y\rangle \longrightarrow 0 \right)$$

and we can use this to build a free resolution of $R/\langle X, Y\rangle$. We consider the homothety chain map

$$\begin{tikzcd}
P^+ = \arrow[d, "X\cdot"', "\Phi^+"] & 0 \arrow[r] & R \arrow[r, "Y\cdot"] \arrow[d, "X\cdot"] & R \arrow[r, "\tau"] \arrow[d, "X\cdot"] & R/\langle Y\rangle \arrow[r] \arrow[d, hook, "X\cdot"] & 0 \\
P^+ = & 0 \arrow[r] & R \arrow[r, "Y\cdot"'] & R \arrow[r, "\tau"'] & R/\langle Y\rangle \arrow[r] & 0
\end{tikzcd}$$

where the homothety map $X\cdot\colon R/\langle Y\rangle \to R/\langle Y\rangle$ is injective since X is a non-zero-divisor on $R/\langle Y\rangle \cong k[X]$. Truncating we get

$$\begin{tikzcd}
P^+ = \arrow[d, "X\cdot"', "\Phi"] & 0 \arrow[r] & R \arrow[r, "Y\cdot"] \arrow[d, "X\cdot"] & R \arrow[r] \arrow[d, "X\cdot"] & 0 \\
P^+ = & 0 \arrow[r] & R \arrow[r, "Y\cdot"'] & R \arrow[r] & 0.
\end{tikzcd}$$

Then $\mathrm{Cone}(\Phi)$ is a free resolution of $R/\langle X, Y\rangle$ by Theorem II.A.7.

$$\mathrm{Cone}(\Phi) = \left(0 \longrightarrow \begin{matrix} R \\ \oplus \\ 0 \end{matrix} \xrightarrow{\left(\begin{smallmatrix} -Y & 0 \\ X & 0 \end{smallmatrix}\right)} \begin{matrix} R \\ \oplus \\ R \end{matrix} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ X & Y \end{smallmatrix}\right)} \begin{matrix} 0 \\ \oplus \\ R \end{matrix} \longrightarrow 0 \right)$$

$$\cong \left(0 \longrightarrow R \xrightarrow{\left(\begin{smallmatrix} -Y \\ X \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} X & Y \end{smallmatrix}\right)} R \longrightarrow 0 \right)$$

DEFINITION II.B.2. Let $x, y, x_1, \ldots, x_n \in R$ be given. The <u>Koszul complex</u> is defined inductively on n.

$$\begin{aligned}
n = 1: &\qquad K^R(x) = \ 0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0 \\
n = 2: &\qquad K^R(x, y) = \mathrm{Cone}\left(K^R(y) \xrightarrow{x} K^R(y)\right) \\
&\qquad \cong\ 0 \longrightarrow R \xrightarrow{\left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & y \end{smallmatrix}\right)} R \longrightarrow 0 \\
n \geq 2: &\qquad K^R(x_1, \ldots, x_n) = \mathrm{Cone}\left(K^R(x_2, \ldots, x_n) \xrightarrow{x_1} K^R(x_2, \ldots, x_n)\right)
\end{aligned}$$

EXAMPLE II.B.3. Having already explicitly written the Koszul complex for one and two elements, we compute $K^R(x, y, z) = \mathrm{Cone}\left(K^R(y, z) \xrightarrow{x} K^R(y, z)\right)$. First we display the appropriate homothety chain map.

$$\begin{array}{ccc} K^R(y,z) & & 0 \longrightarrow R \xrightarrow{\left(\begin{smallmatrix} -z \\ y \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} y & z \end{smallmatrix}\right)} R \longrightarrow 0 \\ \Big\downarrow x & & \Big\downarrow x \qquad \Big\downarrow x \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) \qquad \Big\downarrow x \\ K^R(y,z) & & 0 \longrightarrow R \xrightarrow[\left(\begin{smallmatrix} -z \\ y \end{smallmatrix}\right)]{} R^2 \xrightarrow[\left(\begin{smallmatrix} y & z \end{smallmatrix}\right)]{} R \longrightarrow 0 \end{array}$$

Now we can write down the appropriate cone. $K^R(x,y,z)$ is equal to

$$0 \longrightarrow \begin{matrix} R \\ \oplus \\ 0 \end{matrix} \xrightarrow{\begin{pmatrix} -\left(\begin{smallmatrix} z \\ -y \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \\ x & 0 \end{pmatrix}} \begin{matrix} R^2 \\ \oplus \\ R \end{matrix} \xrightarrow{\begin{pmatrix} -\left(\begin{smallmatrix} y & z \end{smallmatrix}\right) & 0 \\ \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) & \left(\begin{smallmatrix} -z \\ y \end{smallmatrix}\right) \end{pmatrix}} \begin{matrix} R \\ \oplus \\ R^2 \end{matrix} \xrightarrow{\begin{pmatrix} 0 & \left(\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right) \\ x & \left(\begin{smallmatrix} y & z \end{smallmatrix}\right) \end{pmatrix}} \begin{matrix} 0 \\ \oplus \\ R \end{matrix} \longrightarrow 0$$

and thus isomorphic to

$$0 \longrightarrow R \xrightarrow{\left(\begin{smallmatrix} z \\ -y \\ x \end{smallmatrix}\right)} R^3 \xrightarrow{\left(\begin{smallmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{smallmatrix}\right)} R^3 \xrightarrow{\left(\begin{smallmatrix} x & y & z \end{smallmatrix}\right)} R \longrightarrow 0.$$

Notice the presence of binomial coefficients from Pascal's triangle in the exponents of the previous display. These are also present in the $n=1$ and $n=2$ cases of Definition II.B.2. This leads us to the following proposition.

Proposition II.B.4. *Let $\mathbf{x} = x_1, \ldots, x_n \in R$ and $K = K^R(\mathbf{x})$. Then:*

(a) $K^R(\mathbf{x})_i \cong R^{\binom{n}{i}}$.
(b) $\partial_1^K = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} : R^n \to R$.
(c) $\partial_n^K = \begin{pmatrix} (-1)^{n-1}x_n \\ \vdots \\ -x_2 \\ x_1 \end{pmatrix} : R \to R^n$.
(d) *The matrix representing ∂_i^K consists of 0's and $\pm x_k$'s.*

Proof. We will use induction on n.
Base case: The case for $n=1$ is covered in Definition II.B.2.
Inductive case: Set $\mathbf{x}' = x_2, \ldots, x_n$ and $K' = K^R(\mathbf{x}')$. Then we consider each part of the result:

(a) From the inductive hypothesis, we have $K'_i \cong R^{\binom{n-1}{i}}$. Therefore

$$K_i = \begin{matrix} K'_{i-1} \\ \oplus \\ K'_i \end{matrix} \cong \begin{matrix} R^{\binom{n-1}{i-1}} \\ \oplus \\ R^{\binom{n-1}{i}} \end{matrix} \cong R^{\binom{n-1}{i-1}+\binom{n-1}{i}} = R^{\binom{n}{i}}.$$

(b) We use Definition II.B.2 to construct the ∂_1^K map from the inductive hypothesis.

$$\partial_1^K\colon \begin{matrix} K'_0 \\ \oplus \\ K'_1 \end{matrix} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ x_1 & \partial_1^{K'} \end{smallmatrix}\right)} \begin{matrix} 0 \\ \oplus \\ K'_0 \end{matrix} \Rightarrow \partial_1^K\colon \begin{matrix} R \\ \oplus \\ R^{n-1} \end{matrix} \xrightarrow{\left(\begin{smallmatrix} x_1 & x_2 & \cdots & x_n \end{smallmatrix}\right)} \begin{matrix} 0 \\ \oplus \\ R \end{matrix} \Rightarrow \partial_1^K\colon\ R^n \xrightarrow{\left(\begin{smallmatrix} x_1 & x_2 & \cdots & x_n \end{smallmatrix}\right)} R.$$

The first implication comes from the result in part (a) of the proposition, while the second implication uses the isomorphism from $R \oplus R^{n-1}$ to R^n.

(c) We construct ∂_n^K in a similar way as above.

$$\partial_n^K\colon \begin{matrix} K'_{n-1} \\ \oplus \\ 0 \end{matrix} \xrightarrow{\left(\begin{smallmatrix} -\partial_{n-1}^{K'} & 0 \\ x_1 & 0 \end{smallmatrix}\right)} \begin{matrix} K'_{n-2} \\ \oplus \\ K'_{n-1} \end{matrix} \Rightarrow \partial_n^K\colon \begin{matrix} R \\ \oplus \\ 0 \end{matrix} \xrightarrow{\left(\begin{smallmatrix} -(-1)^{n-2}x_n \\ \vdots \\ -x_2 \\ x_1 \end{smallmatrix}\right)} \begin{matrix} R^{n-1} \\ \oplus \\ R \end{matrix} \Rightarrow \partial_n^K\colon\ R \xrightarrow{\left(\begin{smallmatrix} (-1)^{n-1}x_n \\ \vdots \\ -x_2 \\ x_1 \end{smallmatrix}\right)} R^n.$$

(d) For each i, we have

$$\partial_i^K = \begin{pmatrix} -\partial_{i-1}^{K'} & 0 \\ x_1 \cdot \mathrm{id} & \partial_i^{K'} \end{pmatrix}.$$

Then the inductive hypothesis tells us that $-\partial_{i-1}^{K'}$ and $\partial_i^{K'}$ consist only of 0's and $\pm x_2, \pm x_3, \ldots, \pm x_n$. Furthermore, $x_1 \cdot \mathrm{id}$ only consists of 0's and x_1's. Therefore, ∂_i^K only consists of 0's and $\pm x_i$'s for $i \in [n]$. $\square$

Now we consider the question: when is K a resolution? To answer this question, we introduce the following definition.

DEFINITION II.B.5. A sequence $x_n, x_{n-1}, \ldots, x_1 \in R$ is <u>weakly R-regular</u> if:

(1) x_n is a non-zero-divisor on R.
(2) x_{n-1} is a non-zero-divisor on $R/\langle x_n \rangle$.
(3) x_{n-2} is a non-zero-divisor on $R/\langle x_n, x_{n-1} \rangle$.
$\vdots$
(i) x_{n-i+1} is a non-zero-divisor on $R/\langle x_n, \ldots, x_{n-i+2} \rangle$.
$\vdots$
(n) x_1 is a non-zero-divisor on $R/\langle x_n, \ldots, x_2 \rangle$.

EXAMPLE II.B.6. Let A be a commutative ring with identity, and let $R = A[X_1, \ldots, X_d]$. Then $\mathbf{X} = X_d, \ldots, X_1$ is weakly R-regular. We check the conditions of Definition II.B.5:

(1) X_d is a non-zero-divisor on R.
(2) $R/\langle X_d \rangle \cong A[X_1, \ldots, X_{d-1}]$, so X_{d-1} is a non-zero-divisor on $R/\langle X_d \rangle$.

Continuing in this fashion, we see that $\mathbf{X}$ is weakly R-regular. More generally, if $X_{i_1}, \ldots, X_{i_n}$ are distinct variables in $R = A[X_1, \ldots, X_d]$, then $X_{i_1}, \ldots, X_{i_n}$ is also weakly R-regular.

The following theorem says a free resolution of a ring mod a weakly-regular sequence is the Koszul complex applied to that sequence.

THEOREM II.B.7. *If $\mathbf{x} = x_n, \ldots, x_1 \in R$ is weakly R-regular, then $K^R(\mathbf{x})$ is a free resolution of $R/\langle \mathbf{x} \rangle$.*

COROLLARY II.B.8. *If $R = A[X_1, \ldots, X_d]$, then $K^R(X_1, \ldots, X_d)$ is a free resolution of $R/\langle X_1, \ldots, X_d \rangle \cong A$.*

PROOF OF II.B.7. We will use induction on n.

Base case: Let $n = 1$. Then $K^R(x_1) = 0 \longrightarrow R \xrightarrow{x_1} R \longrightarrow 0$. Since x_1 is a non-zero-divisor on R, we then have

$$H_1(K^R(x_1) \cong \operatorname{Ker}\left(R \xrightarrow{x_1} R \right) = 0.$$

Then $K^R(x_1)$ is a free resolution of $H_0(K^R(x_1)) \cong R/\langle x_1 \rangle$ by Lemma II.A.3.

Inductive Case: Let $\mathbf{x}' = x_n, \ldots, x_2$. By definition, $\mathbf{x}'$ is weakly R-regular. The inductive hypothesis tells us that $K' = K^R(\mathbf{x}')$ is a free resolution of $R/\langle \mathbf{x}' \rangle$. Then we claim that $(\langle \mathbf{x}' \rangle : x_1) = \langle \mathbf{x}' \rangle$.

Proof of claim. By Proposition II.A.6, we have $(\langle \mathbf{x}' \rangle : x_1) \supseteq \langle \mathbf{x}' \rangle$. Then we want to show $(\langle \mathbf{x}' \rangle : x_1) \subseteq \langle \mathbf{x}' \rangle$. Let $\alpha \in (\langle \mathbf{x}' \rangle : x_1)$, so $x_1 \cdot \alpha \in \langle \mathbf{x}' \rangle$. Then in $R/\langle \mathbf{x}' \rangle$, $x_1 \overline{\alpha} = 0$. But x_1 is a non-zero-divisor on $R/\langle \mathbf{x}' \rangle$ by condition (n) of Definition II.B.5, so $\overline{\alpha} = 0$ in $R/\langle \mathbf{x}' \rangle$. Therefore, $\alpha \in \langle \mathbf{x}' \rangle$.

Now consider the following free resolutions given by the inductive hypothesis:

$$\begin{tikzcd}
(K')^+ = \arrow[d, "x_1"'] & 0 \arrow[r] & R \arrow[r] \arrow[d, "x_1"'] & R^{n-1} \arrow[r] \arrow[d, "x_1"'] & \cdots \arrow[r] & R^{n-1} \arrow[r] \arrow[d, "x_1"'] & R \arrow[r] \arrow[d, "x_1"'] & R/\langle \mathbf{x}' \rangle = R/(\langle \mathbf{x}' \rangle : x_1) \arrow[r] \arrow[d, hook, "x_1"'] & 0 \\
(K')^+ = & 0 \arrow[r] & R \arrow[r] & R^{n-1} \arrow[r] & \cdots \arrow[r] & R^{n-1} \arrow[r] & R \arrow[r] & R/\langle \mathbf{x}' \rangle \arrow[r] & 0
\end{tikzcd}$$

By Theorem II.A.7, $K = \mathrm{Cone}(\ K' \xrightarrow{x_1} K'\ )$ is a free resolution of $R/\langle \mathbf{x}' \rangle$. $\square$

Remark II.B.9. If $\mathbf{x} = x_1, \dots, x_n \in R$ is weakly R-regular, then $K^R(x_n, \dots, x_1)$ is a free resolution of $R/\langle \mathbf{x} \rangle$ by Theorem II.B.7 and Exercise 2.4.

Remark II.B.10. The ranks of the modules in $K^R(\mathbf{x})$ are symmetric because of the symmetry in Pascal's triangle. This leads us to the next property, called the self-duality of the Koszul complex. Note that it generalizes the self-duality we observed in Example I.E.7.

Theorem II.B.11 (Self-duality of the Koszul complex). *The Koszul complex is self-dual, i.e.,* $K^R(\mathbf{x}) \cong \Sigma^n \operatorname{Hom}_R(K^R(\mathbf{x}), R)$.

Proof. We will use induction on n.

Base Case: Let $n = 1$. Then we can directly write down an isomorphism between the complexes $K^R(x_1)$ and $\Sigma \operatorname{Hom}_R(K^R(x_1), R)$ as follows:

$$\begin{tikzcd}
K = K^R(x_1) = & 0 \arrow[r] & R \arrow[r, "x_1"] & R \arrow[r] & 0 & \\
K^* = \operatorname{Hom}_R(K,R) = & & 0 \arrow[r] & R \arrow[r, "x_1"] & R \arrow[r] & 0 \\
\Sigma K^* = \arrow[d, "\cong"'] & 0 \arrow[r] & R \arrow[r, "-x_1"] \arrow[d, "-1"'] & R \arrow[r] \arrow[d, "1"'] & 0 & \\
K = & 0 \arrow[r] & R \arrow[r, "x_1"] & R \arrow[r] & 0 &
\end{tikzcd}$$

Base Case: Let $n = 2$. Then we can directly write down an isomorphism between the complexes $K^R(x_1, x_2)$ and $\Sigma^2 \operatorname{Hom}_R(K^R(x_1, x_2), R)$ as follows:

$$\begin{tikzcd}
K = K^R(x_1,x_2) = & 0 \arrow[r] & R \arrow[r, "\begin{pmatrix} -y \\ x \end{pmatrix}"] & R^2 \arrow[r, "\begin{pmatrix} x & y \end{pmatrix}"] & R \arrow[r] & 0 & \\
K^* = \operatorname{Hom}_R(K,R) = & & & 0 \arrow[r] & R \arrow[r, "\begin{pmatrix} x \\ y \end{pmatrix}"] & R^2 \arrow[r, "\begin{pmatrix} -y & x \end{pmatrix}"] & R \arrow[r] & 0 \\
\Sigma^2 K^* = \arrow[d, "\cong"'] & 0 \arrow[r] & R \arrow[r, "\begin{pmatrix} x \\ y \end{pmatrix}"] \arrow[d, "-1"'] & R^2 \arrow[r, "\begin{pmatrix} -y & x \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}"'] & R \arrow[r] \arrow[d, "1"'] & 0 & \\
K = & 0 \arrow[r] & R \arrow[r, "\begin{pmatrix} -y \\ x \end{pmatrix}"'] & R^2 \arrow[r, "\begin{pmatrix} x & y \end{pmatrix}"'] & R \arrow[r] & 0 &
\end{tikzcd}$$

For both of the above base cases, a diagram chase can check that the diagrams are commutative.

Inductive Case: Suppose that $\Phi\colon K' = K^R(x_2, \dots, x_n) \to \Sigma^{n-1}(K')^*$ is an isomorphism. Notice that the following diagram commutes:

$$\begin{tikzcd}
K' \arrow[r, "x_1"] \arrow[d, "\cong"', "\Phi"] & K' \arrow[d, "\cong"', "\Phi"] \\
\Sigma^{n-1}(K')^* \arrow[r, "x_1"] & \Sigma^{n-1}(K')^*.
\end{tikzcd}$$

Then

$$\begin{aligned}
K &= \operatorname{Cone}(\, K' \xrightarrow{x_1} K' \,) && \text{(by definition)} \\
&\cong \operatorname{Cone}(\, \Sigma^{n-1}(K')^* \xrightarrow{x_1} \Sigma^{n-1}(K')^* \,) && \text{(by Exercise 2.1 and previous diagram)} \\
&\cong \Sigma^{n-1} \operatorname{Cone}(\, (K')^* \xrightarrow{x_1} (K')^* \,) && \text{(by Lemma II.B.12(a))} \\
&\cong \Sigma^{n-1} \left(\Sigma(\operatorname{Cone}(\, K' \xrightarrow{x_1} K' \,))^* \right) && \text{(by Lemma II.B.12(b))} \\
&\cong \Sigma^n K^*.
\end{aligned}$$

An alternate proof is given later in the chapter which does not utilize Lemma II.B.12. $\square$

Lemma II.B.12. *Let* $\Psi\colon A \to C$ *be a chain map.*

(a) *Define* $\Sigma\Psi\colon \Sigma A \to \Sigma C$*, using the same rule as* Ψ*. Then* $\Sigma\Psi$ *is a chain map and*

$$\operatorname{Cone}(\Sigma\Psi) \cong \Sigma \operatorname{Cone}(\Psi).$$

Moreover, inductively applying the result for positive integers n gives

$$\operatorname{Cone}(\Sigma^n \Psi) \cong \Sigma^n \operatorname{Cone}(\Psi).$$

(b) *The map $\Psi^* \colon C^* \to A^*$ where $(-)^* = \operatorname{Hom}_R(-, R)$ is a chain map and*

$$\operatorname{Cone}(\Psi^*) \cong \Sigma \operatorname{Cone}(\Psi)^*.$$

SKETCH OF PROOF. Consider the following commutative diagram.

$$\begin{tikzcd}
C^* = \arrow[d, "\Psi^*"'] & (C_{-n})^* \arrow[r, "(\partial^C_{1-n})^*"] \arrow[d, "\Psi^*_{-n}"'] & (C_{1-n})^* \arrow[d, "\Psi^*_{1-n}"] \\
A^* = & (A_{-n})^* \arrow[r, "(\partial^A_{1-n})^*"] & (A_{1-n})^*
\end{tikzcd}$$

This shows that $\operatorname{Cone}(\Psi^*)_n = (C_{1-n})^* \oplus (A_{-n})^*$. A similar computation shows that this is $(\Sigma \operatorname{Cone}(\Psi)^*)_n$. In the interest of time, we omit the remaining details of the proof. $\square$

The Exterior Basis for the Koszul Complex.

DEFINITION II.B.13. Let $\mathbf{x} = x_1, \ldots, x_n \in R$ be a weakly R-regular sequence in R. Then $L = L^R(\mathbf{x})$ is the following sequence of R-module homomorphisms, where we label each module with its homological degree.

$$\begin{array}{ccccccccccccccccc}
0 & \longrightarrow & R & \longrightarrow & R^n & \longrightarrow & \cdots & \longrightarrow & R^{\binom{n}{i}} & \longrightarrow & \cdots & \longrightarrow & R^n & \longrightarrow & R & \longrightarrow & 0 \\
 & & n+1 & & n & & & & i & & & & 1 & & 0 & &
\end{array}$$

A basis for $R^{\binom{n}{i}} = L_i$ is $e_\Lambda = e_{\lambda_1, \ldots, \lambda_i}$ where $\Lambda = \{\lambda_1, \ldots, \lambda_i\}$ and $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$. We may also write $\Lambda = \{\lambda_1 < \cdots < \lambda_i\} \subseteq [n] := \{1, \ldots, n\}$. The differentials of the sequence are given by

$$\partial_i^L(e_{\lambda_1, \ldots, \lambda_i}) = \sum_{j=1}^{i} (-1)^{j-1} e_{\lambda_1, \ldots, \widehat{\lambda}_j, \ldots, \lambda_i} \in R^{\binom{n}{i-1}}$$

where e_Γ with $|\Gamma| = i - 1$ is a basis vector in $R^{\binom{n}{i-1}}$ and $\lambda_1, \ldots, \widehat{\lambda}_j, \ldots, \lambda_i = \lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_i$. For instance,

$$\begin{aligned}
\partial_2^L(e_{pq}) &= x_p e_q - x_q e_p, \\
\partial_3^L(e_{pqr}) &= x_p e_{qr} - x_q e_{pr} + x_r e_{pq}, \text{ and} \\
\partial_1^L(e_p) &= x_p e_\emptyset = x_p \cdot 1_R.
\end{aligned}$$

THEOREM II.B.14. *The sequence of R-module homomorphisms L is an R-complex and $L \cong K = K^R(\mathbf{x})$.*

EXAMPLE II.B.15. We give explicitly the R-complex L for sequences of sizes two and three before proving L is an R-complex.

$$L^R(x, y) = \qquad 0 \longrightarrow R \xrightarrow{\left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & y \end{smallmatrix}\right)} R \longrightarrow 0$$

$$\begin{aligned}
e_{12} &\longmapsto xe_2 - ye_1 \\
e_1 &\longmapsto x \\
e_2 &\longmapsto y
\end{aligned}$$

$$
L^R(x,y,z) = \qquad 0 \longrightarrow R \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} R^3 \xrightarrow{(x\ y\ z)} R \longrightarrow 0
$$

$$
\begin{aligned}
e_{123} &\longmapsto xe_{23} - ye_{13} + ze_{12} \\
e_1 &\longmapsto x \\
e_2 &\longmapsto y \\
e_3 &\longmapsto z \\
e_{12} &\longmapsto xe_2 - ye_1 \\
e_{13} &\longmapsto xe_3 - ze_1 \\
e_{23} &\longmapsto ye_3 - ze_2
\end{aligned}
$$

Proof of Theorem II.B.14. To prove L is and R-complex, it suffices to fix an arbitrary i and show that we have $\left(\partial^L_{i-1} \circ \partial^L_i\right)(e_\Lambda) = 0$ for all subsets Λ satisfying $|\Lambda| = i$. To see the argument more concretely, we first observe

$$
\begin{aligned}
\left(\partial^L_2 \circ \partial^L_3\right)(e_{pqr}) &= \partial^L_2\left(x_p e_{qr} - x_q e_{pr} + x_r e_{pq}\right) \\
&= x_p \partial^L_2(e_{qr}) - x_q \partial^L_2(e_{pr}) + x_r \partial^L_2(e_{pq}) \\
&= x_p(x_q e_r - x_r e_q) - x_q(x_p e_r - x_r e_p) + x_r(x_p e_q - x_q e_p) \\
&= 0.
\end{aligned}
$$

In general we have

$$
\begin{aligned}
\left(\partial^L_{i-1} \circ \partial^L_i\right)(e_\Lambda) &= \partial^L_{i-1}\left(\sum_{j=1}^{i} (-1)^{j-1} x_{\lambda_j} e_{\lambda_1,\ldots,\widehat{\lambda}_j,\ldots,\lambda_i}\right) \\
&= \sum_{j=1}^{i} (-1)^{j-1} x_{\lambda_j} \partial^L_{i-1}(e_{\lambda_1,\ldots,\widehat{\lambda}_j,\ldots,\lambda_i}) \\
&= \sum_{j=1}^{i} (-1)^{j-1} x_{\lambda_j} \left[\left(\sum_{\ell=1}^{j-1} (-1)^{\ell-1} x_{\lambda_\ell} e_{\lambda_1,\ldots,\widehat{\lambda}_\ell,\ldots,\widehat{\lambda}_j,\ldots,\lambda_i}\right) + \left(\sum_{\ell=j+1}^{i} (-1)^{\ell-2} x_{\lambda_\ell} e_{\lambda_1,\ldots,\widehat{\lambda}_j,\ldots,\widehat{\lambda}_\ell,\ldots,\lambda_i}\right)\right] \\
&= 0
\end{aligned}
$$

The last equality holds since every basis vector $e_{\lambda_1,\ldots,\widehat{\lambda}_p,\ldots,\widehat{\lambda}_q,\ldots,\lambda_i}$ occurs twice in the sum and of opposite signs. In the case when λ_p is removed first, the coefficient is $(-1)^{p-1+q-2} x_{\lambda_p} x_{\lambda_q}$, and in the case when λ_q is removed first, the coefficient is $(-1)^{q-1+p-1} x_{\lambda_q} x_{\lambda_p}$.

We prove $L \cong K$ by induction on n. The base cases $n = 2, 3$ are done by Example II.B.15 and the case $n = 1$ is routine. For the inductive step set $\mathbf{x}' = x_2, \ldots, x_n$, and $K' = K^R(\mathbf{x}')$ and $L' = L^R(\mathbf{x}')$. By the induction hypothesis $L' \cong K'$ and we let $\Psi\colon L' \to K'$ be an isomorphism, which automatically makes the following diagram commute.

$$
\begin{tikzcd}
L' \arrow[r, "\Psi", "\cong"'] \arrow[d, "x_1"'] & K' \arrow[d, "x_1"] \\
L' \arrow[r, "\cong", "\Psi"'] & K'
\end{tikzcd}
$$

By Exercise 1(b) in homework 2 this diagram shows that $K = \operatorname{Cone}(K' \xrightarrow{x_1} K') \cong \operatorname{Cone}(L' \xrightarrow{x_1} L')$. It therefore suffices to show $L \cong \operatorname{Cone}(L' \xrightarrow{x_1} L')$. We claim the chain map ϕ in the diagram

$$\begin{array}{ccccccc}
\operatorname{Cone}(L' \xrightarrow{x_1} L')_i & = & \begin{array}{c} L'_{i-1} \\ \oplus \\ L'_i \end{array} & \xrightarrow{\begin{pmatrix} -\partial^{L'}_{i-1} & 0 \\ x_1 & \partial^{L'}_i \end{pmatrix}} & \begin{array}{c} L'_{i-2} \\ \oplus \\ L'_{i-1} \end{array} & = & \operatorname{Cone}(L' \xrightarrow{x_1} L')_{i-1} \\
 & & \Big\downarrow{\scriptstyle \phi_i} & & \Big\downarrow{\scriptstyle \phi_{i-1}} & & \\
 & & L_i & \longrightarrow & L_{i-1} & &
\end{array}$$

is the desired isomorphism, where ϕ_i is defined on basis vectors as follows. Basis vectors for $L'_{i-1} \oplus L'_i$ are of the form $\begin{pmatrix} 0 & e_{\lambda_1,\ldots,\lambda_i} \end{pmatrix}^T$ and $\begin{pmatrix} e_{\gamma_2,\ldots,\gamma_i} & 0 \end{pmatrix}^T$ where $2 \leq \lambda_1 < \cdots < \lambda_i \leq n$ and $2 \leq \gamma_2 < \cdots < \gamma_i \leq n$. We bound below by 2 since $L' = L^R(\mathbf{x}')$ and we begin our index for $\Gamma = \{\gamma_2 < \cdots < \gamma_i\}$ with 2 since L'_{i-1} has basis vectors of size $i-1$. It suffices to show the diagram above commutes with respect to these basis vectors. Define ϕ by the following:

$$\phi_i \begin{pmatrix} 0 \\ e_{\lambda_1,\ldots,\lambda_i} \end{pmatrix} = e_{\lambda_1,\ldots,\lambda_i} \qquad \text{and} \qquad \phi_i \begin{pmatrix} e_{\gamma_2,\ldots,\gamma_i} \\ 0 \end{pmatrix} = e_{1,\gamma_2,\ldots,\gamma_i}.$$

We compute

$$\begin{aligned}
\phi_{i-1}\left(\partial_i^{\operatorname{Cone}(x_1)} \begin{pmatrix} 0 \\ e_{\lambda_1,\ldots,\lambda_i} \end{pmatrix}\right) &= \phi_{i-1} \begin{pmatrix} 0 \\ \partial_i^{L'}(e_{\lambda_1,\ldots,\lambda_i}) \end{pmatrix} \\
&= \phi_{i-1} \begin{pmatrix} 0 \\ \sum_{j=1}^i (-1)^{j-1} x_{\lambda_j} e_{\lambda_1,\ldots,\widehat{\lambda}_j,\ldots,\lambda_i}) \end{pmatrix} \\
&= \sum_{j=1}^i (-1)^{j-1} x_{\lambda_j} \phi_{i-1} \begin{pmatrix} 0 \\ e_{\lambda_1,\ldots,\widehat{\lambda}_j,\ldots,\lambda_i} \end{pmatrix} \\
&= \sum_{j=2}^i (-1)^{j-1} x_{\lambda_j} e_{\lambda_1,\ldots,\widehat{\lambda}_j,\ldots,\lambda_i} \\
&= \partial_i^L (e_{\lambda_1,\ldots,\lambda_i}) \\
&= \partial_i^L \left(\phi_i \begin{pmatrix} 0 \\ e_{\lambda_1,\ldots,\lambda_i} \end{pmatrix}\right)
\end{aligned}$$

and similarly

$$\begin{aligned}
\phi_{i-1}\left(\partial_i^{\operatorname{Cone}(x_1)} \begin{pmatrix} e_{\gamma_2,\ldots,\gamma_i} \\ 0 \end{pmatrix}\right) &= \phi_{i-1} \begin{pmatrix} -\partial_{i-1}^{L'}(e_{\gamma_2,\ldots,\gamma_i}) \\ x_1 e_{\gamma_2,\ldots,\gamma_i} \end{pmatrix} \\
&= \phi_{i-1} \begin{pmatrix} -\sum_{j=2}^i (-1)^j x_{\gamma_j} e_{\gamma_2,\ldots,\widehat{\gamma}_j,\ldots,\gamma_i} \\ x_1 e_{\gamma_2,\ldots,\gamma_i} \end{pmatrix} \\
&= \sum_{j=2}^i (-1)^{j-1} x_{\gamma_j} e_{1,\gamma_2,\ldots,\widehat{\gamma}_j,\ldots,\gamma_i} + x_1 e_{\gamma_2,\ldots,\gamma_i} \\
&= \partial_i^{\operatorname{Cone}(x_1)} (e_{1,\gamma_2,\ldots,\gamma_i}) \\
&= \partial_i^{\operatorname{Cone}(x_1)} \left(\phi_i \begin{pmatrix} e_{\gamma_2,\ldots,\gamma_i} \\ 0 \end{pmatrix}\right),
\end{aligned}$$

so ϕ is a chain map and by construction it bijectively maps basis vectors to basis vectors. (The basis vector assignment is 1-1, so it is bijective by the pigeon hole principle.) Therefore ϕ_i is bijective for all $i \in \mathbb{Z}$ and ϕ is therefore an isomorphism of R-complexes. □

FACT II.B.16. *If A is an R-complex, then $\Sigma^n A$ is isomorphic to*

$$\cdots \xrightarrow{-\partial^A_{i-n+1}} A_{i-n} \xrightarrow{-\partial^A_{i-n}} A_{i-n-1} \xrightarrow{-\partial^A_{i-n-1}} \cdots.$$

The following result was already presented as Theorem II.B.11. We give it again for convenience and present another proof in which we use the exterior basis for K and the dual basis for K^*.

Theorem II.B.17 (Self-duality of the Koszul complex). *Let $\mathbf{x} = x_1, \ldots, x_n \in R$ be given, and set $K = K^R(\mathbf{x})$ and $K^* = \operatorname{Hom}_R(K, R)$. Then $\Sigma^n K^* \cong K$.*

Alternate Proof. Recall that

$$\begin{array}{ccc} \operatorname{Hom}_R(R^t, R) & \cong & R^t \\ e_1^* & & e_1 \\ \vdots & \longleftarrow\!\!\shortmid & \vdots \\ e_t^* & & e_t \end{array}$$

with $e_i^*(e_j) = \delta_{ij}$ and $e_i^*(\sum_j \alpha_j e_j) = \alpha_i$. Then we have $(K^*)_{-i} = \left(R^{\binom{n}{i}}\right)^* \cong R^{\binom{n}{i}}$ with dual basis e_Λ^* and

$$e_\Lambda^*(e_\Gamma) = \begin{cases} 0 & \text{if } \Lambda \neq \Gamma \\ 1 & \text{if } \Lambda = \Gamma, \end{cases}$$

where $|\Lambda| = i = |\Gamma|$. Note also that

$$\partial_{-i}^{K^*}(e_\Lambda^*) = \left(\partial_{i+1}^K\right)^*(e_\Lambda^*) = e_\Lambda^* \circ \partial_{i+1}^K \tag{II.B.17.1}$$

for $\Lambda \subseteq [n]$ satisfying $|\Lambda| = i$. Let $\gamma \in [n] \setminus \Lambda$, and define $s(\gamma, \Lambda)$ to be the number of swaps needed to put the list $\gamma, \lambda_1, \ldots, \lambda_i$ in order. In other words, if $\lambda_{j-1} < \gamma < \lambda_j$, then $s(\gamma, \Lambda) = j - 1$.

For example, if $n = 9$ and $\Lambda = \{3 < 5 < 7\}$ and $\gamma = 4$, then $s(\gamma, \Lambda) = 1$.

Continuing the proof, we claim that $\partial_{-i}^{K^*}(e_\Lambda^*) = \sum_{\gamma \in [n] \setminus \Lambda} (-1)^{s(\gamma,\Lambda)} x_\gamma e_{\Lambda \cup \{\gamma\}}^*$ in $K_{i+1}^* = \operatorname{Hom}_R(K_{i+1}, R)$ with basis e_Γ^* for $|\Gamma| = i + 1$. It suffices to show that

$$\partial_{-i}^{K^*}(e_\Lambda^*)(e_\Gamma) = \sum_{\gamma \in [n] \setminus \Lambda} (-1)^{s(\gamma,\Lambda)} x_\gamma e_{\Lambda \cup \{\gamma\}}^*(e_\Gamma). \tag{II.B.17.2}$$

First consider the left hand side. By display (II.B.17.1) for $\Gamma = \{\gamma_1 < \cdots < \gamma_{i+1}\}$, we have

$$\begin{aligned} \partial_{-i}^{K^*}(e_\Lambda^*)(e_\Gamma) &= e_\Lambda^*(\partial_{i+1}^K(e_\Gamma)) \\ &= e_\Lambda^*\left(\sum_{j=1}^{i+1} (-1)^{j-1} x_{\gamma_j} e_{\Gamma \setminus \{\gamma_j\}}\right) \\ &= \sum_{j=1}^{i+1} (-1)^{j-1} x_{\gamma_j} e_\Lambda^*(e_{\Gamma \setminus \{\gamma_j\}}) \\ &= \begin{cases} 0 & \text{if } \Lambda \not\subseteq \Gamma \\ (-1)^{j-1} x_{\gamma_j} & \text{if } \Gamma \setminus \{\gamma_j\} = \Lambda \text{ and } \Lambda \subseteq \Gamma \end{cases}. \end{aligned}$$

Next consider the right hand side of display (II.B.17.2). If $\Lambda \not\subseteq \Gamma$, then $\Lambda \cup \{\gamma\} \neq \Gamma$ for all $\gamma \in [n] \setminus \Lambda$, so $e_{\Lambda \cup \{\gamma\}}^*(e_\Gamma) = 0$. Therefore, the right hand side of (II.B.17.2) is 0 as well. If $\Lambda \subseteq \Gamma$, then there is a unique $\gamma_j \in [n] \setminus \Lambda$ such that $\Lambda \cup \{\gamma_j\} = \Gamma$. Then the right hand side of (II.B.17.2) is equal to $(-1)^{s(\gamma_j,\Lambda)} x_{\gamma_j}$. Notice that $\Lambda = \{\gamma_1 < \cdots < \gamma_{j-1} < \gamma_{j+1} < \cdots\}$, so $s(\gamma_j, \Lambda) = j - 1$.

Now we want to show that following diagram commutes:

$$\begin{tikzcd} (K^*)_{-i} = (\Sigma^n K^*)_{n-i} \arrow[r, "\partial_{n-i}^{\Sigma^n K^*}", "-\partial_{-i}^{K^*}"'] \arrow[d, "\Phi_{n-i}"] & (\Sigma^n K^*)_{n-i-1} = (K^*)_{-(i+1)} \arrow[d, "\Phi_{n-i-1}"] \\ K_{n-i} \arrow[r, "\partial_{n-i}^K"] & K_{n-i-1}. \end{tikzcd}$$

A diagram chase of the above diagram follows:

$$
\begin{tikzcd}
e^*_\Lambda \arrow[r, mapsto] \arrow[dd, mapsto] & \displaystyle -\sum_{\gamma\in[n]\setminus\Lambda}(-1)^{s(\gamma,\Lambda)}x_\gamma e^*_{\Lambda\cup\{\gamma\}} \arrow[d, mapsto] \\
& \displaystyle -\sum_{\gamma\in[n]\setminus\Lambda}(-1)^{s(\gamma,\Lambda)+\sum_{\lambda\in\Lambda\setminus\{\gamma\}}\lambda}x_\gamma e_{[n]\setminus(\Lambda\cup\{\gamma\})} \\
(-1)^{\sum_{\lambda\in\Lambda}\lambda}e_{[n]\setminus\Lambda} \arrow[r, mapsto] & \displaystyle (-1)^{\sum_{\lambda\in\Lambda}\lambda}\sum_{\gamma\in[n]\setminus\Lambda}(-1)^{t(\gamma,\Lambda)-1}x_\gamma e_{([n]\setminus\Lambda)\setminus\{\gamma\}}
\end{tikzcd}
$$

Here, $t(\gamma,\Lambda)$ is the position of γ in $[n]\setminus\Lambda$. Observe that $s(\gamma,\Lambda)+t(\gamma,\Lambda)=\gamma$. Consider the example from earlier in the proof. For instance, if $n=9$ and $\Lambda=\{3<5<7\}$ and $\gamma=4$, then we have $t(\gamma,\Lambda)=3$ because $[n]\setminus\Lambda=\{1<2<4<6<8<9\}$. Then we can see that $s(\gamma,\Lambda)+t(\gamma,\Lambda)=4=\gamma$.

Now we need to check that the two lines in the bottom right corner of the above diagram are equal. It suffices to show that the powers of (-1) are congruent modulo 2:

$$
\begin{aligned}
s(\gamma,\Lambda)+\sum_{\lambda\in\Lambda\setminus\{\gamma\}}\lambda+1 &\overset{?}{\equiv} t(\gamma,\Lambda)-1+\sum_{\lambda\in\Lambda}\lambda \pmod 2\\
s(\gamma,\Lambda)+t(\gamma,\Lambda) &\overset{?}{\equiv} \sum_{\lambda\in\Lambda}\lambda-\sum_{\lambda\in\Lambda\setminus\{\gamma\}}\lambda \pmod 2\\
\gamma &\overset{\checkmark}{\equiv} \gamma \pmod 2.
\end{aligned}
$$

Therefore, this is a chain map. To determine whether Φ is an isomorphism, it suffices to show that Φ is one-to-one and onto. Since Φ maps the dual basis of $\Sigma^n K^*$ to the exterior basis of K and the bases are the same size, Φ describes a basis bijection. Therefore, the induced map is an isomorphism. $\square$

CHAPTER II.C

Application: Depth Sensitivity of the Koszul Complex

In this chapter, assume that R is noetherian.

REMARK II.C.1. If $\mathbf{x} = x_1, \ldots, x_n \in R$ is weakly R-regular, then $K = K^R(\mathbf{x})$ is a resolution of $R/\langle \mathbf{x} \rangle$. Therefore, $H_i(K) = 0$ for all $i \neq 0$ and $H_0(K) \cong R/(\mathbf{x})$.

The question that comes up now is what happens when $\mathbf{x}$ is not weakly R-regular? In other words, which homology modules still vanish when $\mathbf{x}$ in not weakly R-regular? The answer is not immediately obvious, but we will see that $H_i(K^R(x))$ may be non-zero for some $i > 0$. For example, consider the Koszul complex on one element:

$$K^R(x) = \qquad 0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0.$$

Then $H_1(K^R(x)) \cong \operatorname{Ker}\left(R \xrightarrow{x} R \right) = 0$ if and only if x is a non-zero-divisor on R. This suggests that there is a connection between the existence of weakly R-regular sequences and vanishing of homology modules of $K^R(x)$. As another example, we consider the Koszul complex on two elements below.

EXAMPLE II.C.2. Let $x, y \in R$ and set $K = K^R(x, y)$ and $K' = K^R(y)$. Assume that y is a non-zero-divisor on R, then $H_i(K') = 0$ for all $i > 0$. We consider what happens to the homology modules of K if x is a zero-divisor on $R/\langle y \rangle$. We have the following short exact sequence by Theorem I.D.20:

$$0 \longrightarrow K' \longrightarrow K \longrightarrow \Sigma K' \longrightarrow 0.$$

We consider the long exact sequence of homology modules that arises from the above short exact sequence. In particular, we consider the case for $i \geq 2$ and separately the case for $i = 1$. For $i \geq 2$, we have

$$\cdots \longrightarrow \underbrace{H_i(K')}_{=0} \longrightarrow H_i(K) \longrightarrow \underbrace{H_{i-1}(K')}_{=0} \xrightarrow{x} H_{i-1}(K') \longrightarrow \cdots.$$

Therefore, by Fact I.B.2(c), we have $H_i(K) = 0$. For $i = 1$, we have

$$\begin{tikzcd}
\cdots \arrow[r] & \underbrace{H_1(K')}_{=0} \arrow[r] & H_1(K) \arrow[r] & H_0(K') \arrow[r, "x"] \arrow[d, "\cong"'] & H_0(K') \arrow[r] \arrow[d, "\cong"'] & \cdots \\
& & & R/\langle y \rangle \arrow[r, "x", "\text{not 1-1}"'] & R/\langle y \rangle .
\end{tikzcd}$$

Then $H_1(K) \cong \operatorname{Ker}\left(R/\langle y \rangle \xrightarrow{x} R/\langle y \rangle \right) \neq 0$, since x is a zero-divisor on $R/\langle y \rangle$.

REMARK II.C.3. The point of this chapter is that we can say exactly which $s \in \mathbb{N}$ satisfy $H_i(K) = 0$ for all $i > s$. The method of doing so is in terms of "depth". Conversely, if we know which s satisfies $H_i(K) = 0$ for all $i > s$, then we can calculate the depth.

To define depth, we need the following theorem which we state without proof.

THEOREM II.C.4 (Rees). *Let $I \lneq R$.*

(a) *The following are equivalent:*
 (i) *There exists a sequence $\mathbf{y} = y_1, \ldots, y_m \in I$ that is weakly R-regular, and*
 (ii) $\operatorname{Ext}^i_R(R/I, R) = 0$ *for all $i < m$.*

(b) *There exists a maximal weakly R-regular sequence in I. In other words, there exists a weakly R-regular sequence $\mathbf{y} = y_1, \ldots, y_m \in I$ such that for all $x \in I$, the sequence $y_1, \ldots, y_m, x$ is not weakly R-regular.*

(c) *Every maximal weakly R-regular sequence has the same length and the length is*

$$m = \min\left\{ i \geq 0 \;\middle|\; \operatorname{Ext}^i_R(R/I, R) \neq 0 \right\}.$$

DEFINITION II.C.5. The <u>depth of I in R</u>, denoted $\operatorname{depth}_I(R)$, is the length of any maximal weakly R-regular sequence in $I \lneq R$. When the ring and the ideal are understood, we will denote the depth as δ.

EXAMPLE II.C.6. We verify the conclusion of Theorem II.C.4(c) in the special case of $R = A[X_1, \ldots, X_d]$, where A is a nonzero noetherian commutative ring with identity, and $I = \langle X_1, \ldots, X_n \rangle$. Notice that I is weakly R-regular. Then $K = K^R(X_1, \ldots, X_n)$ is a free resolution of R/I. Therefore

$$\operatorname{Ext}^i_R(R/I, R) = H_{-i}(K^*) = H_{n-i}(\Sigma^n K^*) \cong H_{n-i}(K) = \begin{cases} 0 & \text{if } n > i \\ R/I & \text{if } n = i. \end{cases}$$

The isomorphism above comes from the self-duality property of K. Then we can see that

$$\min\left\{ i \geq 0 \;\middle|\; \operatorname{Ext}^i_R(R/I, R) \neq 0 \right\} = n,$$

which is also the length of the maximal weakly R-regular sequence $X_1, \ldots, X_n \in I$.

REMARK II.C.7. One can see δ visually as follows:

$$\underbrace{\operatorname{Ext}^0_R(R/I, R), \operatorname{Ext}^1_R(R/I, R), \ldots, \operatorname{Ext}^{\delta-1}_R(R/I, R)}_{=0}, \underbrace{\operatorname{Ext}^{\delta}_R(R/I, R)}_{\neq 0}.$$

So one may think of δ as the number of initial vanishing Ext modules.

THEOREM II.C.8 (Auslander-Buchsbaum). *Let $\mathbf{x} = x_1, \ldots, x_n \in I$ such that $I = \langle \mathbf{x} \rangle \lneq R$ and assume $\mathbf{y} = y_1, \ldots, y_m \in I$ is weakly R-regular. Then $H_i\left(K^R(\mathbf{x})\right) = 0$ for all $i > n - m$ and*

$$H_{n-m}\left(K^R(\mathbf{x})\right) \cong \operatorname{Ext}^m_R(R/I, R).$$

COROLLARY II.C.9 (Depth-sensitivity of the Koszul complex). *In the context of Theorem II.C.8 we have $H_i\left(K^R(\mathbf{x})\right) = 0$ for all $i > n - \delta$ and $H_{n-\delta}\left(K^R(\mathbf{x})\right) \neq 0$, i.e.,*

$$\delta = n - \max\left\{ i \geq 0 \;\middle|\; H_i\left(K^R(\mathbf{x})\right) \neq 0 \right\}.$$

REMARK II.C.10. One can once again see δ visually as follows:

$$\underbrace{H_n\left(K^R(\mathbf{x})\right), H_{n-1}\left(K^R(\mathbf{x})\right), \ldots, H_{n-\delta+1}\left(K^R(\mathbf{x})\right)}_{=0}, \underbrace{H_{n-\delta}\left(K^R(\mathbf{x})\right)}_{\neq 0}.$$

So one can think of δ as the number of "initial" vanishings of $H_i\left(K^R(\mathbf{x})\right)$ when counting from degree n.

EXAMPLE II.C.11. With this example we verify the conclusions of Theorem II.C.4 and Corollary II.C.9 for the ideal $I = \langle XY, XZ, YZ \rangle \lneq R$ where $R = k[X, Y, Z]$, i.e., we will show

$$\begin{aligned} &\delta = \operatorname{depth}_I(R) = 2, \\ &\operatorname{Ext}^0_R(R/I, R) = \operatorname{Ext}^1_R(R/I, R) = 0 \text{ and } \operatorname{Ext}^2_R(R/I, R) \neq 0, \text{ and} \\ &H_3(K) = H_2(K) = 0 \text{ and } H_1(K) \neq 0 \end{aligned}$$

where we set $K = K^R(XY, XZ, YZ)$.

To show that $\delta = 2$ we build a weakly regular sequence. Begin with the non-zero-divisor $XY \in I$. Then

$$\frac{R}{\langle XY \rangle} \cong \frac{k[X, Y]}{\langle XY \rangle}[Z]$$

and since $Z, X+Y \in I$ are each non-zero-divisors in the above quotient and I is an ideal, we know $XZ+YZ = (X+Y)Z \in I$ is also a non-zero-divisor. One can then check that $XY, XZ+YZ$ is a maximal weakly R-regular sequence in I, so $\delta = 2$.

To compute Ext modules we first need a projective resolution of R/I:

$$P = \qquad 0 \longrightarrow R^2 \xrightarrow[\partial_2^P]{\left(\begin{smallmatrix} -Z & -Z \\ Y & 0 \\ 0 & X \end{smallmatrix}\right)} R^3 \xrightarrow[\partial_1^P]{(XY \; XZ \; YZ)} R \longrightarrow 0.$$

Taking the dual we obtain

$$P^* \cong \qquad \underset{}{0} \longrightarrow \underset{0}{R} \xrightarrow[\left(\partial_1^P\right)^*]{\left(\begin{smallmatrix} XY \\ XZ \\ YZ \end{smallmatrix}\right)} \underset{-1}{R^3} \xrightarrow[\left(\partial_2^P\right)^*]{\left(\begin{smallmatrix} -Z & Y & 0 \\ -Z & 0 & X \end{smallmatrix}\right)} \underset{-2}{R^2} \longrightarrow 0.$$

Since each entry in the matrix defining $\left(\partial_1^P\right)^*$ is a non-zero-divisor on R, we have

$$\operatorname{Ext}_R^0(R/I, R) \cong \operatorname{Ker}\begin{pmatrix} XY \\ XZ \\ YZ \end{pmatrix} = 0$$

so P^* is exact in degree 0. Alternatively, one can prove this using the isomorphisms

$$\operatorname{Ext}_R^0(R/I, R) \cong \operatorname{Hom}_R(R/I, R) \overset{(1)}{\cong} \{r \in R \mid rI = 0\} \overset{(2)}{=} 0,$$

where (2) holds since I contains a non-zero-divisor and the isomorphism in (1) is the evaluation map $\phi \mapsto \phi(1)$.

In degree -2 we observe that

$$\operatorname{Ext}_R^2(R/I, R) = \frac{R^2}{\left\langle \left(\begin{smallmatrix} -Z \\ -Z \end{smallmatrix}\right), \left(\begin{smallmatrix} Y \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ X \end{smallmatrix}\right) \right\rangle}.$$

We also note that

$$\left\langle \begin{pmatrix} -Z \\ -Z \end{pmatrix}, \begin{pmatrix} Y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix} \right\rangle \subseteq \mathfrak{m} \oplus \mathfrak{m} \subseteq R \oplus R = R^2$$

where $\mathfrak{m} = \langle X, Y, Z \rangle \lneq R$ is a maximal ideal. Thus there exists a surjection

$$\frac{R^2}{\left\langle \left(\begin{smallmatrix} -Z \\ -Z \end{smallmatrix}\right), \left(\begin{smallmatrix} Y \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ X \end{smallmatrix}\right) \right\rangle} \twoheadrightarrow \frac{R \oplus R}{\mathfrak{m} \oplus \mathfrak{m}} \cong \left(\frac{R}{\mathfrak{m}}\right)^2 \cong k^2 \neq 0$$

where the inequality holds because k is a field. Hence, P^* is not exact in this degree.

To show P^* is exact in degree -1 it suffices to show that $\operatorname{Ker}\left(\left(\partial_2^P\right)^*\right) \subseteq \operatorname{Im}\left(\left(\partial_1^P\right)^*\right)$. We observe that if $\begin{pmatrix} f & g & h \end{pmatrix}^T \in \operatorname{Ker}\left(\left(\partial_2^P\right)^*\right)$, then

$$Yg = Zf = Xh \tag{II.C.11.1}$$

which implies there exist $f_1, g_1, h_1 \in R$ such that

$$f = XYf_1, \ g = XZg_1, \ \text{and } h = YZh_1.$$

Substituting this into (II.C.11.1) we obtain

$$XYZg_1 = XYZf_1 = XYZh_1$$

which implies $f_1 = g_1 = h_1$. Hence

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = f_1 \begin{pmatrix} XY \\ XZ \\ YZ \end{pmatrix} \in \operatorname{Im}\left(\left(\partial_1^P\right)^*\right)$$

so P^* is exact in degree -1.

We now study the following Koszul complex.

$$0 \longrightarrow \underset{3}{R} \xrightarrow[\partial_3^K]{\left(\begin{smallmatrix} YZ \\ -XY \\ XY \end{smallmatrix}\right)} \underset{2}{R^3} \xrightarrow[\partial_2^K]{\left(\begin{smallmatrix} -XZ & -YZ & 0 \\ XY & 0 & -YZ \\ 0 & XY & XZ \end{smallmatrix}\right)} \underset{1}{R^3} \xrightarrow[\partial_1^K]{(XY \ XZ \ YZ)} R \longrightarrow 0$$

For exactness in degree 3, argue as in the computation of $\operatorname{Ext}_R^0(R/I, R)$. To prove exactness in degree 2 it suffices to show $\operatorname{Ker}\left(\partial_2^K\right) \subseteq \operatorname{Im}\left(\partial_3^K\right)$ and this argument is analogous to the argument used to compute $\operatorname{Ext}_R^1(R/I, R)$.

To see that $H_1(K)$ is non-zero, we observe

$$\begin{aligned}
\operatorname{Ker}\left(\partial_1^K\right) &= \operatorname{Ker}\begin{pmatrix} XY & XZ & YZ \end{pmatrix} \\
&= \operatorname{Im}\left(\partial_2^P\right) \\
&= \left\langle \begin{pmatrix} -Z \\ Y \\ 0 \end{pmatrix}, \begin{pmatrix} -Z \\ 0 \\ X \end{pmatrix} \right\rangle \\
&\supsetneq \left\langle \begin{pmatrix} -XZ \\ XY \\ 0 \end{pmatrix}, \begin{pmatrix} -YZ \\ 0 \\ XY \end{pmatrix}, \begin{pmatrix} 0 \\ -YZ \\ XZ \end{pmatrix} \right\rangle \\
&= \operatorname{Im}\left(\partial_2^K\right).
\end{aligned}$$

Example II.C.12. Set $R = k[X,Y]/\langle XY\rangle$ and define $I = \langle x, y\rangle \lneq R$ where $x = \overline{X} \in R$ and $y = \overline{Y} \in R$. We again will verify the conclusions of Theorem II.C.4 and Corollary II.C.9. One can conclude $\delta = 1$ by verifying that $x+y$ is a maximal weakly R-regular sequence in I. It remains to show that

$$\operatorname{Ext}^0_R(R/I, R) = 0 \text{ and } \operatorname{Ext}^1_R(R/I, R) \neq 0,$$

and

$$H_2(K) = 0 \text{ and } H_1(K) \neq 0,$$

where we set $K = K^R(x,y)$. Check that a (truncated) projective resolution of R/I is

$$P = \qquad \cdots \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & y \end{smallmatrix}\right)} R \longrightarrow 0$$

and applying $\operatorname{Hom}_R(-, R)$ we obtain

$$P^* = \qquad \underset{}{0} \longrightarrow \underset{0}{R} \xrightarrow[(\partial_1^P)^*]{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)} \underset{-1}{R^2} \xrightarrow[(\partial_2^P)^*]{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)} \cdots.$$

Compare this with P to conclude that P^* is exact in every degree below -1. Since $x+y \in I$ is a non-zero-divisor we have

$$\operatorname{Ext}^0_R(R/I, R) = \operatorname{Hom}_R(R/I, R) = 0.$$

In degree 1 we have

$$\operatorname{Ker}\left(\left(\partial_2^P\right)^*\right) = \operatorname{Ker}\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix} = \operatorname{Im}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \left\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle \supsetneq \left\langle \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \operatorname{Im}\left(\left(\partial_1^P\right)^*\right),$$

so $\operatorname{Ext}^1_R(R/I, R) \neq 0$.

The Koszul complex $K = K^R(x,y)$ is familiar.

$$0 \longrightarrow \underset{2}{R} \xrightarrow[\partial_2^K]{\left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)} \underset{1}{R^2} \xrightarrow[\partial_1^K]{\left(\begin{smallmatrix} x & y \end{smallmatrix}\right)} \underset{0}{R} \longrightarrow 0$$

One can show $H_2(K) = 0$ by showing that $\operatorname{Ker}\left(\partial_2^K\right) = 0$ as we did in the computation of $\operatorname{Ext}^0_R(R/I, R)$. We see that $H_1(K) \neq 0$ since

$$\operatorname{Ker}\left(\partial_1^K\right) = \operatorname{Im}\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix} = \left\langle \begin{pmatrix} y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix} \right\rangle \supsetneq \left\langle \begin{pmatrix} -y \\ x \end{pmatrix} \right\rangle = \operatorname{Im}\left(\partial_2^K\right).$$

Sketch of Proof of Theorem II.C.8. Let $K = K^R(\mathbf{x})$. To prove the result, we would induct on m. In the interest of time, we will only show the base cases for $m = 0$ and $m = 1$; the inductive step follows like the $m = 1$ case.

First consider the base case for $m = 0$. Then we have

$$\begin{aligned}
H_n(K) &\cong \operatorname{Ker}\left(R \xrightarrow[\partial_n^K]{\begin{pmatrix} \pm x_n \\ \vdots \\ -x_2 \\ x_1 \end{pmatrix}} R^n \right) \\
&= \{ r \in R \mid x_i r = 0 \ \forall\, i = 1, \ldots, n \} \\
&= \{ r \in R \mid Ir = 0 \} \\
&\cong \operatorname{Hom}_R(R/I, R) \\
&\cong \operatorname{Ext}^0_R(R/I, R).
\end{aligned}$$

The isomorphism from $\operatorname{Hom}_R(R/I, R)$ to $\{r \in R \mid Ir = 0\}$ is given by sending $\phi \in \operatorname{Hom}_R(R/I, R)$ to $\phi(\overline{1})$. This gives us our result for $m = 0$.

Next consider the base case for $m = 1$. Notice that $H_n(K) \cong \operatorname{Hom}_R(R/I, R)$ by the $m = 0$ case. (In the inductive step, this observation would be replaced by the inductive hypothesis.) By Theorem II.C.4(a) and since $m = 1$, we get $\operatorname{Hom}_R(R/I, R) \cong \operatorname{Ext}^0_R(R/I, R) = 0$. Now consider the following short exact sequence with $\overline{R} = R/\langle y_1 \rangle$:

$$0 \longrightarrow R \xrightarrow{y_1} R \longrightarrow \overline{R} \longrightarrow 0. \tag{II.C.12.1}$$

This induces the following short exact sequence on Koszul complexes, with $\overline{K} = K^{\overline{R}}(\overline{\mathbf{x}})$:

$$0 \longrightarrow K \xrightarrow{y_1} K \longrightarrow \overline{K} \longrightarrow 0.$$

By Theorem I.D.16, this induces the following long exact sequence of homologies:

$$0 \longrightarrow \underbrace{H_n(K)}_{=0} \xrightarrow{y_1} \underbrace{H_n(K)}_{=0} \longrightarrow H_n(\overline{K}) \longrightarrow H_{n-1}(K) \xrightarrow{y_1 = 0} H_{n-1}(K) \longrightarrow \cdots.$$

By Homework 3, the assumption $y_1 \in \langle \mathbf{x} \rangle$ implies $y_1 \cdot H_{n-1}(K) = 0$, which tells us the map in the second row of the above sequence is the zero map. Therefore by Fact I.B.2(d), the above sequence simplifies to

$$0 \longrightarrow H_n(\overline{K}) \xrightarrow{\cong} H_{n-1}(K) \longrightarrow 0.$$

Then $H_{n-1}(K) \cong H_n(\overline{K}) \cong \operatorname{Hom}_R(\overline{R}/\overline{I}, \overline{R}) \cong \operatorname{Hom}_R(R/I, \overline{R})$. Now consider the long exact sequence in $\operatorname{Ext}^i_R(R/I, -)$ associated to equation II.C.12.1:

$$0 \longrightarrow \underbrace{\operatorname{Hom}_R(R/I, R)}_{=0} \xrightarrow{y_1} \underbrace{\operatorname{Hom}_R(R/I, R)}_{=0} \longrightarrow \operatorname{Hom}_R(R/I, \overline{R}) \longrightarrow \operatorname{Ext}^1_R(R/I, R) \xrightarrow{y_1 = 0} \operatorname{Ext}^1_R(R/I, R) \longrightarrow \cdots.$$

As in the previous long exact sequence, Fact I.B.2(d) implies that $\operatorname{Ext}^1_R(R/I, R) \cong \operatorname{Hom}_R(R/I, \overline{R})$, so $\operatorname{Ext}^1_R(R/I, R) \cong H_{n-1}(K)$. This gives us our result for $m = 1$. $\square$

Remark II.C.13. Let $\delta = \operatorname{depth}_I(R)$. Then from Theorem II.C.4 and Theorem II.C.8, we have

$$\begin{gathered}
\overbrace{\operatorname{Ext}^0_R(R/I, R), \ldots, \operatorname{Ext}^{\delta-1}_R(R/I, R)}^{=0}, \overbrace{\operatorname{Ext}^{\delta}_R(R/I, R)}^{\neq 0}, \overbrace{\operatorname{Ext}^{\delta+1}_R(R/I, R), \ldots}^{?} \\
\underbrace{H_n(K^R(\mathbf{x})), \ldots, H_{n-m+1}(K^R(\mathbf{x}))}_{=0}, \underbrace{H_{n-m}(K^R(\mathbf{x}))}_{\neq 0}, \underbrace{H_{n-m-1}(K^R(\mathbf{x})), \ldots}_{?}.
\end{gathered}$$

We would like to know what happens to the end of each of the sequences in the above remark. The following fact tells us that all homologies past δ are nonzero. The proof of this fact requires some localization and Nakayama's Lemma, which are outside the scope of this course.

FACT II.C.14 (Rigidity of Koszul homology). *Let $I = \langle \mathbf{x} \rangle \lneq R$ and $\delta = \operatorname{depth}_I(R)$. Then $H_i(K^R(\mathbf{x})) \neq 0$ for all $i = 0, \ldots, \delta$.*

However, there is not as clear-cut an answer for the sequence in $\operatorname{Ext}^i_R(R/I, R)$ for $i > \delta$. The following example shows two different cases.

EXAMPLE II.C.15. Let $\mathbf{X} = X_1, \ldots, X_m$ and $\mathbf{Y} = Y_1, \ldots, Y_n$.

(a) Consider $R = k[\mathbf{X}, \mathbf{Y}]/\langle \mathbf{X} \rangle^2$. Set $x_i = \overline{X_i} \in R$ and $y_j = \overline{Y_j} \in R$ and $I = \langle \mathbf{x}, \mathbf{y} \rangle$. Notice that $\mathbf{y}$ is a maximal weakly R-regular sequence in I so $\delta = n$ because $R \cong \frac{k[\mathbf{X}]}{\langle \mathbf{X} \rangle^2}[\mathbf{Y}]$. Then:

$$\begin{aligned}
&\operatorname{Ext}^i_R(R/I,R) : \overbrace{\operatorname{Ext}^0_R(R/I,R), \ldots, \operatorname{Ext}^{n-1}_R(R/I,R)}^{=0}, \overbrace{\operatorname{Ext}^n_R(R/I,R)}^{\neq 0}, \overbrace{\operatorname{Ext}^{n+1}_R(R/I,R), \ldots, \operatorname{Ext}^{m+n}_R(R/I,R), \ldots}^{\neq 0} \\
&H_i(K^R(\mathbf{x},\mathbf{y})) : \underbrace{H_{m+n}(K^R(\mathbf{x},\mathbf{y})), \ldots, H_{m+1}(K^R(\mathbf{x},\mathbf{y}))}_{=0}, \underbrace{H_m(K^R(\mathbf{x},\mathbf{y}))}_{\neq 0}, \underbrace{H_{m-1}(K^R(\mathbf{x},\mathbf{y})), \ldots, H_0(K^R(\mathbf{x},\mathbf{y}))}_{\neq 0}
\end{aligned}$$

Therefore, we have an example of a sequence in Ext in which $\operatorname{Ext}^i_R \neq 0$ for all $i \geq \delta$.

(b) Consider $R = k[\mathbf{X}, \mathbf{Y}]/\langle X_1^2, \ldots, X_m^2 \rangle$. Set $x_i = \overline{X_i} \in R$ and $y_j = \overline{Y_j} \in R$ and $I = \langle \mathbf{x}, \mathbf{y} \rangle$. Notice that $\mathbf{y}$ is a maximal weakly R-regular sequence in I so $\delta = n$ because $R \cong \frac{k[\mathbf{X}]}{\langle X_1^2, \ldots, X_m^2 \rangle}[\mathbf{Y}]$. Then we have the following.

$$\begin{aligned}
&\operatorname{Ext}^i_R(R/I,R) : \overbrace{\operatorname{Ext}^0_R(R/I,R), \ldots, \operatorname{Ext}^{n-1}_R(R/I,R)}^{=0}, \overbrace{\operatorname{Ext}^n_R(R/I,R)}^{\neq 0}, \overbrace{\operatorname{Ext}^{n+1}_R(R/I,R), \ldots, \operatorname{Ext}^{m+n}_R(R/I,R), \ldots}^{=0} \\
&H_i(K^R(\mathbf{x},\mathbf{y})) : \underbrace{H_{m+n}(K^R(\mathbf{x},\mathbf{y})), \ldots, H_{m+1}(K^R(\mathbf{x},\mathbf{y}))}_{=0}, \underbrace{H_m(K^R(\mathbf{x},\mathbf{y}))}_{\neq 0}, \underbrace{H_{m-1}(K^R(\mathbf{x},\mathbf{y})), \ldots, H_0(K^R(\mathbf{x},\mathbf{y}))}_{\neq 0}
\end{aligned}$$

Therefore, we also have an example of a sequence in Ext in which $\operatorname{Ext}^i_R = 0$ for all $i > \delta$.

The next result shows one more connection between depth and the topic of this course.

THEOREM II.C.16 (Auslander-Buchsbaum). *Let $R = k[X_1, \ldots, X_d]$ and $\mathbf{f} = f_1, \ldots, f_n \in R$, where each f_i is a non-constant homogeneous polynomial. Let $I = \langle \mathbf{f} \rangle \lneq R$ and $\overline{R} = R/I$ and $x_i = \overline{X_i} \in \overline{R}$ and $m = \langle \mathbf{x} \rangle \lneq \overline{R}$ and $\Delta = \operatorname{depth}_m(\overline{R})$.*

(a) *There exists a free resolution $0 \to F_{d-\Delta} \to F_{d-\Delta-1} \to \cdots \to F_0 \to \overline{R} \to 0$.*

(b) *If $0 \to G_n \to \cdots \to G_0 \to R \to 0$ is a free resolution over R, then $n \geq d - \Delta$. Furthermore, $0 \to \operatorname{Ker}\left(\partial^G_{d-\Delta-1}\right) \to G_{d-\Delta-1} \to \cdots \to G_0 \to \overline{R} \to 0$ is exact and $\operatorname{Ker}\left(\partial^G_{d-\Delta-1}\right)$ is projective. By a result of Serre, $\operatorname{Ker}\left(\partial^G_{d-\Delta-1}\right)$ is also free.*

The slogan here is that the "projective dimension" of $\overline{R}$ over R is $\dim(R) - \operatorname{depth}_I\left(\overline{R}\right) = d - \Delta$.

CHAPTER II.D

The Taylor Resolution

In this chapter, assume that $R = k[X_1, \dots, X_d]$. We want to find explicit resolutions of R/I where I is a monomial ideal.

Recall II.D.1. A monomial in R is $\mathbf{X}^{\mathbf{e}} = X_1^{e_1} \cdots X_d^{e_d} \in R$, where $\mathbf{e} = (e_1, \dots, e_d) \in \mathbb{N}^d$. It is noteworthy that our definition of a monomial requires the coefficient to be 1.

Definition II.D.2. A monomial ideal in R is an ideal generated by monomials. We will use the notation $[\![R]\!]$ to represent the set of all monomials in R and $[\![I]\!] = I \cap [\![R]\!]$ to represent the set of all monomials in I.

Example II.D.3.
(a) The ideal $\mathfrak{X} = m = \langle \mathbf{X} \rangle = \langle X_1, \dots, X_d \rangle \lneq R$ is a monomial ideal.
(b) The ideal $\langle XY, XZ, YZ \rangle \lneq R = k[X, Y, Z]$ is a monomial ideal.

Definition II.D.4 (Taylor resolution). Let $\mathbf{f} = f_1, \dots, f_n \in [\![R]\!]$. Then the Taylor resolution of $\mathbf{f}$ is

$$T = T^R(\mathbf{f}) = \ (0 \longrightarrow R \xrightarrow{\partial_n^T} R^n \xrightarrow{\partial_{n-1}^T} \cdots \xrightarrow{\partial_{i+1}^T} R^{\binom{n}{i}} \xrightarrow{\partial_i^T} \cdots \xrightarrow{\partial_2^T} R^n \xrightarrow{\partial_1^T} R \longrightarrow 0)$$

where the basis is the same as the exterior basis for the Koszul complex

$$\{e_{j_1,\dots,j_i} \mid 1 \leq j_1 < \cdots < j_i \leq n\} \subset R^{\binom{n}{i}} = T_i$$

and

$$\partial_i^T(e_{j_1,\dots,j_i}) = \sum_{p=1}^{i} (-1)^{p-1} \frac{\operatorname{lcm}(f_{j_1}, \dots, f_{j_i})}{\operatorname{lcm}(f_{j_1}, \dots, \widehat{f_{j_p}}, \dots, f_{j_i})} \, e_{j_1,\dots,\widehat{j_p},\dots,j_i}.$$

Proposition II.D.5. *The Taylor resolution $T^R(\mathbf{f})$ is an R-complex which satisfies $H_0(T^R(\mathbf{f})) \cong R/\langle \mathbf{f} \rangle$.*

Example II.D.6. We give the following two examples of Taylor resolutions. The first is familiar.
(a) If $\mathbf{f} = X_1, \dots, X_n$, then $T^R(\mathbf{f}) = K^R(X_1, \dots, X_n)$, since

$$\frac{\operatorname{lcm}(X_{j_1}, \dots, X_{j_i})}{\operatorname{lcm}(X_{j_1}, \dots, \widehat{X}_{j_p}, \dots, X_{j_i})} = \frac{X_{j_1} \cdots X_{j_p} \cdots X_{j_i}}{X_{j_1} \cdots \widehat{X}_{j_p} \cdots X_{j_i}} = X_{j_p}.$$

(b) Let $I = \langle XY, XZ, YZ \rangle$. Then since I is generated by three elements, the outline for the Taylor resolution of I is

$$T^R(XY, XZ, YZ): \ (0 \longrightarrow R \xrightarrow{\partial_3^T} R^3 \xrightarrow{\partial_2^T} R^3 \xrightarrow{\partial_1^T} R \longrightarrow 0).$$

Next, we determine ∂_j^T for $j = 1, 2, 3$. For ∂_1^T, it is true in general that $e_i \in R^n$ maps to $f_i \in R$ for all i, since

$$e_i \mapsto (-1)^{1-1} \frac{\operatorname{lcm}(f_i)}{\operatorname{lcm}(\ )} e_\emptyset = 1 \cdot \frac{f_i}{1} \cdot 1 = f_i.$$

Therefore, we have $\partial_1^T = \begin{pmatrix} XY & XZ & YZ \end{pmatrix}$. For ∂_2^T, we see the following:

$$e_{12} \mapsto \frac{\mathrm{lcm}(f_1, f_2)}{\mathrm{lcm}(f_2)} e_2 - \frac{\mathrm{lcm}(f_1, f_2)}{\mathrm{lcm}(f_1)} e_1 = \frac{XYZ}{XZ} e_2 - \frac{XYZ}{XY} e_1 = Ye_2 - Ze_1 = \begin{pmatrix} -Z \\ Y \\ 0. \end{pmatrix}$$

$$e_{13} \mapsto \frac{\mathrm{lcm}(f_1, f_3)}{\mathrm{lcm}(f_3)} e_3 - \frac{\mathrm{lcm}(f_1, f_3)}{\mathrm{lcm}(f_1)} e_1 = \frac{XYZ}{YZ} e_3 - \frac{XYZ}{XY} e_1 = Xe_3 - Ze_1 = \begin{pmatrix} -Z \\ 0 \\ X. \end{pmatrix}$$

$$e_{23} \mapsto \frac{\mathrm{lcm}(f_2, f_3)}{\mathrm{lcm}(f_3)} e_3 - \frac{\mathrm{lcm}(f_2, f_3)}{\mathrm{lcm}(f_2)} e_2 = \frac{XYZ}{YZ} e_3 - \frac{XYZ}{XZ} e_2 = Xe_3 - Ye_2 = \begin{pmatrix} 0 \\ -Y \\ X. \end{pmatrix}$$

Therefore, we have

$$\partial_2^T = \begin{pmatrix} -Z & -Z & 0 \\ Y & 0 & -Y \\ 0 & X & X \end{pmatrix}.$$

For ∂_3^T, we see that

$$\begin{aligned} e_{123} \mapsto & \frac{\mathrm{lcm}(f_1, f_2, f_3)}{\mathrm{lcm}(f_2, f_3)} e_{23} - \frac{\mathrm{lcm}(f_1, f_2, f_3)}{\mathrm{lcm}(f_1, f_3)} e_{13} + \frac{\mathrm{lcm}(f_1, f_2, f_3)}{\mathrm{lcm}(f_1, f_2)} e_{12} \\ &= \frac{XYZ}{XYZ} e_{23} - \frac{XYZ}{XYZ} e_{13} + \frac{XYZ}{XYZ} e_{12} = e_{23} - e_{13} + e_{12} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \end{aligned}$$

Therefore, we have $\partial_3^T = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^T$.

REMARK II.D.7. We will see later that $T^R(\mathbf{f})$ is always a resolution of $R/\langle \mathbf{f} \rangle$ (under monomial assumptions). However, Examples II.C.11 and II.D.6 show that $T^R(\mathbf{f})$ might not be minimal.

PROOF OF PROPOSITION II.D.5. We first check that $\partial_{i-1}^T(\partial_i^T(e_{j_1,\dots,j_i})) = 0$ for all i. We have

$$\begin{aligned} \partial_{i-1}^T(\partial_i^T(e_{j_1,\dots,j_i})) &= \sum_{p=1}^{i} (-1)^{p-1} \frac{\mathrm{lcm}(f_{j_1}, \dots, f_{j_i})}{\mathrm{lcm}(f_{j_1}, \dots, \widehat{f}_{j_p}, \dots, f_{j_i})} \, \partial_{i-1}^T(e_{j_1,\dots,\widehat{j}_p,\dots,j_i}) \\ &= \sum_{p=1}^{i} (-1)^{p-1} \frac{\mathrm{lcm}(f_{j_1}, \dots, f_{j_i})}{\mathrm{lcm}(f_{j_1}, \dots, \widehat{f}_{j_p}, \dots, f_{j_i})} \left[\sum_{q=1}^{p-1} (-1)^{q-1} \frac{\mathrm{lcm}(f_{j_1}, \dots, \widehat{f}_{j_p}, \dots, f_{j_i})}{\mathrm{lcm}(f_{j_1}, \dots, \widehat{f}_{j_q}, \dots, \widehat{f}_{j_p}, \dots, f_{j_i})} e_{j_1,\dots,\widehat{j}_q,\dots,\widehat{j}_p,\dots,j_i} \right. \\ &\qquad \left. + \sum_{q=p+1}^{i} (-1)^{q} \frac{\mathrm{lcm}(f_{j_1}, \dots, \widehat{f}_{j_p}, \dots, f_{j_i})}{\mathrm{lcm}(f_{j_1}, \dots, \widehat{f}_{j_p}, \dots, \widehat{f}_{j_q}, \dots, f_{j_i})} e_{j_1,\dots,\widehat{j}_p,\dots,\widehat{j}_q,\dots,j_i} \right] \end{aligned}$$

Then the coefficients of the two inner sums exactly cancel, as in the proof of Theorem II.B.14. Therefore, $T^R(\mathbf{f})$ is an R-complex.

Also, we have

$$H_0(T) = \frac{\mathrm{Ker}\left(R \longrightarrow 0 \right)}{\mathrm{Im}\left(R^n \xrightarrow[\partial_1^T]{(f_1,\dots,f_n)} R \right)} = \frac{R}{\langle \mathbf{f} \rangle}$$

which is the desired result. $\square$

THEOREM II.D.8. *The Taylor resolution $T^R(\mathbf{f})$ is a free resolution of $R/\langle \mathbf{f} \rangle$.*

PROOF. We will use induction on n. First consider the base case for $n = 1$, where

$$T^R(\mathbf{f}) = (0 \longrightarrow R \xrightarrow{f_1} R \longrightarrow 0).$$

Since $0 \neq f_1 \in R = k[\mathbf{X}]$, then f_1 is a non-zero-divisor of R. Therefore, $H_i(T) = 0$ for all $i \neq 0$. As in Lemma II.A.3, $T^R(\mathbf{f})$ is a free resolution of $R/\langle \mathbf{f} \rangle$.

Now assume that the Taylor resolution on sequences of length $n-1$ resolve appropriately. Set $\mathbf{f}' = f_2, \ldots, f_n$ and $I' = \langle \mathbf{f}' \rangle$. We consider the colon ideal as in Example II.A.5(c)

$$(I' : f_1) = (\langle \mathbf{f}' \rangle : f_1) = \{g \in R \mid g f_1 \in \langle \mathbf{f}' \rangle\} = \langle g_2, \ldots, g_n \rangle,$$

where $g_i = \mathbf{X}^{(\mathbf{a}_i - \mathbf{a}_1)_+} = X_1^{(a_{i1}-a_{11})_+} \cdots X_d^{(a_{id}-a_{1d})_+}$ and where

$$(a_{ij} - a_{1j})_+ = \begin{cases} 0 & \text{if } a_{ij} - a_{1j} \leq 0 \\ a_{ij} - a_{1j} & \text{if } a_{ij} - a_{1j} \geq 0. \end{cases}$$

Set $\mathbf{g} = g_2, \ldots, g_n$, so $\langle \mathbf{g} \rangle = (I' : f_1)$. By the inductive hypothesis, $T^R(\mathbf{f}')$ is a free resolution of $R/\langle \mathbf{f}' \rangle$ and $T^R(\mathbf{g})$ is a free resolution of $R/\langle \mathbf{g} \rangle$. Our goal from here is to construct a chain map $\Psi^+ : T^R(\mathbf{g})^+ \to T^R(\mathbf{f})^+$ such that Ψ_{-1} is the map $R/\langle \mathbf{g} \rangle = R/(I' : f_1) \xrightarrow{f_1} R/I' = R/\langle \mathbf{f}' \rangle$ and $\operatorname{Cone}(\Psi) \cong T^R(\mathbf{f})$. Then by Theorem II.A.7, $\operatorname{Cone}(\Psi)$ will be a free resolution of $R/\langle \mathbf{f} \rangle$. Our setup for Ψ comes from filling in the question mark in the following diagram:

$$\begin{tikzcd}
T^R(\mathbf{g})^+ = \arrow[d, dashed, "\Psi^+"] & \cdots \arrow[r] & R^n \arrow[r] \arrow[d, dashed, "?"] & R \arrow[r] \arrow[d, dashed, "f_1"] & R/\langle \mathbf{g} \rangle \arrow[r] \arrow[d, hook, "f_1"] & 0 \\
T^R(\mathbf{f}')^+ = & \cdots \arrow[r] & R^n \arrow[r] & R \arrow[r] & R/I' \arrow[r] & 0.
\end{tikzcd}$$

For example, let $\mathbf{f} = X^2, XY, Y^2$. Then $\mathbf{f}' = XY, Y^2$ and $(\langle \mathbf{f}' \rangle : f_1) = (\langle XY, Y^2 \rangle : X^2) = \langle Y, Y^2 \rangle$, so $\mathbf{g} = Y, Y^2$. Then

$$\begin{tikzcd}
T^R(Y,Y^2)^+ = \arrow[d, dashed, "\Psi^+"] & 0 \arrow[r] & R \arrow[r, "\left(\begin{smallmatrix} -Y \\ 1 \end{smallmatrix}\right)"] \arrow[d, dashed, "(**)"] & R^2 \arrow[r, "(Y\ Y^2)"] \arrow[d, dashed, "(*)"] & R \arrow[r, "\pi"] \arrow[d, dashed, "X^2"] & R/\langle \mathbf{g} \rangle \arrow[r] \arrow[d, hook, "X^2"] & 0 \\
T^R(XY,Y^2)^+ = & 0 \arrow[r] & R \arrow[r, "\left(\begin{smallmatrix} -Y \\ X \end{smallmatrix}\right)"'] & R^2 \arrow[r, "(XY\ Y^2)"'] & R \arrow[r, "\tau"] & R/\langle \mathbf{f}' \rangle \arrow[r] & 0.
\end{tikzcd}$$

In order to determine $(*)$ and $(**)$, we chase the respective parts of the above diagram in order to make the diagram commute. For $(*)$, we have

$$\begin{tikzcd}
e_2 \arrow[r, mapsto] \arrow[d, mapsto] & Y \arrow[d, mapsto] & e_3 \arrow[r, mapsto] \arrow[d, mapsto] & Y^2 \arrow[d, mapsto] \\
Xe_2 \arrow[r, mapsto] & X^2Y & X^2e_3 \arrow[r, mapsto] & X^2Y^2 \\
e_2 \arrow[r, mapsto] & XY, & e_3 \arrow[r, mapsto] & Y^2.
\end{tikzcd}$$

Therefore $(*) = \left(\begin{smallmatrix} X & 0 \\ 0 & X^2 \end{smallmatrix}\right)$ makes the diagram commute. For $(**)$, we have

$$\begin{tikzcd}
e_{23} \arrow[r, mapsto] \arrow[d, mapsto] & e_3 - Ye_2 \arrow[d, mapsto] \\
Xe_{23} \arrow[r, mapsto] & X^2e_3 - XYe_2 \\
e_{23} \arrow[r, mapsto] & Xe_3 - Ye_2.
\end{tikzcd}$$

Therefore $(**) = X$ makes the diagram commute.

Continuing the proof, we prove two claims.

Claim.

$$g_j = \frac{\operatorname{lcm}(f_1, f_j)}{f_1}. \tag{II.D.8.1}$$

Proof. We prove this claim by comparing the exponents of X_q on each side of the equation. The exponent for X_q on the right hand side is $\max(a_{1q}, a_{jq}) - a_{1q}$, while the exponent for X_q on the left hand side is $(a_{jq} - a_{1q})_+$. There are two cases:

$$\begin{aligned}
&\text{If } a_{jq} \geq a_{1q}, \text{ then } \max(a_{1q}, a_{jq}) - a_{1q} = a_{jq} - a_{1q} = (a_{jq} - a_{1q})_+ \\
&\text{If } a_{jq} \leq a_{1q}, \text{ then } \max(a_{1q}, a_{jq}) - a_{1q} = a_{1q} - a_{1q} = 0 = (a_{jq} - a_{1q})_+
\end{aligned}$$

This proves the first claim. ✓

Claim.

$$f_1 \cdot \mathrm{lcm}(g_{j2}, \ldots, g_{ji}) = \mathrm{lcm}(f_1, f_{j2}, \ldots, f_{ji}). \tag{II.D.8.2}$$

Proof. Using Claim II.D.8.1, we have

$$\begin{aligned}
f_1 \cdot \mathrm{lcm}(g_{j2}, \ldots, g_{ji}) &= f_1 \cdot \mathrm{lcm}\left(\frac{\mathrm{lcm}(f_1, f_{j_2})}{f_1}, \ldots, \frac{\mathrm{lcm}(f_1, f_{j_i})}{f_1}\right) \\
&= \frac{f_1}{f_1} \cdot \mathrm{lcm}\left(\mathrm{lcm}(f_1, f_{j_2}), \ldots, \mathrm{lcm}(f_1, f_{j_i})\right) \\
&= \mathrm{lcm}(f_1, f_{j2}, \ldots, f_{ji}).
\end{aligned}$$

This proves the second claim. ✓

Now we define

$$\Psi_i(e_{j_1,\ldots,j_i}) = \frac{f_1 \cdot \mathrm{lcm}(g_{j_1}, \ldots, g_{j_i})}{\mathrm{lcm}(f_{j_1}, \ldots, f_{j_i})}\, e_{j_1,\ldots,j_i}.$$

By Claim II.D.8.2, the coefficients of the above definition are

$$\frac{f_1 \cdot \mathrm{lcm}(g_{j_1}, \ldots, g_{j_i})}{\mathrm{lcm}(f_{j_1}, \ldots, f_{j_i})} = \frac{\mathrm{lcm}(f_1, f_{j_1}, \ldots, f_{j_i})}{\mathrm{lcm}(f_{j_1}, \ldots, f_{j_i})},$$

which means they are elements of R. We want to show that Ψ is a chain map and that $\mathrm{Cone}(\Psi) \cong T^R(\mathbf{f})$. In order to check that Ψ is a chain map, consider the following diagram:

$$\begin{tikzcd}
R^{\binom{n}{i}} \arrow[r, "\partial_i^{T^R(\mathbf{g})}"] \arrow[d, "\Psi_i"] & R^{\binom{n}{i-1}} \arrow[d, "\Psi_{i-1}"] \\
R^{\binom{n}{i}} \arrow[r, "\partial_i^{T^R(\mathbf{f}')}"] & R^{\binom{n}{i-1}}.
\end{tikzcd}$$

A diagram chase of the above diagram follows:

$$\begin{tikzcd}
e_{j_1,\ldots,j_i} \arrow[r, mapsto] \arrow[ddd, mapsto] & \displaystyle\sum_{p=1}^{i}(-1)^{p-1}\frac{\mathrm{lcm}(g_{j_1}, \ldots, g_{j_i})}{\mathrm{lcm}(g_{j_1}, \ldots, \widehat{g_{j_p}}, \ldots, g_{j_i})} e_{j_1,\ldots,\widehat{j_p},\ldots,j_i} \arrow[d, mapsto] \\
& \displaystyle\sum_{p=1}^{i}(-1)^{p-1}\frac{\mathrm{lcm}(g_{j_1}, \ldots, g_{j_i})}{\mathrm{lcm}(g_{j_1}, \ldots, \widehat{g_{j_p}}, \ldots, g_{j_i})} \cdot \frac{f_1 \cdot \mathrm{lcm}(g_{j_1}, \ldots, \widehat{g_{j_p}}, \ldots, g_{j_i})}{\mathrm{lcm}(f_{j_1}, \ldots, \widehat{f_{j_p}}, \ldots, f_{j_i})} e_{j_1,\ldots,\widehat{j_p},\ldots,j_i} \\
& \\
\displaystyle\frac{f_1 \cdot \mathrm{lcm}(g_{j_1}, \ldots, g_{j_i})}{\mathrm{lcm}(f_{j_1}, \ldots, f_{j_i})} e_{j_1,\ldots,j_i} \arrow[r, mapsto] & \displaystyle\frac{f_1 \cdot \mathrm{lcm}(g_{j_1}, \ldots, g_{j_i})}{\mathrm{lcm}(f_{j_1}, \ldots, f_{j_i})} \sum_{p=1}^{i}(-1)^{p-1}\frac{\mathrm{lcm}(f_{j_1}, \ldots, f_{j_i})}{\mathrm{lcm}(f_{j_1}, \ldots, \widehat{f_{j_p}}, \ldots, f_{j_i})} e_{j_1,\ldots,\widehat{j_p},\ldots,j_i}.
\end{tikzcd}$$

Notice that once we cancel factors in both sums in the bottom right corner, we see that they are equal. Therefore, Ψ is a chain map.

In order to check that $\mathrm{Cone}(\Psi) \cong T^R(\mathbf{f})$, consider the following diagram:

$$\begin{tikzcd}
\mathrm{Cone}(\Psi)_i = \arrow[d, "\Phi_i"] & \begin{matrix} T^R(\mathbf{g})_{i-1} \\ \oplus \\ T^R(\mathbf{f}')_i \end{matrix} \arrow[rr, "\begin{pmatrix} -\partial_{i-1}^{T^R(\mathbf{g})} & 0 \\ \Psi_{i-1} & \partial_i^{T^R(\mathbf{f}')} \end{pmatrix}"] & & \begin{matrix} T^R(\mathbf{g})_{i-2} \\ \oplus \\ T^R(\mathbf{f}') \end{matrix} & = \mathrm{Cone}(\Psi)_{i-1} \arrow[d, "\Phi_{i-1}"] \\
T^R(\mathbf{f})_i = & R^{\binom{n}{i}} \arrow[rr, "\partial_i^{T^R(\mathbf{f})}"] & & R^{\binom{n}{i-1}} & = T^R(\mathbf{f})_{i-1}
\end{tikzcd}$$

where Φ_i is defined on basis vectors as

$$\Phi_i \begin{pmatrix} 0 \\ e_{j_1,\dots,j_i} \end{pmatrix} = e_{j_1,\dots,j_i}, \text{ and}$$

$$\Phi_i \begin{pmatrix} e_{j_2,\dots,j_i} \\ 0 \end{pmatrix} = e_{1,j_2,\dots,j_i}.$$

We now show that Φ is a chain map by checking commutativity of the above diagram on the basis vectors for $\mathrm{Cone}(\Psi)$:

$$\begin{tikzcd}
\begin{pmatrix} 0 \\ e_{j_1,\dots,j_i} \end{pmatrix} \arrow[r, mapsto] \arrow[d, mapsto] & \begin{pmatrix} 0 \\ \partial_i^{T^R(\mathbf{f}')}(e_{j_1,\dots,j_i}) \end{pmatrix} = \displaystyle\sum_{p=1}^{i} (-1)^{p-1} \frac{\mathrm{lcm}(f_{j_1},\dots,f_{j_i})}{\mathrm{lcm}(f_{j_1},\dots,\widehat{f_{j_p}},\dots,f_{j_i})} \begin{pmatrix} 0 \\ e_{j_1,\dots,\widehat{j_p},\dots,j_i} \end{pmatrix} \arrow[d, mapsto] \\
e_{j_1,\dots,j_i} \arrow[r, mapsto] & \displaystyle\sum_{p=1}^{i} (-1)^{p-1} \frac{\mathrm{lcm}(f_{j_1},\dots,f_{j_i})}{\mathrm{lcm}(f_{j_1},\dots,\widehat{f_{j_p}},\dots,f_{j_i})} e_{j_1,\dots,\widehat{j_p},\dots,j_i},
\end{tikzcd}$$

and

$$\begin{tikzcd}
\begin{pmatrix} e_{j_2,\dots,j_i} \\ 0 \end{pmatrix} \arrow[r, mapsto] \arrow[dd, mapsto] & \begin{pmatrix} \partial_{i-1}^{T^R(\mathbf{g})}(e_{j_2,\dots,j_i}) \\ \Psi_{i-1}(e_{j_2,\dots,j_i}) \end{pmatrix} = \begin{matrix} \left[-\displaystyle\sum_{p=2}^{i} (-1)^{p} \frac{\mathrm{lcm}(g_{j_2},\dots,g_{j_i})}{\mathrm{lcm}(g_{j_2},\dots,\widehat{g_{j_p}},\dots,g_{j_i})} \begin{pmatrix} e_{j_2,\dots,\widehat{j_p},\dots,j_i} \\ 0 \end{pmatrix} \right] \\ + \left[\dfrac{f_1 \cdot \mathrm{lcm}(g_{j_2},\dots,g_{j_i})}{\mathrm{lcm}(f_{j_2},\dots,f_{j_i})} \begin{pmatrix} 0 \\ e_{j_2,\dots,j_i} \end{pmatrix} \right] \end{matrix} \arrow[d, mapsto] \\
& \begin{matrix} \left[\displaystyle\sum_{p=2}^{i} (-1)^{p-1} \frac{\mathrm{lcm}(g_{j_2},\dots,g_{j_i})}{\mathrm{lcm}(g_{j_2},\dots,\widehat{g_{j_p}},\dots,g_{j_i})} e_{1,j_2,\dots,\widehat{j_p},\dots,j_i} \right] \\ + \left[\dfrac{f_1 \cdot \mathrm{lcm}(g_{j_2},\dots,g_{j_i})}{\mathrm{lcm}(f_{j_2},\dots,f_{j_i})} e_{j_2,\dots,j_i} \right] \end{matrix} \\
e_{1,j_2,\dots,j_i} \arrow[r, mapsto] & \begin{matrix} \left[\dfrac{\mathrm{lcm}(f_1,f_{j_2},\dots,f_{j_i})}{\mathrm{lcm}(f_{j_2},\dots,f_{j_i})} e_{j_2,\dots,j_i} \right] \\ + \left[\displaystyle\sum_{p=2}^{i} (-1)^{p-1} \frac{\mathrm{lcm}(f_1,f_{j_2},\dots,f_{j_i})}{\mathrm{lcm}(f_1,f_{j_2},\dots,\widehat{f_{j_p}},\dots,f_{j_i})} e_{1,j_2,\dots,\widehat{j_p},\dots,j_i} \right]. \end{matrix}
\end{tikzcd}$$

To show that Φ is a chain map, it suffices to show that the two lines in the bottom right corner are equal. By Claim II.D.8.2, the coefficients for the $e_{j_2,\dots,j_i}$ basis element are equal. For the terms inside the summation, we just need to multiply by $\frac{f_1}{f_1}$ and use Claim II.D.8.2 to see that they are equal:

$$\frac{f_1}{f_1} \cdot \frac{\mathrm{lcm}(g_{j_2},\dots,g_{j_i})}{\mathrm{lcm}(g_{j_2},\dots,\widehat{g_{j_p}},\dots,g_{j_i})} = \frac{\mathrm{lcm}(f_1,f_{j_2},\dots,f_{j_i})}{\mathrm{lcm}(f_1,f_{j_2},\dots,\widehat{f_{j_p}},\dots,f_{j_i})}.$$

This shows that the diagram commutes, so Φ is chain map. Furthermore, it is straightforward to show that Φ induces a bijection between bases, so Φ is an isomorphism. Therefore, $\mathrm{Cone}(\Phi) \cong T^R(\mathbf{f})$, so Theorem II.A.7 tells us that $T^R(\mathbf{f})$ is a free resolution of $R/\langle \mathbf{f} \rangle$. $\square$

CHAPTER II.E

A Colloquial Presentation of Two Resolutions

Parlor Trick. Let $R = k[X,Y,Z]$ and $I = \langle XY, XZ, YZ\rangle$, and consider the resolution

$$0 \longrightarrow R^2 \xrightarrow[\partial_2]{\left(\begin{smallmatrix} -Z & -Z \\ Y & 0 \\ 0 & X \end{smallmatrix}\right)} R^3 \xrightarrow{(XY\ XZ\ YZ)} R \longrightarrow 0.$$

Look at the three 2×2 minors (i.e., sub-determinants) of ∂_2:

$$\text{By deleting the first row: } \begin{vmatrix} Y & 0 \\ 0 & X \end{vmatrix} = XY,$$

$$\text{By deleting the second row: } \begin{vmatrix} -Z & -Z \\ 0 & X \end{vmatrix} = -XZ,$$

$$\text{By deleting the third row: } \begin{vmatrix} -Z & -Z \\ Y & 0 \end{vmatrix} = YZ.$$

Notice here that these three minors are the generators of I. This is a special case of the Hilbert-Burch resolution; see Theorem II.E.10 below.

To further motivate our use of minors in this chapter, consider a resolution

$$0 \longrightarrow R^{\beta_m} \xrightarrow{\partial_m} \cdots \longrightarrow R^{\beta_0} \longrightarrow 0.$$

In particular, recall that ∂_m is one-to-one, so the columns of the matrix A representing ∂_m are linearly independent over R. If R is a field, then ∂_m is one-to-one if and only if some size-β_m minor of A is non-zero. Our goal in this chapter is to find similar conditions for the case where R is not a field.

Definition II.E.1. Let m, n be positive integers and

$$M_{m\times n}(R) = \{m \times n \text{ matrices } (a_{ij}) \mid \text{ all } a_{ij} \in R\} \cong\ R^{mn}.$$

For all positive integers $r \leq \min(m,n)$, a <u>size-r minor</u> of a matrix $A \in M_{m\times n}(R)$ is the determinant of an $r \times r$ matrix obtained by deleting some number of rows and columns of A. We also call this an <u>$r \times r$ subdeterminant</u> of A. If this deletion leaves rows $\mathbf{i} = i_1, \ldots, i_r$ and columns $\mathbf{j} = j_1, \ldots, j_r$, then the corresponding size-r minor is of the form

$$\begin{vmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_r} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_r,j_1} & a_{i_r,j_2} & \cdots & a_{i_r,j_r} \end{vmatrix}$$

and is denoted $[i_1, \ldots, i_r \mid j_1, \ldots, j_r]_A = [\mathbf{i} \mid \mathbf{j}]_A$. Also, define $I_r(A)$ to be the ideal of R generated by all size-r minors of A.

Example II.E.2. Let $R = k[X,Y,Z]$ and let

$$A = \begin{pmatrix} -Z & -Z \\ Y & 0 \\ 0 & X \end{pmatrix}.$$

From our computations in the Parlor Trick at the beginning of this chapter, $I_2(A) = \langle XY, XZ, YZ\rangle$. Furthermore, $I_1(A) = \langle X, Y, Z\rangle$, since this is the ideal generated by all 1×1 minors of A. We will later be able to use Proposition II.E.5 to show that each ideal $I_r(A)$ is independent of the choice of basis.

LEMMA II.E.3. *Let m, r be positive integers such that $r \leq m$ and $A \in M_{r\times m}(R)$ and $B \in M_{m\times r}(R)$. Then $|AB| \in I_r(A) \cap I_r(B)$.*

PROOF. Notice that the rows of AB are linear combinations of the rows of B. Since $B \in M_{m\times r}(R)$, each row of B has r entries and since $AB \in M_{r\times r}(R)$, AB has r rows. Then

$$|AB| = \begin{vmatrix} \text{linear combination of rows of } B \\ \text{linear combination of rows of } B \\ \vdots \\ \text{linear combination of rows of } B \end{vmatrix} = \text{big linear combination of } \begin{vmatrix} \text{row of } B \\ \text{row of } B \\ \vdots \\ \text{row of } B \end{vmatrix} \in I_r(B).$$

Similarly, notice that the columns of AB are linear combinations of the columns of A. By a similar argument, $|AB| \in I_r(A)$. □

There is an alternate proof of Lemma II.E.3 using the Cauchy-Binet Formula.

LEMMA II.E.4. *Let m, n, p, r be positive integers such that $r \leq \min(m, n, p)$ and $A \in M_{n\times p}(R)$ and $B \in M_{p\times m}(R)$. Then $I_r(AB) \subseteq I_r(A) \cap I_r(B)$.*

PROOF. It suffices to show that each size-r minor of AB is in both $I_r(A)$ and $I_r(B)$. We can find that $[\mathbf{i} \mid \mathbf{j}]_{AB} = |A_{\mathbf{i}}B_{\mathbf{j}}|$ by expanding the corresponding matrices, where $A_{\mathbf{i}} \in M_{r\times p}(R)$ consists of rows $i_1, \ldots, i_r$ of A and $B_{\mathbf{j}} \in M_{p\times r}$ consists of columns $j_1, \ldots, j_r$ of B. Then by Lemma II.E.3,

$$|A_{\mathbf{i}}B_{\mathbf{j}}| \in I_r(A_{\mathbf{i}}) \cap I_r(B_{\mathbf{j}}).$$

Since all minors of $A_{\mathbf{i}}$ are also minors of A and all minors of $B_{\mathbf{j}}$ are also minors of B, we have

$$[\mathbf{i} \mid \mathbf{j}]_{AB} = |A_{\mathbf{i}}B_{\mathbf{j}}| \in I_r(A_{\mathbf{i}}) \cap I_r(B_{\mathbf{j}}) \subseteq I_r(A) \cap I_r(B).$$

□

The above two lemmas allow for the following proposition. The slogan is if A and B differ only by a change of basis, then $I_r(A)$ and $I_r(B)$ are equal.

PROPOSITION II.E.5. *Let m, n, r be positive integers such that $r \leq \min(m, n)$ and $A \in M_{m\times n}(R)$ and $U \in M_{m\times m}(R)^\times$ and $V \in M_{n\times n}(R)^\times$. Set $B = UAV$. Then $I_r(A) = I_r(B)$. Note that B is defined so that the following diagram commutes:*

$$\begin{tikzcd} R^n \arrow[r, "A"] & R^m \arrow[d, "\cong"', "U"] \\ R^n \arrow[u, "V", "\cong"'] \arrow[r, "B"'] & R^m. \end{tikzcd}$$

PROOF. Since U and V are invertible, we write $A = U^{-1}BV^{-1}$. Then we use Lemma II.E.4 to show both inclusions:

$$\begin{aligned} I_r(B) &= I_r(UAV) \subseteq I_r(U) \cap I_r(A) \cap I_r(V) \subseteq I_r(A), \text{ and} \\ I_r(A) &= I_r(U^{-1}BV^{-1}) \subseteq I_r(U^{-1}) \cap I_r(B) \cap I_r(V^{-1}) \subseteq I_r(B). \end{aligned}$$

Therefore, $I_r(A) = I_r(B)$. □

DEFINITION II.E.6. Let r, m, n be positive integers such that $r \leq \min(m, n)$, let $f \in \operatorname{Hom}_R(R^n, R^m)$, and let $A \in M_{m\times n}(R)$ represent f with respect to some bases. Then $I_r(f) = I_r(A)$. By Proposition II.E.5, this definition is independent of the choice of bases. Furthermore, if $s > \min(m, n)$, then $I_s(f) = 0$ and if $s \leq 0$, then $I_s(f) = R$.

PROPOSITION II.E.7. *Let r, m, n be positive integers such that $r \leq \min(m, n)$ and let $f \in \operatorname{Hom}_R(R^n, R^m)$. Then*

$$I_0(f) \supseteq I_1(f) \supseteq I_2(f) \supseteq \cdots \supseteq I_r(f) \supseteq \cdots \supseteq \underbrace{I_{\min(m,n)+1}(f)}_{=0}.$$

PROOF. Expanding any size-r minor along a single row or column allows us to write it as a linear combination of size-$(r-1)$ minors. Therefore $I_r(f) \subseteq I_{r-1}(f)$ for all r. □

The proof of the following result is outside of the scope of this class. We will use it to verify the two resolutions of interest in Theorems II.E.10 and II.E.30 below.

THEOREM II.E.8 (Buchsbaum-Eisenbud). *Assume that R is noetherian and consider an R-complex*

$$F = \left(0 \longrightarrow R^{\beta_m} \xrightarrow{\partial_m^F} R^{\beta_{m-1}} \xrightarrow{\partial_{m-1}^F} \cdots \xrightarrow{\partial_1^F} R^{\beta_0} \longrightarrow 0 \right).$$

For all $i = 1, \ldots, m$, set

$$r_i = \sum_{j=i}^{m} (-1)^{j-i} \beta_j = \beta_i - \beta_{i+1} + \cdots + (-1)^{m-i} \beta_m.$$

Then F is a resolution of $H_0(F)$ if and only if $\operatorname{depth}_{I_{r_i}(\partial_i^F)}(R) \geq i$ *for all $i = 1, \ldots, m$.*

EXAMPLE II.E.9. Let $R = k[X, Y, Z]$ and $I = \langle XY, XZ, YZ \rangle$ and consider the resolution

$$F = \left(0 \longrightarrow R^2 \xrightarrow{\left(\begin{smallmatrix} -Z & -Z \\ Y & 0 \\ 0 & X \end{smallmatrix}\right)} R^3 \xrightarrow{(XY \ XZ \ YZ)} R \longrightarrow 0 \right).$$

Notice here that $\beta_2 = 2$, $\beta_1 = 3$, and $\beta_0 = 1$. We check $\operatorname{depth}_{I_{r_i}(\partial_i^F)}(R) \geq i$ for $i = 1, 2$:

$i = 2:$ $r_2 = \beta_2 = 2$, so $I_{r_2}(\partial_2^F) = I_2 \left(\begin{smallmatrix} -Z & -Z \\ Y & 0 \\ 0 & X \end{smallmatrix}\right) = \langle XY, XZ, YZ \rangle = I$. Then

$$\operatorname{depth}_{I_{r_2}(\partial_2^F)}(R) = \operatorname{depth}_I(R) = 2 \geq 2.$$

$i = 1:$ $r_1 = \beta_1 - \beta_2 = 3 - 2 = 1$, so $I_{r_1}(\partial_1^F) = I_1 \begin{pmatrix} XY & XZ & YZ \end{pmatrix} = \langle XY, XZ, YZ \rangle = I$. Then

$$\operatorname{depth}_{I_{r_1}(\partial_1^F)}(R) = \operatorname{depth}_I(R) = 2 \geq 1.$$

Therefore by Theorem II.E.8, F is a resolution of R/I.

We next give a result by Hilbert and Burch, an unnumbered example, and the proof of the result.

THEOREM II.E.10 (Hilbert-Burch). *Assume that $R = k[X_1, \ldots, X_d]$.*

(a) *Let $f \in \operatorname{Hom}_R(R^n, R^{n+1})$ for $n \geq 1$, and let $B \in M_{(n+1)\times n}(R)$ represent f with respect to the standard bases. For $i = 1, \ldots, n+1$, set $f_i = |B_i|$, where B_i is obtained from B by deleting the i^{th} row. In other words, we have*

$$f_i = [1, \ldots, \widehat{i}, \ldots, n+1 \mid 1, \ldots, n]_B.$$

Assume $\operatorname{depth}_{I_n(f)}(R) \geq 2$. *Then*

$$0 \longrightarrow R^n \xrightarrow{f} R^{n+1} \xrightarrow{(f_1 \ \ -f_2 \ \ \cdots \ \ (-1)^n f_{n+1})} R \longrightarrow 0$$

is a free resolution of $R/I_n(f)$. Also, for any non-zero-divisor $a \in R$, we get a resolution

$$0 \longrightarrow R^n \xrightarrow{f} R^{n+1} \xrightarrow[h]{(af_1 \ \ -af_2 \ \ \cdots \ \ (-1)^n af_{n+1})} R \longrightarrow 0.$$

of $R/aI_n(f)$.

(b) *Conversely, if $I \leq R$ with $I \neq 0$ such that there exists a free resolution*

$$0 \longrightarrow R^{\beta_2} \xrightarrow{\partial_2^F} R^{\beta_1} \xrightarrow{\partial_1^F} R \longrightarrow 0,$$

of R/I, then $\beta_1 = \beta_2 + 1$ and there exists a non-zero-divisor $a \in R$ such that $I = aI_{\beta_2}(\partial_2^F)$.

EXAMPLE. We saw in the II.E opening this chapter that the complex

$$0 \longrightarrow R^2 \xrightarrow[\partial_2]{\left(\begin{smallmatrix} -Z & -Z \\ Y & 0 \\ 0 & X \end{smallmatrix}\right)} R^3 \xrightarrow[\partial_1]{(XY \ \ XZ \ \ YZ)} R \longrightarrow 0$$

resolves the R-module R/I where $I = (XY\ XZ\ YZ)$. We also saw that the 2×2 minors of ∂_2 are generators of I. By the theorem we also have that

$$0 \longrightarrow R^2 \xrightarrow[\partial_2]{\left(\begin{smallmatrix} -Z & -Z \\ Y & 0 \\ 0 & X \end{smallmatrix}\right)} R^3 \xrightarrow[\partial_1]{(\,X^2Y^2Z \quad X^2YZ^2 \quad XY^2Z^2\,)} R \longrightarrow 0$$

is a resolution for $R/(X^2Y^2Z,\ X^2YZ^2,\ XY^2Z^2)$.

Proof. (a) It suffices to prove the last statement of (a). We want to apply Buchsbaum-Eisenbud to the sequence

$$0 \longrightarrow R^n \xrightarrow{f} R^{n+1} \xrightarrow{h} R \longrightarrow 0, \tag{II.E.10.1}$$

which we claim is an R-complex. If $A = (a_{ij})$ is the matrix representing f, then we have

$$\begin{aligned} hf &= \begin{pmatrix} af_1 & -af_2 & \cdots & (-1)^n af_{n+1} \end{pmatrix} (a_{ij}) \\ &= \begin{pmatrix} a\,(f_1a_{11} - f_2a_{21} + \cdots + (-1)^n f_{n+1}a_{n+1,1}) \\ a\,(f_1a_{12} - f_2a_{22} + \cdots + (-1)^n f_{n+1}a_{n+1,2}) \\ \vdots \\ a\,(f_1a_{1,n+1} - f_2a_{2,n+1} + \cdots + (-1)^n f_{n+1}a_{n+1,n+1}) \end{pmatrix}^T \end{aligned}$$

which is a row vector of zeros, since, for instance, the first entry is the product of a and the determinant

$$\begin{vmatrix} a_{11} & a_{11} & a_{12} & \cdots & a_{1,n+1} \\ a_{21} & a_{21} & a_{22} & \cdots & a_{2,n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n+1,1} & a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n+1} \end{vmatrix},$$

which contains a repeated column. Hence (II.E.10.1) is an R-complex. In the context of the result by Buchsbaum-Eisenbud, if $i = 2$ then $r_2 = \beta_2 = n$ and

$$\operatorname{depth}_{I_{r_2}(f)}(R) = \operatorname{depth}_{I_n(f)}(R) \geq 2$$

where the inequality holds by assumption. If $i = 1$, then $r_1 = \beta_1 - \beta_2 = (n+1) - n = 1$ and

$$I_{r_1}(h) = I_1\begin{pmatrix} af_1 & \cdots & af_{n+1} \end{pmatrix} = \langle af_1, \ldots, af_{n+1} \rangle = a \cdot \langle f_1, \ldots, f_{n+1} \rangle = a \cdot I_n(f).$$

Hence it now suffices to show that $I_{r_1}(h) = a \cdot I_n(f)$ contains a non-zero-divisor on R. Since $\operatorname{depth}_{I_n(f)}(R) \geq 2$, there exists a non-zero-divisor $b \in I_n(f)$. Since a is a non-zero divisor, so is the product $ab \in a \cdot I_n(f)$.

(b) Assume R/I has a resolution

$$0 \longrightarrow R^{\beta_2} \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R \longrightarrow R/I \longrightarrow 0 \tag{II.E.10.2}$$

and that $I \neq 0$. By Theorem I.C.8, we know $\beta_1 = \beta_2 + 1$ (here we use our assumption that $R = k[X_1, \ldots, X_d]$). Then since $r_2 = \beta_2$ and $r_1 = \beta_1 - \beta_2 = 1$, by Buchsbaum-Eisenbud we have

$$\operatorname{depth}_{I_{\beta_2}(\partial_2)}(R) \geq 2$$

and

$$\operatorname{depth}_{I_1(\partial_1)}(R) \geq 1.$$

Note that since the determinant of a one-by-one matrix is equal to the lone entry, the ideal $I_1(\partial_1)$ is generated by the entries in the matrix representing ∂_1 and hence $I_1(\partial_1) = \operatorname{Im}(\partial_1) = I$. Consider the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & R^{\beta_2} \arrow[r, "\partial_2"] & R^{\beta_1} \arrow[r] & \operatorname{Im}(\partial_1) \arrow[r] & 0 \\ 0 \arrow[r] & R^{\beta_2} \arrow[r, "\partial_2"'] \arrow[u, "\mathrm{id}"] & R^{\beta_1} \arrow[r] \arrow[u, "\mathrm{id}"] & I_{\beta_2}(\partial_2) \arrow[r] \arrow[u, dashed, "\exists!\phi"'] & 0 \end{tikzcd}$$

The top row is exact since (II.E.10.2) is exact, the bottom row is exact by Hilbert-Burch (a), the first square commutes by construction, and the existence of ϕ follows from the exactness of the rows. Then by the Snake Lemma we have that ϕ is an isomorphism, i.e., $I_{\beta_2}(\partial_2) \cong I$.

We need to show that there is a non-zero-divisor $a \in R$ such that $I = a \cdot I_{\beta_2}(\partial_2)$. Since $\operatorname{depth}_{I_{\beta_2}(\partial_2)}(R) \geq 2$, we know there is a weakly R-regular sequence $g, h \in I_{\beta_2}(\partial_2)$. Since ϕ is a homomorphism we have

$$g \cdot \phi(h) = \phi(gh) = h \cdot \phi(g)$$

and we know $g, h \neq 0$ since the sequence is weakly R-regular. We claim $\phi(g) \in \langle g \rangle$. In $\overline{R} = R/\langle g \rangle$ we have

$$h \cdot \overline{\phi(g)} = \overline{g \cdot \phi(h)} = 0$$

and since h is a non-zero-divisor on $\overline{R}$ by assumption, we also have

$$\overline{\phi(g)} = 0 \in \overline{R},$$

so $\phi(g) \in \langle g \rangle$. Thus since for all $\zeta \in I_{\beta_2}(\partial_2)$ we have

$$g \cdot \phi(\zeta) = \zeta \cdot \phi(g)$$

we also have

$$\phi(\zeta) = \zeta \cdot \frac{\phi(g)}{g}$$

where $\phi(g)/g \in R$ by the claim. Set $a = \phi(g)/g$ note we have shown that $\phi(\zeta) = \zeta \cdot a$ for all $\zeta \in I_{\beta_2}(\partial_2)$, i.e.,

$$I = \operatorname{Im}(\phi) = a \cdot I_{\beta_2}(\partial_2).$$

$\square$

Buchsbaum-Eisenbud.

EXAMPLE II.E.11. Set $R = k[X, Y, Z]$ and consider the matrix

$$A = \begin{pmatrix} 0 & X & Y \\ -X & 0 & Z \\ -Y & -Z & 0 \end{pmatrix}.$$

Then we construct generators of an ideal I by taking the square roots of the 2×2 minors of A obtained by deleting the i^{th} row and i^{th} column, $i = 1, 2, 3$, i.e., $I = \langle Z, Y, X \rangle$. Observe that

$$\begin{pmatrix} Z & -Y & X \end{pmatrix} \cdot \begin{pmatrix} 0 & X & Y \\ -X & 0 & Z \\ -Y & -Z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

and we thus have the following R-complex.

$$0 \longrightarrow R \xrightarrow[\left(\begin{smallmatrix} Z \\ -Y \\ X \end{smallmatrix}\right)]{} R^3 \xrightarrow[\left(\begin{smallmatrix} 0 & X & Y \\ -X & 0 & Z \\ -Y & -Z & 0 \end{smallmatrix}\right)]{} R^3 \xrightarrow[\left(\begin{smallmatrix} Z & -Y & X \end{smallmatrix}\right)]{} R \longrightarrow 0$$

It is the Koszul complex $K^R(Z, -Y, X)$! We show how to build resolutions from matrices like A in Theorem II.E.30 below.

DEFINITION II.E.12. A matrix $A \in M_{n \times n}(R)$ is alternating if $A^T = -A$ and $a_{ii} = 0$ for all $i = 1, \ldots, n$. We denote the set of $n \times n$ alternating matrices in $M_{n \times n}(R)$ by

$$\operatorname{Alt}_n(R) = \{\text{alternating } A \in M_{n \times n}(R)\}.$$

Notice that if 2 is a unit in R (e.g. if $R \supseteq \mathbb{Q}$), then $A^T = -A$ implies that $a_{ii} = 0$ for all $i = 1, \ldots, n$ since $a_{ii} = -a_{ii}$ implies that $2a_{ii} = 0$.

EXAMPLE II.E.13. Let $R = k[X, Y, Z]$. The following two matrices are both alternating:

$$A = \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 & X & Y \\ -X & 0 & Z \\ -Y & -Z & 0 \end{pmatrix}.$$

Furthermore, notice that $|A| = X^2$ and

$$|B| = -X \begin{vmatrix} -X & Z \\ -Y & 0 \end{vmatrix} + Y \begin{vmatrix} -X & 0 \\ -Y & -Z \end{vmatrix} = -XYZ + XYZ = 0.$$

THEOREM II.E.14 (Cayley). *Let $A \in \operatorname{Alt}_n(R)$. If n is even, then there exists $f \in R$ such that $|A| = f^2$. If n is odd, then $|A| = 0 = 0^2$.*

Before we can prove the above theorem, we need a few more tools.

PROPOSITION II.E.15. *If n is odd and $A \in \mathrm{Alt}_n(R)$, then $|A| = 0$.*

PROOF. We split the proof into two cases. For the first case, suppose that 2 is a unit in R. Then we use that n is odd to get

$$|A| = \left|A^T\right| = |-A| = (-1)^n |A| = -|A|.$$

Therefore, $2|A| = 0$. Since 2 is a unit, this implies $|A| = 0$.

For the second case, we do not assume that 2 is a unit in R. Let $A = (a_{ij})$ and set

$$S = \mathbb{Z}[X_{ij} \mid i = 2, \ldots, n, \ j = i+1, \ldots, n],$$

so S is a polynomial ring in $\binom{n-1}{2}$ variables. Then $S \subseteq \mathbb{Q}(X_{ij}) = \mathrm{Frac}(S)$, where $\mathrm{Frac}(S)$ is the field of fractions of S. Define

$$X = \begin{pmatrix} 0 & X_{12} & X_{13} & \cdots & X_{1n} \\ -X_{12} & 0 & X_{23} & \cdots & X_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -X_{1n} & -X_{2n} & -X_{3n} & \cdots & 0 \end{pmatrix} \in \mathrm{Alt}_n(S) \subseteq \mathrm{Alt}_n(\mathrm{Frac}(S)).$$

Then X falls into the first case, so $|X| = 0$. Define a ring homomorphism $\phi : S \to R$ by $\phi(X_{ij}) = a_{ij}$. Then $|A| = \phi(|X|) = \phi(0) = 0$. $\square$

LEMMA II.E.16. *If n is even, then*

$$\begin{vmatrix} 0 & b_{12} & 0 & 0 & \cdots & 0 \\ -b_{12} & 0 & b_{23} & 0 & \cdots & 0 \\ 0 & -b_{23} & 0 & b_{34} & \cdots & 0 \\ 0 & 0 & -b_{34} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{vmatrix} = b_{12}^2 b_{34}^2 \cdots b_{n-1,n}^2 = \prod_{\substack{i=1 \\ i \text{ odd}}}^{n-1} b_{i,i+1}^2.$$

PROOF. We prove this by induction. For the base case, consider $n = 2$. Then

$$\begin{vmatrix} 0 & b_{12} \\ -b_{12} & 0 \end{vmatrix} = b_{12}^2.$$

Now suppose that the result holds for $(n-2) \times (n-2)$ matrices of the given form. Then expand along the first column, then along the first row to obtain the first equality in the next display.

$$\begin{vmatrix} 0 & b_{12} & 0 & 0 & \cdots & 0 \\ -b_{12} & 0 & b_{23} & 0 & \cdots & 0 \\ 0 & -b_{23} & 0 & b_{34} & \cdots & 0 \\ 0 & 0 & -b_{34} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{vmatrix} = b_{12}^2 \begin{vmatrix} 0 & b_{34} & \cdots & 0 \\ -b_{34} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix} = b_{12}^2 b_{34}^2 \cdots b_{n-1,n}^2$$

The second equality follows from the inductive hypothesis. $\square$

LEMMA II.E.17. *Let $\mathbb{Q} \subseteq K$ be a field extension and let $A \in \mathrm{Alt}_n(K)^\times$. Then there exists $C \in M_{n\times n}(K)^\times$ such that $B = C^T A C \in \mathrm{Alt}_n(K)^\times$ has the form*

$$B = \begin{pmatrix} 0 & b_{12} & 0 & 0 & \cdots & 0 \\ -b_{12} & 0 & b_{23} & 0 & \cdots & 0 \\ 0 & -b_{23} & 0 & b_{34} & \cdots & 0 \\ 0 & 0 & -b_{34} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

PROOF. If n is odd, then $|A| = 0$ by Proposition II.E.15, which contradicts that A is invertible. Therefore n must be even. Define $f : K^n \times K^n \to K$ by $f(v,w) = v^T A w \in M_{1\times 1}(K) \cong K$. Then f is bilinear and

$$f(w,v) = w^T A v = (w^T A v)^T = v^T A^T w = -v^T A w = -f(v,w),$$

so f is skew-symmetric. Therefore we have $f(v,v) = 0$ because 2 is a unit in $\mathbb{Q} \subseteq K$.

Claim (1). *There exists $v, w \in K^n$ such that $f(v,w) \neq 0$.*

Proof. We show that there exist $1 \le i, j \le n$ such that $f(e_i, e_j) \neq 0$. Notice that

$$f(e_i, e_j) = e_i^T A e_j = a_i^T \operatorname{Col}(A, j) = a_{ij}.$$

Since A is invertible, we must have $a_{ij} \neq 0$ for some $1 \le i, j \le n$, so $f(e_i, e_j) \neq 0$ for those i and j. ✓

Now replace v with $\frac{1}{f(v,w)}\, v$ to assume that $f(v,w) = 1$. If there exists $\alpha \in K$ such that $v = \alpha w$, then

$$1 = f(v,w) = f(\alpha w, w) = \alpha f(w,w) = 0,$$

which is clearly contradictory. This implies that v and w are linearly independent. Set $V_1 = \operatorname{Span}_K(v,w) \subseteq K^n$, so a basis for V_1 is $\{v, w\}$. Set

$$V_2 = \text{“}V_1^{\perp_f}\text{”} = \{y \in K^n \mid f(y,z) = 0 \;\forall\, z \in V_1\} \subseteq K^n.$$

Claim (2). $K^n = V_1 \oplus V_2$.

Proof. We need to prove that $V_1 \cap V_2 = 0$ and for all $t \in K^n$, $t = z + y$ for some $z \in V_1$ and $y \in V_2$.

(1) We first show that the intersection is trivial. Let $u \in V_1 \cap V_2$, so $u = av + bw$ for $a, b \in K$. Then

$$0 = f(u,v) = f(av + bw, v) = a\underbrace{f(v,v)}_{=0} + b\underbrace{f(w,v)}_{=-1} = -b,$$

so $b = 0$. By a similar argument, $a = 0$. Therefore, $u = av + bw = 0$.

(2) Let $t \in K^n$. We want to find $z \in V_1$ and $y \in V_2$ such that $t = z + y$. Set $z = f(t,w)v - f(t,v)w \in \operatorname{Span}(v,w) = V_1$. Then $t = z + (t - z)$, so it suffices to show that $t - z \in V_2$. To do this, it suffices to show that $f(t - z, v) = 0$ and $f(t - z, w) = 0$:

$$\begin{aligned} f(t - z, v) &= f(t - (f(t,w)v - f(t,v)w), v) \\ &= f(t,v) - f(t,w)\underbrace{f(v,v)}_{=0} + f(t,v)\underbrace{f(w,v)}_{=-1} \\ &= f(t,v) - f(t,v) = 0. \end{aligned}$$

By a similar argument, $f(t - z, w) = 0$. ✓

Let $v_3, \dots, v_n$ be a basis of V_2, so $v, w, v_3, \dots, v_n$ is a basis of K^n. Define $f_2 = f|_{V_2 \times V_2} : V_2 \times V_2 \to K$, and let $B_1 = (b_{ij}) \in M_{(n-2)\times(n-2)}(K)$ be the matrix representing f_2 with respect to the basis $v_3, \dots, v_n$. Then

$$b_{ij} = f(v_{i+2}, v_{j+2}) = -f(v_{j+2}, v_{i+2}) = -b_{ji},$$

so $B_1 \in \operatorname{Alt}_n(K)$. Set $P = (v|w|v_3|\cdots|v_n)$.

Claim (3).

$$P^T A P = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & B_1 \end{array}\right)$$

Proof. Consider each entry for the above matrix as follows:

$$\begin{aligned} (P^T A P)_{ij} &= e_i^T P^T A P e_j \\ &= (Pe_i)^T A (Pe_j) \\ &= \operatorname{Col}(P,i)^T A \operatorname{Col}(P,j). \end{aligned}$$

If $i, j \ge 3$, then

$$(P^T A P)_{ij} = v_i^T A v_j = f(v_i, v_j) = b_{i-2,j-2}.$$

The other cases for i and j are computed similarly. ✓

In particular, since P and A are invertible, then $P^T A P$ is also invertible, so B_1 must be invertible since it is a submatrix of $P^T A P$. We can repeat this process as many times as needed to find $Q \in M_{n\times n}(K)^\times$ such that $C = QP$ and $B = C^T A C$ has the desired form. □

Lemma II.E.18. *Assume that R is a unique factorization domain and let $g, h \in R$ be such that $h \neq 0$ and $(g/h)^2 \in R$. Then $g/h \in R$.*

PROOF. Since R is a unique factorization domain, we can assume that g/h is in lowest terms. In other words, g and h have no common prime factors. Set $f = g^2/h^2 \in R$, so $h^2 f = g^2$. If h is not a unit, then it has a prime factor p, so

$$p \mid h \quad \Rightarrow \quad p \mid g^2 \quad \Rightarrow \quad p \mid g.$$

This is a contradiction since g and h have no common prime factors. Therefore, h is a unit, so $g/h \in R$. $\square$

Now we are finally ready to prove the result by Cayley.

PROOF OF CAYLEY'S THEOREM. If n is odd, then the conclusion follows from Proposition II.E.15, so we assume n is even. We proceed by way of cases. As a special case we consider

$$R_0 = \mathbb{Z}[X_{ij} \mid i = 1, \ldots, n-1;\ j = i+1, \ldots, n]$$

and

$$X = \begin{pmatrix} 0 & X_{12} & \cdots & X_{1n} \\ -X_{12} & 0 & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -X_{1n} & -X_{2n} & \cdots & 0 \end{pmatrix}.$$

By Lemma II.E.17 there exists a matrix $C \in M_{n\times n}(K)^\times$ such that $C^T X C = B$ is tridiagonal and alternating, where $K = \mathrm{frac}(R_0)$. Then by Lemma II.E.16 we have

$$(b_{12} b_{34} \cdots b_{n-1,n})^2 = |B| = |C^T||X||C| = |C|^2 |X|$$

and since C is invertible we have

$$|X| = \frac{1}{|C|^2}(b_{12} b_{34} \cdots b_{n-1,n})^2 \in K.$$

Since $|X| \in R_0$, by Lemma II.E.18 we have

$$f = \frac{b_{12} b_{34} \cdots b_{n-1,n}}{|C|} \in R_0,$$

which proves this case.

In the general case, we let $A \in \mathrm{Alt}_n(R)$ and consider the ring homomorphism $\phi\colon R_0 \to R$ given by $\phi(X_{ij}) = a_{ij}$. There exists an element $F \in R_0$ such that $|X| = F^2$ and we observe that

$$|A| = \phi(|X|) = \phi(F^2) = \phi(F)^2.$$

Taking $f = \phi(F)$, this proves the general case. $\square$

NOTE II.E.19. Let $A \in \mathrm{Alt}_n(R)$ and $f \in R$ such that $f^2 = |A|$. Then f is not unique, not even up to a sign, in general. For instance, if $x \in R$ such that $x^2 = 0$, then for all $\alpha \in R$ we have

$$\begin{vmatrix} 0 & x \\ -x & 0 \end{vmatrix} = x^2 = 0 = (\alpha x)^2.$$

This motivates the following question: how does one choose f well?

PROPOSITION II.E.20. *Let R be an integral domain and let $f, g \in R$ such that $f^2 = g^2$. Then $f = \pm g$.*

PROOF. This follows from the equality $0 = g^2 - f^2 = (g - f)(g + f)$. $\square$

PROPOSITION II.E.21. *Let D be an integral domain and set $R_0 = D[X_1, \ldots, X_d]$.*

(a) *If $f \in R_0$ is such that $0 \neq f^2$ is homogeneous of degree n, then f is homogeneous, n is even, and* $\deg(f) = n/2$.

(b) *If $f, g \in R_0$ are such that $0 \neq fg$ is homogeneous of degree n, then f and g are each homogeneous and* $\deg(f) + \deg(g) = n$.

PROOF. (b) We write $f = f_i + \cdots + f_j$ and $g = g_p + \cdots + g_q$ such that $i \leq j$, $p \leq q$, and where f_ℓ and g_m are homogeneous of degree ℓ and m, respectively, and $f_i, f_j, g_p, g_q \neq 0$. Then since we are in a domain we know $f_i g_p$ and $f_j g_q$ are each nonzero. Note in the product fg these are the terms of lowest and highest possible degree, respectively. Since fg is homogeneous this implies $i = j$ and $p = q$, implying f and g are each homogeneous and $\deg(f) + \deg(g) = \deg(fg)$.

(a) This follows directly from part (b). $\square$

NOTE II.E.22. Assume $q \in \mathbb{N}$ and $n = 2q$, and consider the ring R_0 and the matrix $X \in \mathrm{Alt}_n(R)$ as in the proof of Cayley's Theorem. Let $f \in R_0$ be such that $f^2 = |X|$. Then

$$f^2 = |X| = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) x_{1,\sigma(1)} x_{2,\sigma(2)} \cdots x_{n,\sigma(n)} \tag{II.E.22.1}$$

where

$$x_{ij} = \begin{cases} X_{ij} & \text{if } i < j \\ 0 & \text{if } i = j \\ -X_{ij} & \text{if } i > j. \end{cases}$$

Consider $\sigma_0 = (1\ 2)(3\ 4) \cdots (n{-}1\ \ n) \in S_n$ with $\mathrm{sgn}(\sigma_0) = (-1)^q$. Then the term of the sum in (II.E.22.1) associated with σ_0 is

$$\begin{aligned} (-1)^q X_{12}(-X_{12}) X_{34}(-X_{34}) \cdots X_{n-1,n}(-X_{n-1,n}) &= (-1)^{2q}(X_{12}X_{34} \cdots X_{n-1,n})^2 \\ &= (X_{12}X_{34} \cdots X_{n-1,n})^2. \end{aligned}$$

One can check that σ_0 is the unique element of S_n such that its associated term uses $(X_{12}X_{34} \cdots X_{n-1,n})^2$. Therefore f must contain $\pm X_{12}X_{34} \cdots X_{n-1,n}$ and we multiply f by -1 if necessary to assume f contains $X_{12}X_{34} \cdots X_{n-1,n}$.

DEPFINITION II.E.23. Using the notation of Note II.E.22, the pfaffian of X is $\mathrm{pf}\, X = f$ where the coefficient of $X_{12}X_{34} \cdots X_{n-1,n}$ is 1. Note $\mathrm{pf}(X)^2 = |X|$. Let $A \in \mathrm{Alt}_n(R)$ and define the ring homomorphism $\phi\colon R_0 \to R$ by $\phi(X_{ij}) = a_{ij}$. The pfaffian of A is $\mathrm{pf}(A) = \phi(\mathrm{pf}(X))$.

PROPOSITION II.E.24. *Observe that in the notation of Definition II.E.23 we have*

$$\mathrm{pf}(A)^2 = \phi(\mathrm{pf}(X))^2 = \phi(\mathrm{pf}(X)^2) = \phi(|X|) = |A|.$$

The next fact we state without proof.

FACT II.E.25. *Let $q \in \mathbb{N}$ and $n = 2q$, and consider R_0 and X as in Note II.E.22. Then*

$$\mathrm{pf}(X) = \frac{1}{2^q \cdot q!} \cdot \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{q} X_{\sigma(2i-1),\sigma(2i)}.$$

Set

$$\Pi_n = \{(i_1, j_1, i_2, j_2, \ldots, i_q, j_q) \in \mathbb{N}^n \mid \{i_1, j_1, \ldots, i_q, j_q\} = [n];\ i_1 < i_2 < \cdots < i_q;\ i_m < j_m,\ \forall m = 1, \ldots, q\}.$$

Then we have

$$\begin{aligned} \mathrm{pf}(X) &\overset{(1)}{=} \sum_{(i_1,\ldots,j_q) \in \Pi_n} \mathrm{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ i_1 & j_1 & i_2 & j_2 & \cdots & i_q & j_q \end{pmatrix} X_{i_1,j_1} X_{i_2,j_2} \cdots X_{i_q,j_q} \\ &\overset{(2)}{=} \sum_{j=1, j \neq i}^{n} (-1)^{i+j+1+\theta(j-i)} \cdot a_{ij}\, \mathrm{pf}(A_{ij}) \end{aligned}$$

where

$$\theta(j-i) = \begin{cases} 0 & \text{if } j - i > 0 \\ 1 & \text{if } j - i < 0 \end{cases}$$

and $A_{ij} \in \mathrm{Alt}_{n-2}(R)$ is obtained from A by deleting the i^{th} row and column as well as the j^{th} row and column.

EXAMPLE II.E.26. Consider the ring $\mathbb{Z}[a, b, c, x, y, z]$ and the alternating matrix

$$X = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & x & y \\ -b & -x & 0 & z \\ -c & -y & -z & 0 \end{pmatrix}.$$

Expanding $|X|$ along the first column we have

$$|X| = a \cdot \begin{vmatrix} a & b & c \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix} - b \cdot \begin{vmatrix} a & b & c \\ 0 & x & y \\ -y & -z & 0 \end{vmatrix} + c \cdot \begin{vmatrix} a & b & c \\ 0 & x & y \\ -x & 0 & z \end{vmatrix}.$$

Expanding each of these determinants along the top row we find

$$\begin{aligned} |X| &= a^2z^2 - 2abyz + 2acxz - 2bcxy + b^2y^2 + c^2x^2 \\ &= (az - by + cx)^2. \end{aligned}$$

Since $X_{12} = a$ and $X_{34} = z$ we want the coefficient of az to be $+1$, as we have above. Hence $\operatorname{pf}(X) = az - by + cx$.

Now we demonstrate equalities (1) and (2) from Fact II.E.25 for X. First, note that

$$\Pi_4 = \{(1,2,3,4),\ (1,3,2,4),\ (1,4,2,3)\}$$

and we compute the signs of the corresponding elements of S_4.

$$\begin{aligned} &\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1) && \operatorname{sgn} = 1 \\ &\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = (2\ 3) && \operatorname{sgn} = -1 \\ &\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = (2\ 4\ 3) && \operatorname{sgn} = 1 \end{aligned}$$

Therefore equality (1) gives

$$\operatorname{pf}(X) = X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23} = az - by + cx$$

which agrees with our initial computation.

Second, we confirm (2) for this example. We choose $i = 3$ Then (2) gives

$$\begin{aligned} \operatorname{pf}(X) &= \sum_{j=1, j\neq 3}^{4} (-1)^{3+j+1+\theta(j-3)} a_{3j} \operatorname{pf}(A_{3j}) \\ &= (-1)^{3+1+1+1} a_{31} \operatorname{pf}\left(\begin{smallmatrix} 0 & y \\ -y & 0 \end{smallmatrix}\right) + (-1)^{3+2+1+1} a_{32} \operatorname{pf}\left(\begin{smallmatrix} 0 & c \\ -c & 0 \end{smallmatrix}\right) + (-1)^{3+4+1} a_{34} \operatorname{pf}\left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right) \\ &= -by + xc + za, \end{aligned}$$

which also agrees with our initial computation.

Definition II.E.27. Let $A \in \operatorname{Alt}_n(R)$. For all $i = 1, \ldots, n$, let A_i denote the $(n-1)\times(n-1)$ alternating matrix found by deleting the i^{th} row and i^{th} column of A. Set

$$\operatorname{Pf}_{n-1}(A) = \langle \operatorname{pf}(A_1), \ldots, \operatorname{pf}(A_n) \rangle.$$

Then define $P \in M_{1\times n}(R)$ by

$$P = P(A) = \begin{pmatrix} \operatorname{pf}(A_1) & -\operatorname{pf}(A_2) & \cdots & (-1)^{n-1}\operatorname{pf}(A_n) \end{pmatrix}$$

and

$$F = F(A) = \left(0 \longrightarrow R \xrightarrow{P^T} R^n \xrightarrow{A} R^n \xrightarrow{P} R \longrightarrow 0 \right).$$

Example II.E.28.
(a) If n is even, then $\operatorname{Pf}_{n-1}(A) = 0$ since by Proposition II.E.15, we have $\operatorname{pf}(A_i) = 0$ for all $i = 1, \ldots, n$.
(b) Let $n = 3$. Then $F(A)$ is the Koszul complex found in Example II.E.11.

Proposition II.E.29. *If $A \in \operatorname{Alt}_n(R)$, then $F(A)$ is an R-complex with $H_0(F(A)) = R/\operatorname{Pf}_{n-1}(A)$.*

PROOF. Let $i \in [n]$ and set

$$\widetilde{A} = \begin{bmatrix} 0 & \mathrm{Row}(A,i) \\ \mathrm{Col}(A,i) & A \end{bmatrix} = \left(\begin{array}{c|ccccc} 0 & a_{i1} & \cdots & a_{ii} & \cdots & a_{in} \\ \hline a_{1i} = -a_{i1} & a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{ii} = -a_{ii} & a_{i1} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{ni} = -a_{in} & a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{array}\right) \in \mathrm{Alt}_{n+1}(R).$$

Since A is alternating, we have $a_{ii} = 0$. Therefore $\widetilde{A}$ has a repeated column, so $|A| = 0$; since $\widetilde{A}$ is alternating, it follows that $\mathrm{pf}(\widetilde{A}) = 0$. Using Fact II.E.25, we have

$$0 = \mathrm{pf}(\widetilde{A}) = \sum_{j=2}^{n+1} (-1)^j \widetilde{a}_{ij} \,\mathrm{pf}(\widetilde{A}_{1j}) = \sum_{j=1}^{n} (-1)^{j-1} a_{ij} \,\mathrm{pf}(A_j) = \mathrm{Row}(A,i) \cdot P^T.$$

This is true for any i, so $AP^T = 0$. Also

$$0 = (AP^T)^T = PA^T = -PA,$$

so $PA = 0$ as well. Therefore, $F(A)$ is an R-complex. Furthermore, notice that

$$H_0(F(A)) = R/\operatorname{Im}(P) = R/\operatorname{Pf}_{n-1}(A).$$

□

THEOREM II.E.30 (Buchsbaum-Eisenbud). *Assume R is local and noetherian with maximal ideal $\mathfrak{m}$.*

(a) *Let $A \in \mathrm{Alt}_n(R)$ such that $I = \mathrm{Pf}_{n-1}(A)$ satisfies $\operatorname{depth}_I(R) \geq 3$ and $a_{ij} \in \mathfrak{m}$. Then $F(A)$ is a free resolution of R/I and n is odd.*

(b) *Conversely, if $I \subsetneq R$ satisfies $\operatorname{depth}_I(R) \geq 3$ and R/I has a free resolution*

$$0 \longrightarrow R \longrightarrow R^n \longrightarrow R^n \longrightarrow R \longrightarrow 0,$$

then there exists $A \in \mathrm{Alt}_n(R)$ such that $I = \mathrm{Pf}_{n-1}(A)$ and $F(A)$ is a free resolution of R/I. In particular, n is odd.

NOTE II.E.31. For $A \in \mathrm{Alt}_n(R)$, verifying that $\operatorname{depth}_{\mathrm{Pf}_{n-1}(A)}(R) \geq 3$ can be hard.

PROOF OF BUCHSBAUM-EISENBUD(a). We use the Buchsbaum Eisenbud acyclicity criterion II.E.8, beginning with the $i = 3$ and $i = 1$ cases.

$i = 3$: In this case, we have $r_3 = 1$. Then

$$I_{r_3}(\partial_3^F) = I_{r_3}(P^T) = I_1(P^T) = \langle P_{ij} \rangle = \langle \mathrm{pf}(A_1), \ldots, \mathrm{pf}(A_n) \rangle = I.$$

By assumption, I satisfies $\operatorname{depth}_I(R) \geq 3$, so $\operatorname{depth}_{I_{r_3}(\partial_3^F)}(R) \geq 3$ as well.

$i = 1$: In this case, we have $r_1 = n - (n-1) = 1$, so $I_{r_1}(\partial_1^F) = I_1(P) = I$ via the same argument as for the $i = 3$ case. Therefore by assumption, $\operatorname{depth}_{I_{r_1}(\partial_1^F)}(R) \geq 3 \geq 1$.

$i = 2$: In this case, we have $r_2 = n - 1$, so $I_{r_2}(\partial_2^F) = I_{n-1}(A)$. This ideal is related to $\mathrm{Pf}_{n-1}(A)$, but they are not equal. In order to complete this case, we need to show that

$$\operatorname{depth}_{\mathrm{Pf}_{n-1}(A)}(R) = \operatorname{depth}_{I_{n-1}(A)}(R).$$

For this, we need to build up a few more results.

LEMMA II.E.32. *Set $R = \mathbb{Z}[X_{ij} \mid i,j = 1, \ldots, n]$ and $X = (X_{ij})$. Then $|X|$ is prime in R.*

PROOF. Notice that $|X|$ is a homogeneous polynoimal of degree n. A result of Gauss tells us that R is a unique factorization domain, so it suffices to show that $|X|$ is irreducible in R. By Proposition II.E.21, if $|X|$ factors in R, then it factors as $|X| = fg$ where f and g are both homogeneous polynomials. Furthermore, since $\mathbb{Z}$ is an integral domain, $\deg_{X_{ij}}$ is additive on products. In particular, X_{11} appears in $|X|$, so X_{11} must appear in f or in g, and moreover

$$1 = \deg_{X_{11}}(|X|) = \deg_{X_{11}}(f) + \deg_{X_{11}}(g).$$

By symmetry, we can assume without loss of generality that $\deg_{X_{11}}(f) = 1$ and $\deg_{X_{11}}(g) = 0$.

CLAIM. $\deg_{X_{1j}}(g) = 0$, *i.e., X_{1j} does not appear in g for any $j = 1, \ldots, n$.*

Proof. By way of contradiction, suppose that $\deg_{X_{1j}}(g) > 0$. Then $\deg_{X_{1j}}(f) = 0$ and $\deg_{X_{1j}}(g) = 1$. Then we can rewrite $|X|$ as

$$|X| = fg = (f_0 + f_1X_{11})(g_0 + g_1X_{1j}),$$

where X_{11} does not appear in f_i and X_{1j} does not appear in g_i for $i = 0, 1$. Multiplying out the above product gives us

$$|X| = f_0g_0 + f_0g_1X_{1j} + f_1g_0X_{11} + f_1g_1X_{11}X_{1j}.$$

Notice that $f_1 \neq 0$ and $g_1 \neq 0$, so the final term in the above equation is non-zero. Since X_{11} does not appear in g, then X_{11} does not appear in g_i for $i = 0, 1$ and since X_{1j} does not appear in f, then X_{1j} does not appear in f_i for $i = 0, 1$. Therefore we have

$$|X| = \underbrace{f_0g_0}_{\text{no } X_{11}X_{1j}} + \underbrace{\underbrace{f_1g_0}_{\text{no } X_{1j}} X_{11}}_{\text{no } X_{11}X_{1j}} + \underbrace{\underbrace{f_0g_1}_{\text{no } X_{11}} X_{1j}}_{\text{no } X_{11}X_{1j}} + \underbrace{f_1g_1X_{11}X_{1j}}_{X_{11}X_{1j} \text{ appears}}.$$

Therefore there is no cancellation, so $|X|$ has a term with $X_{11}X_{1j}$, which contradicts the fact that $|X|$ only contains terms of the form $X_{1*}X_{2*}\cdots X_{n*}$. Therefore, X_{1j} does not appear in g for any $j = 1, \dots, n$. ✓

Through a similar argument, we can show g has no X_{ij} for any $i, j = 1, \dots, n$. So g is a constant polynomial. Furthermore, the terms of $|X|$ each have a coefficient of ± 1, so $g = \pm 1$. Therefore $|X|$ is irreducible. □

Lemma II.E.33. *Let $A \in M_{n\times n}(R)$ and*

$$\operatorname{Adj}(A)_{ji} = (-1)^{i+j}[1, \dots, \widehat{i}, \dots, n \mid 1, \dots, \widehat{j}, \dots, n]_A,$$

where we recall that $[1, \dots, \widehat{i}, \dots, n \mid 1, \dots, \widehat{j}, \dots, n]_A$ denotes the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of A. It follows that $A \cdot \operatorname{Adj}(A) = |A|\, I_n = \operatorname{Adj}(A) \cdot A$. Assume that $|A| = 0$. Then for all i, j, p, q,

$$\operatorname{Adj}(A)_{ij} \operatorname{Adj}(A)_{pq} = \operatorname{Adj}(A)_{iq} \operatorname{Adj}(A)_{pj}.$$

In particular,

$$\operatorname{Adj}(A)_{ij} \operatorname{Adj}(A)_{ji} = \operatorname{Adj}(A)_{ii} \operatorname{Adj}(A)_{jj}.$$

Proof. The conclusion is trivial if $\operatorname{Adj}(A) = 0$, so assume without loss of generality that $\operatorname{Adj}(A) \neq 0$. First we prove a special case. Assume R is an integral domain and let $K = \operatorname{Frac}(R)$. Since $\operatorname{Adj}(A) \neq 0$ there exists some non-zero size-$(n-1)$ minor of A. The columns used for that minor must be linearly independent over K and therefore $\dim_K \operatorname{Col}_K(A) \geq n - 1$. Since the determinant of A is zero, the columns of A are linearly dependent, i.e., $\dim_K \operatorname{Col}_K(A) \leq n - 1$. Hence $\dim_K \operatorname{Col}_K(A) = n - 1$ and therefore by rank-nullity any two vectors in $\operatorname{Null}_K(A)$ are linearly dependent over K. Now since $A \cdot \operatorname{Adj}(A) = |A| \cdot I_n = 0$, we know every column of $\operatorname{Adj}(A)$ is in $\operatorname{Null}_K(A)$, i.e., every two columns of $\operatorname{Adj}(A)$ are linearly dependent over K. Thus every size-2 minor of $\operatorname{Adj}(A)$ using rows i, p and columns j, q gives the desired result.

Now we prove the general case. Set $R_1 = \mathbb{Z}[X_{ij} \mid i, j = 1, \dots, n]$ and set $X = (X_{ij}) \in M_{n\times n}(R_1)$. By Lemma II.E.32 we know $|X| \in R_1$ is prime and therefore $R_2 = R_1/\langle |X| \rangle$ is an integral domain. Set $x_{ij} = \overline{X_{ij}} \in R_2$ and $x = (x_{ij}) \in M_{n\times n}(R_2)$, and note $|x| = \overline{|X|} = 0$ in R_2. Now, by the special case we have

$$\operatorname{Adj}(x)_{ij} \operatorname{Adj}(x)_{pq} = \operatorname{Adj}(x)_{iq} \operatorname{Adj}(x)_{pj}.$$

Recall the ring homomorphism $\phi\colon R_1 \to R$ given by $\phi(X_{ij}) = a_{ij}$ and note that $\phi(|X|) = |A| = 0$. Therefore there exists a unique, well-defined ring homomorphism $\overline{\phi}$ making the following diagram commute

$$\begin{tikzcd}
R_1 \arrow[r, "\phi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\curvearrowright"] & R \\
R_2 \arrow[ur, dashed, "\exists!\overline{\phi}"'] &
\end{tikzcd}$$

where π is the natural surjection. Furthermore, we have $\overline{\phi}(x_{ij}) = \phi(X_{ij}) = a_{ij}$ for all i, j and

$$\operatorname{Adj}(A)_{ij} = \phi(\operatorname{Adj}(X)_{ij}) = \overline{\phi}(\operatorname{Adj}(x)_{ij}).$$

Thus we conclude as follows.

$$\begin{aligned}
\operatorname{Adj}(A)_{ij}\operatorname{Adj}(A)_{pq} &= \overline{\phi}(\operatorname{Adj}(x)_{ij})\overline{\phi}(\operatorname{Adj}(x)_{pq})\\
&= \overline{\phi}(\operatorname{Adj}(x)_{ij}\operatorname{Adj}(x)_{pq})\\
&= \overline{\phi}(\operatorname{Adj}(x)_{iq}\operatorname{Adj}(x)_{pj})\\
&= \overline{\phi}(\operatorname{Adj}(x)_{iq})\overline{\phi}(\operatorname{Adj}(x)_{pj})\\
&= \operatorname{Adj}(A)_{iq}\operatorname{Adj}(A)_{pj}
\end{aligned}$$

□

LEMMA II.E.34. *Let $A \in \operatorname{Alt}_n(R)$ with $n = 2q+1$ for some $q \in \mathbb{N}$. Assume $\operatorname{Pf}_{n-1}(A) \neq 0$. Then*

$$\operatorname{rad}(\operatorname{Pf}_{n-1}(A)) = \operatorname{rad}(I_{n-1}(A)).$$

PROOF. For the forward containment, it suffices to show that $\operatorname{pf}(A_i)^2 \in I_{n-1}(A)$ for each $i = 1, \dots, n$. Since $\operatorname{pf}(A_i)^2 = |A_i|$, which is a size-$(n-1)$ minor of A, this is immediate.

For the reverse containment, we need to show that for every $i, j \in [n]$ we have $(\operatorname{Adj}(A)_{ij})^2 \in \operatorname{Pf}_{n-1}(A)$. Since n is odd, we know $|A| = 0$. Then by Lemma II.E.32 we have

$$\begin{aligned}
(\operatorname{Adj}(A)_{ij})^2 &= -\operatorname{Adj}(A)_{ij}\operatorname{Adj}(A)_{ji}\\
&= -\operatorname{Adj}(A)_{ii}\operatorname{Adj}(A)_{jj}\\
&= -|A_i||A_j|\\
&= -\operatorname{pf}(A_i)^2\operatorname{pf}(A_j)^2 \in \operatorname{Pf}_{n-1}(A)
\end{aligned}$$

where the first equality holds since A is alternating. □

PROPOSITION II.E.35. *Let $I, J \leq R$ be ideals such that $\operatorname{rad}(I) = \operatorname{rad}(J)$. Then $\operatorname{depth}_I(R) = \operatorname{depth}_J(R)$.*

PROOF. If $I = R$, then $\operatorname{rad}(J) = \operatorname{rad}(I) = R$, so $J = R$. Therefore we assume without loss of generality that I and J are proper ideals. Let $\mathbf{f} = f_1, \dots, f_n \in I$ be a weakly R-regular sequence.

CLAIM (1). *If $r_1, \dots, r_n \in R$ such that $\sum_{i=1}^n f_i r_i = 0$, then $r_i \in \langle \mathbf{f} \rangle$ for all $i \in [n]$.*

PROOF. We induct on n. Suppose $n = 1$. Then $f_1 r_1 = 0$ implies $r_1 = 0 \in \langle \mathbf{f} \rangle$, since f_1 is a non-zero-divisor. We therefore proceed with the inductive step. Then by assumption we have $f_n r_n = -\sum_{i=1}^{n-1} f_i r_i$, implying $f_n \overline{r_n} = 0$ in the ring $R/\langle f_1, \dots, f_{n-1}\rangle$. Since f_n is a non-zero-divisor on $R/\langle f_1, \dots, f_{n-1}\rangle$, this implies $r_n \in \langle f_1, \dots, f_{n-1}\rangle \subseteq \langle \mathbf{f} \rangle$. So let $s_1, \dots, s_{n-1} \in R$ be such that $r_n = \sum_{i=1}^{n-1} f_i s_i$. Substituting we have

$$0 = \sum_{i=1}^{n} f_i r_i = \sum_{i=1}^{n-1} f_i r_i + f_n \cdot \sum_{i=1}^{n-1} f_i s_i = \sum_{i=1}^{n-1} f_i(r_i + f_n s_i).$$

Then the inductive hypothesis implies

$$r_i + f_n s_i \in \langle f_1, \dots, f_{n-1}\rangle \subseteq \langle \mathbf{f} \rangle$$

for each $i = 1, \dots, n-1$. Then since each $f_n s_i \in \langle \mathbf{f} \rangle$, it follows that $r_i \in \langle \mathbf{f} \rangle$ for $i = 1, \dots, n-1$. ✓

CLAIM (2). *Let $m \in \mathbb{Z}_{m\geq 1}$. Then $\mathbf{f}^{(m)} := f_1^m, f_2, \dots, f_n$ is a weakly R-regular sequence.*

PROOF. We induct on m. The base case $m = 1$ holds by assumption, so we proceed with the inductive step. Assume $m \geq 2$ and $\mathbf{f}^{(m-1)}$ is weakly R-regular. Since f_1 is a non-zero-divisor on R by assumption, it follows that f_1^m is a non-zero-divisor on R as well. For $i \geq 2$ set $\overline{R} = R/\langle f_1^m, f_2, \dots, f_{i-1}\rangle$ and we need to show that f_i is a non-zero-divisor on $\overline{R}$. Let $\overline{r} \in \overline{R}$ be such that $f_i \overline{r} = 0 \in \overline{R}$. Then $r \in R$ satisfies $f_i r \in \langle f_1^m, f_2, \dots, f_{i-1}\rangle$ and we let $t_1, \dots, t_{i-1} \in R$ such that

$$f_i r = t_1 f_1^m + \sum_{j=2}^{i-1} t_j f_j = (t_1 f_1) f_1^{m-1} + \sum_{j=2}^{i-1} t_j f_j. \tag{II.E.35.1}$$

Since $\mathbf{f}^{(m-1)}$ is weakly R-regular we know $r \in \left\langle f_1^{m-1}, f_2, \dots, f_{i-1}\right\rangle$ and thus there exist $u_1, \dots, u_{i-1} \in R$ such that

$$r = u_1 f_1^{m-1} + \sum_{j=2}^{i-1} u_j f_j.$$

Rearranging (II.E.35.1) and substituting we obtain the following.

$$\begin{aligned}0 &= f_i r - t_1 f_1^m - \sum_{j=2}^{i-1} t_j f_j \\ &= u_1 f_i f_1^{m-1} + \sum_{j=2}^{i-1} f_i u_j f_j - t_1 f_1^m - \sum_{j=2}^{i-1} t_j f_j \\ &= f_1^{m-1}(u_1 f_i - t_1 f_1) + \sum_{j=2}^{i-1} f_j(f_i u_j - t_j)\end{aligned}$$

Since $f_1^{m-1}, f_2, \ldots, f_{i-1}$ is weakly R-regular, by Claim (1) we have

$$u_1 f_i - f_1 t_1, f_i u_j - t_j \in \left\langle f_1^{m-1}, f_2, \ldots, f_{i-1} \right\rangle$$

for all $j = 2, \ldots, i-1$. Therefore $u_1 f_i \in \langle f_1, f_2, \ldots, f_{i-1}\rangle$ and since $f_1, \ldots, f_{i-1}$ is weakly R-regular, we know $u_1 \in \langle f_1, \ldots, f_{i-1}\rangle$. Hence $u_1 = \sum_{j=1}^{i-1} v_j f_j$ for some $v_1, \ldots, v_{i-1} \in R$ and we have

$$\begin{aligned}r &= u_1 f_1^{m-1} + \sum_{j=2}^{i-1} u_j f_j \\ &= f_1^{m-1} \sum_{j=1}^{i-1} v_j f_j + \sum_{j=2}^{i-1} u_j f_j \\ &= v_1 f_1^m + \sum_{j=2}^{i-1} f_j (f_1^{m-1} v_j + u_j) \in \langle f_1^m, f_2, \ldots, f_{i-1}\rangle .\end{aligned}$$

Therefore $\overline{r} = 0 \in \overline{R}$ and thus f_i is a non-zero-divisor on $\overline{R}$. ✓

Claim (3). *Let $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 1}$. Then $f_1^{m_1}, f_2^{m_2}, \ldots, f_n^{m_n}$ is weakly R-regular.*

Proof. We again induct on n and note the base case $n = 1$ is done by Claim (2). For the inductive step, note that $f_1^{m_1}$ is a non-zero-divisor on R and $f_2, \ldots, f_n$ is weakly regular on $R/\langle f_1^{m_1}\rangle$ by Claim (2). Therefore by the inductive hypothesis $f_2^{m_2}, \ldots, f_n^{m_n}$ is weakly regular on $R/\langle f_1^{m_1}\rangle$. ✓

Claim (4). *Finally, we claim $\operatorname{depth}_I(R) \leq \operatorname{depth}_J(R)$, which will complete the proof by symmetry.*

Proof. Let $n = \operatorname{depth}_I(R)$ and let $\mathbf{f} = f_1, \ldots, f_n \in I$ be weakly R-regular. Since $\operatorname{rad}(I) = \operatorname{rad}(J)$, there exist $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 1}$ such that $f_i^{m_i} \in J$ for $i = 1, \ldots, n$. Then by Claim (3) we know $f_1^{m_1}, \ldots, f_n^{m_n} \in J$ is weakly R-regular. By the definition of depth we have

$$\operatorname{depth}_J(R) \geq n = \operatorname{depth}_I(R).$$

✓

□

Proof of Buchsbaum-Eisenbud (Continued). Recall that the only remaining case is for $i = 2$, and we needed to show that $\operatorname{depth}_{I_{n-1}(A)}(R) \overset{?}{\geq} 2$. By Lemma II.E.34, we have that

$$\operatorname{rad}(\operatorname{Pf}_{n-1}(A)) = \operatorname{rad}(I_{n-1}(A))$$

and by Proposition II.E.35 this implies

$$\operatorname{depth}_{\operatorname{Pf}_{n-1}(A)}(R) = \operatorname{depth}_{I_{n-1}(A)}(R).$$

Therefore,

$$\operatorname{depth}_{I_{n-1}(A)}(R) = \operatorname{depth}_{\operatorname{Pf}_{n-1}(A)}(R) \geq 3 \geq 2.$$

□

Part III

Differential Graded Algebra Resolutions

We have covered several different types of resolutions so far, and some of them are a lot nicer than others! However, each of them has a downside as well.

(1) Taylor resolutions for monomial ideals have a closed formula! ☺ However, these resolutions are most likely not minimal, so they are not efficient. ☹
(2) A minimal resolution for an ideal always exist! ☺ However, they can be difficult to compute. ☹
(3) DG algebra resolutions have an extra ring structure, so they convey more information than other types of resolutions! ☺ However, they are not usually minimal. ☹

In this part, we will first discuss some definitions, properties, and examples relevant to DG algebra resolutions, then talk about several applications of DG algebra resolutions.

CHAPTER III.A

Definitions, Properties, and Examples

Throughout this chapter, assume that R is a commutative ring with identity. Recall that if X is an R-complex and $0 \neq x \in X_i$, then the homological degree of x is $|x| = i$.

Definition III.A.1. A commutative <u>differential graded R-algebra</u> (DG R-algebra) is an R-complex

$$A = \left(\cdots \xrightarrow{\partial_2^A} A_1 \xrightarrow{\partial_1^A} A_0 \longrightarrow 0 \right)$$

equipped with a binary operation $\mu_{ij} : A_i \times A_j \to A_{i+j}$ (we will write $\mu_{ij}(a, b) = ab$) satisfying the following properties.

- μ_{ij} is R-bilinear. Therefore, μ_{ij} is also distributive. In particular, $0 \cdot b = 0 = b \cdot 0$ for all $b \in A$.
- μ_{ij} is unital, i.e., there exists $1 \in A_0$ such that $1 \cdot a = a = a \cdot 1$ for all $a \in A_i$.
- μ_{ij} is associative.
- μ_{ij} is graded commutative, i.e., for all $a, b \in A \setminus \{0\}$ one has $ba = (-1)^{|a|\cdot|b|}ab$ and $a^2 = 0$ whenever $|a|$ is odd. The second condition is automatic if 2 is a unit in R.
- μ_{ij} satisfies the *Leibniz rule*, i.e., for all $a, b \in A \setminus \{0\}$ one has $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$.

The convention for determining signs is that if we switch the order of two factors, multiply that term by $(-1)^{\text{product of degrees}}$.

Note III.A.2. Let A be a complex of free R-modules, with $A_i = R^{\beta_i}$ for all $i \geq 0$ and $A_i = 0$ for all $i < 0$. Let B_i be a basis of A_i over R.

(a) Any function $f_{ij} : B_i \times B_j \to A_{i+j}$ extends uniquely to an R-bilinear function $\mu_{ij} : A_i \times A_j \to A_{i+j}$ so that $f_{ij} = \mu_{ij}\big|_{B_i \times B_j}$ as in Exercise I.B.10. Therefore to define μ_{ij} it suffices to specify it on the basis vectors.

(b) The operation μ_{ij} is unital in general if and only if it is unital on the basis vectors, and similarly for associativity, graded commutativity, and the Leibniz rule. Exercises I.B.11 and I.B.12 show this for the unital, associative, and graded commutative properties.

(c) There are a few ways to make the Leibniz rule easier to verify:

Claim. *The Leibniz rule is automatic for products of the form $1 \cdot b$ and $b \cdot 1$.*

Proof. Since $1 \cdot b = b$, then $\partial(1 \cdot b) = \partial(b)$. On the other hand,

$$\underbrace{\partial(1)}_{=0} \cdot b + \underbrace{(-1)^{|1|}}_{=1} 1 \cdot \partial(b) = \partial(b)$$

because $|1| = 0$ and because the mapping $\partial : A_0 \to 0$ satisfies $\partial(1) = 0$. ✓

Claim. *The Leibniz rule is automatic for a^2 when $|a|$ is odd.*

Proof. Since $a^2 = 0$, then $\partial(a^2) = \partial(0) = 0$. On the other hand,

$$\begin{aligned}\partial(a) \cdot a + \underbrace{(-1)^{\partial(a)}}_{=-1} a \cdot \partial(a) &= \partial(a) \cdot a - a\partial(a) \\ &= \partial(a) \cdot a - \underbrace{(-1)^{|a||\partial(a)|}}_{=1} \partial(a) \cdot a \\ &= \partial(a) \cdot a - \partial(a) \cdot a = 0\end{aligned}$$

✓

Claim. *The Leibniz rule holds for ab if and only if the Leibniz rule holds for ba.*

PROOF. We show that if Leibniz rule hold for ab, then Leibniz rule holds for ba. The other implication is by symmetry. We will use that $|\partial(a)| = |a| - 1$. We have

$$\begin{aligned}
\partial(ba) &= (-1)^{|a||b|}\partial(ab) \\
&= (-1)^{|a||b|}[\partial(a)b + (-1)^{|a|}a\partial(b)] \\
&= (-1)^{|a||b|}\partial(a)b + (-1)^{|a||b|+|a|}a\partial(b) \\
&= (-1)^{|a||b|+|\partial(a)||b|}b\partial(a) + (-1)^{|a||b|+|a|+|a||\partial(b)|}\partial(b)a \\
&= \underbrace{(-1)^{|a||b|+|a||b|-|b|}}_{=(-1)^{|b|}}b\partial(a) + \underbrace{(-1)^{|a||b|+|a|+|a||b|-|a|}}_{=1}\partial(b)a \\
&= \partial(b)a + (-1)^{|b|}b\partial(a).
\end{aligned}$$

✓

EXAMPLE III.A.3. Let $x, y \in R$ and set $K = K^R(x,y)$ with exterior basis

$$K = \left(0 \longrightarrow \underset{e_{12}}{R} \xrightarrow{\left(\begin{smallmatrix}-y\\ x\end{smallmatrix}\right)} \underset{\substack{e_1\\ e_2}}{R^2} \xrightarrow{(x\ y)} \underset{1=e_\emptyset}{R} \longrightarrow 0 \right).$$

The rules for multiplication on K are as follows:

- $1 \cdot e_* = e_* = e_* \cdot 1$ for $* \in \{1, 2, 12, \emptyset\}$.
- $e_i^2 = 0$ for $i = 1, 2$.
- $e_1e_2 = e_{12} = -e_2e_1$.
- $e_ie_{12} = 0 = e_{12}e_i$ and $e_{12}^2 = 0$ for degree reasons.

With these definitions, Note III.A.2 implies that the unital, R-bilinear, and graded commutative properties are automatically satisfied. To show associatitivity, we need to show $a(bc) \stackrel{?}{=} (ab)c$ for basis vectors a, b, c. We split this into two cases below.

(a) If $b = 1$, then $a(1 \cdot c) = ac = (a \cdot 1) \cdot c$. If $a = 1$ or $c = 1$, then we are done similarly.
(b) If $|a|, |b|, |c| > 1$, then $|a(bc)| \geq 3$, so $a(bc) = 0 = (ab)c$.

We also need to check the Leibniz rule for the basis vectors. By our rules for multiplication above and by symmetry, we need only check the following.

$$\partial(e_1e_2) = \partial(e_{12}) = xe_2 - ye_1 = \partial(e_1)e_2 + (-1)^{|e_1|}e_1\partial(e_2) \checkmark$$
$$\partial(e_2e_{12}) = 0 = ye_{12} - xe_2^2 - ye_{12} = ye_{12} - xe_2^2 + ye_2e_1 = ye_{12} - e_2(xe_2 - ye_1) = \partial(e_2)e_{12} - e_2\partial(e_{12}) \checkmark$$
$$\partial(e_{12}^2) = \partial(0) = 0 = (xe_2 - ye_1)e_{12} + e_{12}(xe_2 - ye_1) = \partial(e_{12})e_{12} + e_{12}\partial(e_{12}) \checkmark$$

Our next result expands on Example III.A.3 by showing that all resolutions of R/I with length at most 2 have DG algebra structures. A homework exercise deals with resolutions of length 3.

THEOREM III.A.4. *Let $I \leq R$ be an ideal such that R/I has a resolution of the form*

$$F = \left(0 \longrightarrow R^n \longrightarrow R^m \longrightarrow R \longrightarrow 0 \right).$$

Then F has the structure of a DG algebra.

PROOF. Let $1 \in R$ and $e_1, \ldots, e_m \in R^m$ and $f_1, \ldots, f_n \in R^n$ be bases of R, R^m, and R^n, respectively. Define multiplication on the basis vectors in the following way.

$$\begin{aligned}
&1 \cdot a = a = a \cdot 1 \\
&e_i^2 = 0, \quad \forall i \\
&e_if_j = 0 = f_je_i, \quad \forall i, j \\
&f_if_j = 0, \quad \forall i, j
\end{aligned}$$

It remains to define e_ie_j for $i \neq j$. The Leibniz rule dictates that

$$\partial(e_ie_j) = \partial(e_i)e_j - e_i\partial(e_j) = x_ie_j - x_je_i,$$

where $x_i = \partial(e_i) \in R$. Now we observe

$$\partial(x_i e_j - x_j e_i) = x_i x_j - x_j x_i = 0$$

and therefore $x_i e_j - x_j e_i \in \operatorname{Im}\left(\partial_2^F\right)$. Thus there exists some $\gamma_{ij} \in F_2 = R^n$ such that $\partial(\gamma_{ij}) = x_i e_j - x_j e_i$. Since ∂_2^F is injective, this γ_{ij} is unique and we define $e_i e_j = \gamma_{ij}$, i.e., $e_i e_j$ is the unique element of F_2 such that $\partial(e_i e_j) = x_i e_j - x_j e_i$.

As in Example III.A.3, associativity follows for degree reasons. To show graded commutivity we need to show that $e_j e_i = -e_i e_j$ for all i, j, for which it suffices to show that $\partial(e_j e_i) = -\partial(e_i e_j)$. By definition we have

$$\partial(e_j e_i) = x_j e_i - x_i e_j = -(x_i e_j - x_j e_i) = -\partial(e_i e_j),$$

as desired. For the most part, the Leibniz rule is satisfied by definition. For instance, we have $f_i f_j \in F_4 = 0$ and $\partial(f_i f_j) \in F_3 = 0$, so the Leibniz rule is satisfied. What about $e_i f_j$? For degree reasons we have $\partial(e_i f_j) = 0$ and we therefore need to show $0 = \partial(e_i) f_j - e_i \partial(f_j)$. Again noting that ∂_2^F is injective, it suffices to show that $\partial\left(\partial(e_i) f_j - e_i \partial(f_j)\right) = 0$. The Leibniz rule in degree 1 is satisfied, so we have

$$\partial\left(\partial(e_i) f_j - e_i \partial(f_j)\right) = [\partial(\partial(e_i)) f_j + \partial(e_i)\partial(f_j)] - [\partial(e_i)\partial(f_j) - e_i \partial(\partial(f_j))] = 0,$$

as desired. $\square$

Next, we demonstrate some general properties of DG algebras.

Proposition III.A.5. *Let A be a DG algebra. Then A_0 is a commutative ring with identity under the operations from A.*

Proof. Since A_0 is an R-module, it is an additive abelian group. Since A is a DG algebra, the multiplication $A_0 \times A_0 \to A_0$ is well-defined. It is also associative, unital, and distributive by assumption. Finally, we see that for any $a, b \in A_0$ we have

$$ba = (-1)^{|a||b|} ab = (-1)^{0 \cdot 0} ab = ab.$$

$\square$

The following result shows that the homology modules of a DG algebra each have more than just a module structure. We will use this in our applications to show that certain collections of homology modules form graded commutative rings.

Theorem III.A.6. *Assume A is a DG algebra. Then*

$$H(A) := \left(\cdots \xrightarrow{0} H_1(A) \xrightarrow{0} H_0(A) \longrightarrow 0 \right)$$

is also a DG algebra. Therefore $\bigoplus_{i=0}^{\infty} H_i(A)$ is a graded commutative ring with identity.

Proof Sketch. Recall that $Z_i(A) = \operatorname{Ker}\left(\partial_i^A\right)$. One first shows that

$$Z(A) = \left(\cdots \xrightarrow{0} Z_1(A) \xrightarrow{0} Z_0(A) \xrightarrow{0} 0 \right)$$

is a DG sub-algebra of A, i.e., that $Z(A)$ is a subcomplex of A such that the multiplication on A induces a multiplication on $Z(A)$ making $Z(A)$ into a DG algebra. One does this using a DG subalgebra test, which is analogous to the familiar subring and subgroup tests: we need to show $Z(A)$ is a subcomplex that is closed under multiplication with $1 \in Z_0(A)$. It is a subcomplex since $\partial_i^A|_{Z_i(A)} = 0$ for all i by definition of $Z_i(A)$. We also know $1 \in A_0 = Z_0(A)$ and if we let $z, w \in Z(A)$, then it follows that $zw \in Z(A)$ since

$$\partial(zw) = \underbrace{\partial(z)}_{=0} w \pm z \underbrace{\partial(w)}_{=0} = 0.$$

Next, recalling that $B_i(A) = \operatorname{Im}\left(\partial_{i+1}^A\right)$, one sets

$$B(A) = \left(\cdots \xrightarrow{0} B_1(A) \xrightarrow{0} B_0(A) \longrightarrow 0 \right)$$

and proves this is a DG ideal of $Z(A)$, i.e., that $B(A)$ is a subcomplex of $Z(A)$ that absorbs multiplication by elements of $Z(A)$. The subcomplex condition is straightforward since $B_i(A) \subseteq Z_i(A)$. Let $b \in B(A)$ and $z \in Z(A)$ and in order to show that $zb, bz \in B(A)$, it suffices to show that $bz \in B(A)$ (because of graded commutivity). Let $a \in A$ such that $b = \partial(a)$ and observe that

$$\partial(az) = \partial(a)z + \underbrace{(-1)^{|a|}a\partial(z)}_{=0} = bz,$$

so $bz \in B(A)$.

For the third and final step, one shows that $H(A) = Z(A)/B(A)$ is a DG algebra with differential and multiplication induced from $Z(A)$. Certainly the 0-differential on $H(A)$ makes it into an R-complex. Most of the work is done showing that the multiplication

$$\begin{aligned} H_i(A) \times H_j(A) &\longrightarrow H_{i+j}(A) \\ (\overline{a}, \overline{b}) &\longmapsto \overline{ab} \end{aligned}$$

is well-defined. If $\overline{a} = \overline{a'}$ and $\overline{b} = \overline{b'}$, then $a - a', b - b' \in B(A)$. Therefore $a - a' = \partial(c)$ and $b - b' = \partial(d)$ for some $c, d \in Z(A)$. Thus $a = a' + \partial(c)$ and $b = b' + \partial(d)$, and since $B(A)$ is a DG ideal we have

$$ab = a'b' + \underbrace{a'\partial(d) + \partial(c)b' + \partial(c)\partial(d)}_{\in B(A)},$$

so $\overline{ab} = \overline{a'b'}$. To show that this makes $H(A)$ a DG algebra, one checks that all other DG algebra axioms are inherited from the corresponding axioms on $Z(A)$. □

Example III.A.7. Let $R = k[X, Y]/\langle XY \rangle$ and $x = \overline{X}$ and $y = \overline{Y}$ and

$$K = K^R(x, y) = \left(0 \longrightarrow R \xrightarrow{\binom{-y}{x}} R^2 \xrightarrow{(x\ y)} R \longrightarrow 0 \right).$$

Recall that $H_0(K) = R/\langle x, y \rangle \cong k$, $H_1(K) \cong k$, and $H_2(K) = 0$. Then

$$H = H(K) \cong \left(0 \longrightarrow 0 \longrightarrow \underset{\varepsilon}{k} \xrightarrow{0} \underset{1}{k} \longrightarrow 0 \right)$$

and

$$\begin{aligned} \bigoplus_{i=0}^{\infty} H_i(K) &= k \cdot 1 \oplus k \cdot \varepsilon && \text{(since } \varepsilon^2 = 0) \\ &\cong k[Z]/\langle Z^2 \rangle && \text{(where } \overline{Z} \sim \varepsilon). \end{aligned}$$

Exterior DG Algebra Structure on the Koszul Complex. Our next goal is to extend Example III.A.3 again by showing that every Koszul complex has a DG algebra structure.

Definition III.A.8. Let $[n] = \{1, \ldots, n\}$ for a positive integer n and $\mathbf{x} = x_1, \ldots, x_n \in R$ and $K = K^R(\mathbf{x})$. Let e_Λ and e_Γ be basis vectors of K, where $\Lambda, \Gamma \subseteq [n]$. Define

$$e_\Lambda e_\Gamma = \begin{cases} 0 & \text{if } \Lambda \cap \Gamma \neq \emptyset \\ \operatorname{sgn}(\Lambda, \Gamma) e_{\Lambda \cup \Gamma} & \text{if } \Lambda \cap \Gamma = \emptyset, \end{cases}$$

where $\operatorname{sgn}(\Lambda, \Gamma)$ is the sign of the permutation used to put $\Lambda \cup \Gamma$ into strictly increasing order. To understand the two cases in the display, notice that if $\Lambda = \{\lambda_1 < \cdots < \lambda_j\}$ and $\Gamma = \{\gamma_1 < \cdots < \gamma_k\}$, then $|e_\Lambda| = j$ and $|e_\Gamma| = k$, so we must have $|e_\Lambda e_\Gamma| = j + k$, that is, in order for $e_\Lambda e_\Gamma$ to possibly be a non-zero multiple of $e_{\Lambda \cup \Gamma}$, we must have $\Lambda \cap \Gamma \neq \emptyset$.

Example III.A.9. Consider the following three products of basis vectors:

- Let $\Lambda = \{1 < 3\}$ and $\Gamma = \{2 < 3\}$. Then $e_{13}e_{23} = 0$ since $\Lambda \cap \Gamma \neq \emptyset$.
- Let $\Lambda = \{1 < 3\}$ and $\Gamma = \{2 < 4\}$, and consider $e_{13}e_{24}$. Then

$$\Lambda \cup \Gamma = \{1 < 3, 2 < 4\} = \begin{pmatrix} 2 & 3 \end{pmatrix} \{1 < 2 < 3 < 4\},$$

so the permutation used to put $\Lambda \cup \Gamma$ in order is odd (i.e., $\operatorname{sgn}(\Lambda, \Gamma) = -1$). Therefore $e_{13}e_{24} = -e_{1234}$.

- Consider $e_{24}e_{13}$. Then as in the preceding bullet, we find that $e_{24}e_{13} = -e_{1234}$. Let's check graded commutativity for this product:

$$e_{24}e_{13} \stackrel{?}{=} (-1)^{|e_{13}||e_{24}|}e_{13}e_{24} = (-1)^{2\cdot 2}e_{13}e_{24} = e_{13}e_{24} = -e_{1234} = e_{24}e_{13}.\checkmark$$

Also, we check the Leibniz rule for the first product above. On the one hand, $\partial(e_{13}e_{23}) = \partial(0) = 0$. On the other hand, we have

$$\begin{aligned}\partial(e_{13})e_{23} + e_{13}\partial(e_{23}) &= (x_1e_3 - x_3e_1)e_{23} + e_{13}(x_2e_3 - x_3e_2)\\ &= x_1\underbrace{e_3e_{23}}_{=0} - x_3\underbrace{e_1e_{23}}_{=e_{123}} + x_2\underbrace{e_{13}e_3}_{=0} - x_3\underbrace{e_{13}e_2}_{=-e_{123}}\\ &= -x_3e_{123} + x_3e_{123} = 0.\checkmark\end{aligned}$$

Next, we work to make our treatment of $\operatorname{sgn}(\Lambda, \Gamma)$ rigorous.

Definition III.A.10. Let $\mathbf{i} = (i_1, \ldots, i_n)$ where all of the i_k are distinct positive integers. Define

$$S(\mathbf{i}) = \{(p, q) \in [n] \times [n] \mid p < q \text{ and } i_p > i_q\}.$$

In words, $S(\mathbf{i})$ counts the entries of $\mathbf{i}$ not in strictly ascending order. Also define

$$\sigma(\mathbf{i}) = (-1)^{|S(\mathbf{i})|}.$$

Example III.A.11.
(a) Let $\mathbf{i} = (1, 3, 2, 4)$. The second and third positions are out of order, so $S(\mathbf{i}) = \{(2, 3)\}$ and $\sigma(\mathbf{i}) = (-1)^1 = -1$.
(b) Let $\mathbf{j} = (2, 4, 1, 3)$. Then there are three pairs that are out of order, so

$$S(\mathbf{j}) = \{(1, 3), (2, 3), (2, 4)\} \text{ and } \sigma(\mathbf{j}) = (-1)^3 = -1.$$

Proposition III.A.12. *Let $\mathbf{i}$ be as in Definition III.A.10 and let $\tau \in S_n$. Define $\tau \cdot \mathbf{i} = (i_{\tau(1)}, \ldots, i_{\tau(n)})$. Write τ as a product of adjacent transpositions $\tau = \tau_1 \cdots \tau_\ell$. Then*

$$\sigma(\tau \cdot \mathbf{i}) = (-1)^\ell \sigma(\mathbf{i}).$$

Proof. We prove this by induction on ℓ.
Base case: Let $\ell = 1$. Then τ is an adjacent transposition, so can be written as $\tau = \begin{pmatrix} x & x+1 \end{pmatrix}$. Then

$$\begin{aligned}S(\tau \cdot \mathbf{i}) &= S(i_1, \ldots, i_{x-1}, i_{x+1}, i_x, i_{x+2}, \ldots, i_n)\\ &= \begin{cases} S(\mathbf{i}) \cup \{(x, x+1)\} & \text{if } i_x < i_{x+1}\\ S(\mathbf{i}) \setminus \{(x, x+1)\} & \text{if } i_{x+1} < i_x.\end{cases}\end{aligned}$$

Then

$$|S(\tau \cdot \mathbf{i})| = \begin{cases} |S(\mathbf{i})| + 1 & \text{if } i_x < i_{x+1}\\ |S(\mathbf{i})| - 1 & \text{if } i_{x+1} < i_x.\end{cases}$$

Therefore

$$\sigma(\tau \cdot \mathbf{i}) = (-1)^{|S(\mathbf{i})| \pm 1} = -(-1)^{|s(\mathbf{i})|} = -\sigma(\mathbf{i}).$$

We omit the inductive case here since it is routine. □

Corollary III.A.13. *Let $\Lambda = \{\lambda_1 < \cdots < \lambda_j\}$ and $\Gamma = \{\gamma_1 < \cdots < \gamma_k\}$. Assume $\Lambda \cap \Gamma = \emptyset$. Then*

$$\operatorname{sgn}(\Lambda, \Gamma) = \sigma(\lambda_1, \ldots, \lambda_j, \gamma_1, \ldots, \gamma_k).$$

Proof. Set $\mathbf{i} = (i_1, \ldots, i_n)$. Let $\tau = \tau_1, \ldots, \tau_\ell$ be as in Proposition III.A.12 so that $\tau \cdot \mathbf{i}$ is in strictly ascending order. Then $S(\tau \cdot \mathbf{i}) = \emptyset$, so $\sigma(\tau \cdot \mathbf{i}) = (-1)^0 = 1$. Therefore,

$$\sigma(\mathbf{i}) = (-1)^\ell \sigma(\tau \cdot \mathbf{i}) = (-1)^\ell = \operatorname{sgn}(\Lambda, \Gamma).$$

□

Now we are in position to verify that every Koszul complex is a DG algebra. We accomplish this in a sequence of theorems.

Theorem III.A.14. *The exterior product on K is associative.*

Proof. It suffices to show that $(e_\Lambda e_\Gamma)e_\Omega = e_\Lambda(e_\Gamma e_\Omega)$ for all $\Lambda, \Gamma, \Omega \subseteq [n]$. If $\Lambda \cap \Gamma \neq \emptyset$, then both sides are zero, and similarly if $\Lambda \cap \Omega \neq \emptyset$ or $\Gamma \cap \Omega \neq \emptyset$. Thus we assume that Λ, Γ, and Ω are pairwise disjoint. Then

$$(e_\Lambda e_\Gamma)e_\Omega = \operatorname{sgn}(\Lambda,\Gamma)\cdot\operatorname{sgn}(\Lambda\cup\Gamma,\Omega)e_{\Lambda\cup\Gamma\cup\Omega}$$
$$e_\Lambda(e_\Gamma e_\Omega) = \operatorname{sgn}(\Gamma,\Omega)\cdot\operatorname{sgn}(\Lambda,\Gamma\cup\Omega)e_{\Lambda\cup\Gamma\cup\Omega}.$$

Notice that the coefficients of the above items are both equal to

$$\sigma(\lambda_1,\dots,\lambda_j,\gamma_1,\dots,\gamma_k,\omega_1,\dots,\omega_\ell),$$

so the two products are equal. □

Theorem III.A.15. *The exterior product on K is graded commutative.*

Proof. If $\Lambda \neq 0$, then $e_\Lambda^2 = 0$. If $\Lambda\cap\Gamma \neq \emptyset$, then $e_\Lambda e_\Gamma = 0 = (-1)^{|\Lambda||\Gamma|}e_\Gamma e_\Lambda$. So assume $\Lambda\cap\Gamma=\emptyset$. Then we claim

$$\operatorname{sgn}(\Gamma,\Lambda) = (-1)^{|\Gamma||\Lambda|}\operatorname{sgn}(\Lambda,\Gamma).$$

To show this, we want to show that the right hand side describes the sign of a permutation that puts $\Gamma\cup\Lambda$ in strictly ascending order. This happens in two steps: first, $(-1)^{|\Lambda||\Gamma|}$ moves all $\gamma\in\Gamma$ to the right of Λ; second, $\operatorname{sgn}(\Lambda,\Gamma)$ puts $\Lambda\cup\Gamma$ into strictly ascending order. Therefore,

$$e_\Gamma e_\Lambda = \operatorname{sgn}(\Gamma,\Lambda)e_{\Gamma\cup\Lambda} = (-1)^{|\Gamma||\Lambda|}\operatorname{sgn}(\Lambda,\Gamma)e_{\Lambda\cup\Gamma} = (-1)^{|\Gamma||\Lambda|}e_\Lambda e_\Gamma.$$

□

Remark. *Even though squares of basis vectors are equal to zero, squares of linear combinations of basis vectors are not necessarily equal to zero. Consider the following two examples assuming $2\neq 0$ in R:*

(a) $(e_1+e_2)^2 = e_1^2 + e_1e_2 + e_2e_1 + e_2^2 = e_1e_2 - e_1e_2 = 0$.
(b) $(e_{12}+e_{34})^2 = e_{12}^2 + e_{12}e_{34} + e_{34}e_{12} + e_{34}^2 = e_{12}e_{34} + e_{12}e_{34} = 2e_{12}e_{34} \neq 0$.

Theorem III.A.16. *The Koszul complex $K = K^R(\mathbf{x})$ is a DG algebra with exterior multiplication.*

Proof. We have already shown that every property other than the Leibniz rule is satisfied, so we want to show that

$$\partial(e_\Lambda e_\Gamma) \stackrel{?}{=} \partial(e_\Lambda)e_\Gamma + (-1)^{|\Lambda|}e_\Lambda\partial(e_\Gamma).$$

Define $\phi(m,\Lambda) = |\{\lambda\in\Lambda \mid \lambda < m\}|$, so

$$\partial(e_\Lambda) = \sum_{\lambda\in\Lambda}(-1)^{\phi(\lambda,\Lambda)}x_\lambda e_{\Lambda\setminus\{\lambda\}}.$$

Then the right hand side of the Leibniz rule looks like

$$\sum_{\lambda\in\Lambda}(-1)^{\phi(\lambda,\Lambda)}x_\lambda e_{\Lambda\setminus\{\lambda\}}e_\Gamma + (-1)^{|\Lambda|}\sum_{\gamma\in\Gamma}(-1)^{\phi(\gamma,\Gamma)}x_\gamma e_\Lambda e_{\Gamma\setminus\{\gamma\}}.$$

We have three cases to consider:

Case 1: Suppose $|\Lambda\cap\Gamma|\geq 2$. Then $|(\Lambda\setminus\{\lambda\})\cap\Gamma|\geq 1$, so $e_{\Lambda\setminus\{\lambda\}}e_\Gamma = 0$ for all $\lambda\in\Lambda$. Similarly, for all $\gamma\in\Gamma$, we have $e_\Lambda e_{\Gamma\setminus\{\gamma\}} = 0$. Therefore, the entire right hand side of the Leibniz rule is equal to 0. Also, the left hand side is $\partial(e_\Lambda e_\Gamma) = \partial(0) = 0$, so the Leibniz rule is satisfied.

Case 2: Suppose $|\Lambda\cap\Gamma| = 1$, say $\lambda_{p_0} = \gamma_{q_0} \in \Lambda\cap\Gamma$. The left hand side of the Leibniz rule is 0, so we want to show the right hand side is also 0. For $\lambda\neq\lambda_{p_0}$, we have $\lambda_{p_0}\in(\Lambda\setminus\{\lambda\})\cap\Gamma$, so $e_{\Lambda\setminus\{\lambda\}}e_\Gamma = 0$. Similarly, for $\gamma\neq\gamma_{q_0}$, we have $e_\Lambda e_{\Gamma\setminus\{\gamma\}} = 0$. So the right hand side reduces to

$$(-1)^{\phi(\lambda_{p_0},\Lambda)}x_{\lambda_{p_0}}e_{\Lambda\setminus\{\lambda_{p_0}\}}e_\Gamma + (-1)^{|\Lambda|+\phi(\gamma_{q_0},\Gamma)}x_{\gamma_{q_0}}e_\Lambda e_{\Gamma\setminus\{\gamma_{q_0}\}}.$$

Notice that $(\Lambda\setminus\{\lambda_{p_0}\})\cup\Gamma = \Lambda\cup\Gamma = \Lambda\cup(\Gamma\setminus\{\gamma_{q_0}\})$ because $\lambda_{p_0}\in\Gamma$ and $\gamma_{q_0}\in\Lambda$. Then the right hand side simplifies to

$$(-1)^{\phi(\lambda_{p_0},\Lambda)}\operatorname{sgn}(\Lambda\setminus\{\lambda_{p_0}\},\Gamma)x_{\lambda_{p_0}}e_{\Lambda\cup\Gamma} + (-1)^{|\Lambda|+\phi(\gamma_{q_0},\Gamma)}\operatorname{sgn}(\Lambda,\Gamma\setminus\{\gamma_{q_0}\})x_{\gamma_{q_0}}e_{\Lambda\cup\Gamma}.$$

We want to show that the two terms in the display above have opposite signs. Consider listing the elements in $(\Lambda\setminus\{\lambda_{p_0}\})\cup\Gamma$:

$$(\lambda_1,\dots,\lambda_{p_0-1},\lambda_{p_0+1},\dots,\lambda_j,\gamma_1,\dots,\gamma_{q_0-1},\gamma_{q_0},\dots,\gamma_k).$$

Notice that $(q_0-1)+(j-p_0)$ adjacent transpositions are needed to move γ_{q_0} to between λ_{p_0-1} and λ_{p_0+1}, and $\phi(\lambda_{p_0},\Lambda)=p_0-1$. Then the sign of the first term is

$$(-1)^{\phi(\lambda_{p_0},\Lambda)}\operatorname{sgn}(\Lambda\setminus\{\lambda_{p_0}\},\Gamma)=(-1)^{(p_0-1)+(q_0-1)+(j-p_0)}\operatorname{sgn}(\Lambda,\Gamma\setminus\{\gamma_{q_0}\})=(-1)^{q_0+j}\operatorname{sgn}(\Lambda,\Gamma\setminus\{\gamma_{q_0}\}).$$

Also, notice that $|\Lambda|=j$ and $\phi(\gamma_{q_0},\Gamma)=q_0-1$, so the sign of the second term is

$$(-1)^{|\Lambda|+\phi(\gamma_{q_0},\Gamma)}\operatorname{sgn}(\Lambda,\Gamma\setminus\{\gamma_{q_0}\})=(-1)^{j+q_0-1}\operatorname{sgn}(\Lambda,\Gamma\setminus\{\gamma_{q_0}\})=-(-1)^{j+q_0}\operatorname{sgn}(\Lambda,\Gamma\setminus\{\gamma_{q_0}\}).$$

Therefore the signs of the two terms are opposites, so the two terms cancel.

Case 3: Suppose $\Lambda\cap\Gamma=\emptyset$. Then $(\Lambda\setminus\{\lambda\})\cap\Gamma=\emptyset=\Lambda\cap(\Gamma\setminus\{\gamma\})$ for all $\lambda\in\Lambda$ and $\gamma\in\Gamma$. The right hand side of the Leibniz rule is

$$\begin{aligned}&\sum_{\lambda\in\Lambda}(-1)^{\phi(\lambda,\Lambda)}x_\lambda e_{\Lambda\setminus\{\lambda\}}e_\Gamma+(-1)^{|\Lambda|}\sum_{\gamma\in\Gamma}(-1)^{\phi(\gamma,\Gamma)}x_\gamma e_\Lambda e_{\Gamma\setminus\{\gamma\}}\\ &\qquad=\sum_{\lambda\in\Lambda}(-1)^{\phi(\lambda,\Lambda)}\operatorname{sgn}(\Lambda\setminus\{\lambda\},\Gamma)x_\lambda e_{(\Lambda\setminus\{\lambda\})\cup\Gamma}+\sum_{\gamma\in\Gamma}(-1)^{|\Lambda|+\phi(\gamma,\Gamma)}\operatorname{sgn}(\Lambda,\Gamma\setminus\{\gamma\})x_\gamma e_{\Lambda\cup(\Gamma\setminus\{\gamma\})}.\end{aligned}$$

The left hand side of the Leibniz rule is

$$\begin{aligned}\partial(e_\Lambda e_\Gamma)&=\operatorname{sgn}(\Lambda,\Gamma)\partial(e_{\Lambda\cup\Gamma})\\ &=\operatorname{sgn}(\Lambda,\Gamma)\sum_{\zeta\in\Lambda\cup\Gamma}(-1)^{\phi(\zeta,\Lambda\cup\Gamma)}x_\zeta e_{(\Lambda\cup\Gamma)\setminus\{\zeta\}}\\ &=\operatorname{sgn}(\Lambda,\Gamma)\left[\sum_{\lambda\in\Lambda}(-1)^{\phi(\lambda,\Lambda\cup\Gamma)}x_\lambda e_{(\Lambda\setminus\{\lambda\})\cup\Gamma}+\sum_{\gamma\in\Gamma}(-1)^{\phi(\gamma,\Lambda\cup\Gamma)}x_\gamma e_{\Lambda\cup(\Gamma\setminus\{\gamma\})}\right].\end{aligned}$$

Now we compare the signs of both terms in the left hand side and right hand side of the Leibniz rule:

$$\underbrace{(-1)^{\phi(\lambda,\Lambda)}}_{\substack{\text{move }\lambda\text{ to the}\\ \text{beginning of }\Lambda}}\ \underbrace{\operatorname{sgn}(\Lambda\setminus\{\lambda\},\Gamma)}_{\substack{\text{order the}\\ \text{leftovers}}}\overset{?}{=}\underbrace{\operatorname{sgn}(\Lambda,\Gamma)}_{\substack{\text{order all}\\ \text{elements}}}\ \underbrace{(-1)^{\phi(\lambda,\Lambda\cup\Gamma)}}_{\substack{\text{move }\lambda\text{ to the}\\ \text{beginning of }\Lambda\cup\Gamma}},\checkmark$$

$$\underbrace{(-1)^{\phi(\gamma,\Gamma)}}_{\substack{\text{move }\gamma\text{ to the}\\ \text{beginning of }\Gamma}}\ \underbrace{(-1)^{|\Lambda|}}_{\substack{\text{move }\gamma\text{ past}\\ \text{all elements of }\Lambda}}\ \underbrace{\operatorname{sgn}(\Lambda,\Gamma\setminus\{\gamma\})}_{\substack{\text{order the}\\ \text{leftovers}}}\overset{?}{=}\underbrace{\operatorname{sgn}(\Lambda,\Gamma)}_{\substack{\text{order all}\\ \text{elements}}}\ \underbrace{(-1)^{\phi(\gamma,\Lambda\cup\Gamma)}}_{\substack{\text{move }\gamma\text{ to the}\\ \text{beginning of }\Lambda\cup\Gamma}}.\checkmark$$

Therefore the Leibniz rule is satisfied. $\square$

Next, we use the results about the Koszul complex to verify that every Taylor resolution is a DG algebra.

Definition III.A.17. Let $R=k[X_1,\dots,X_d]$ and $f_1,\dots,f_n\in[\![R]\!]$. For all $\Lambda=\{\lambda_1<\cdots<\lambda_j\}\subseteq[n]$, set

$$f_\Lambda=\operatorname{lcm}(\{f_\lambda\mid\lambda\in\Lambda\})=\operatorname{lcm}(f_{\lambda_1},\dots,f_{\lambda_j}).$$

For $\Lambda,\Gamma\subseteq[n]$, consider $e_\Lambda,e_\Gamma\in T=T^R(\mathbf{f})$. Define

$$e_\Lambda e_\Gamma=\begin{cases}0 & \text{if }\Lambda\cap\Gamma\neq\emptyset\\ \operatorname{sgn}(\Lambda,\Gamma)\frac{f_\Lambda f_\Gamma}{f_{\Lambda\cup\Gamma}}e_{\Lambda\cup\Gamma} & \text{if }\Lambda\cap\Gamma=\emptyset.\end{cases}$$

Example III.A.18. Consider the Taylor resolution $T=T^R(XY,XZ,YZ)$:

$$0\longrightarrow\underset{e_{123}}{R}\xrightarrow{\begin{pmatrix}1\\-1\\1\end{pmatrix}}\underset{\substack{e_{12}\\ e_{13}\\ e_{23}}}{R^3}\xrightarrow{\begin{pmatrix}-Z&-Z&0\\Y&0&-Y\\0&X&X\end{pmatrix}}\underset{\substack{e_1\\ e_2\\ e_3}}{R^3}\xrightarrow{\begin{pmatrix}XY&XZ&YZ\end{pmatrix}}\underset{1=e_\emptyset}{R}\longrightarrow 0.$$

Then:

$$\begin{aligned}e_2e_1&=-\frac{f_2f_1}{f_{12}}e_{12}=-\frac{XZ\cdot XY}{XYZ}e_{12}=-Xe_{12},\\ e_3e_{13}&=0,\\ e_3e_{12}&=+\frac{f_3f_{12}}{f_{123}}e_{123}=\frac{YZ\cdot XYZ}{XYZ}=YZe_{123}.\end{aligned}$$

Theorem III.A.19. *Multiplication on T is associative.*

Proof. It suffices to show that $(e_\Lambda e_\Gamma)e_\Omega = e_\Lambda(e_\Gamma e_\Omega)$ where $\Lambda, \Gamma, \Omega \subseteq [n]$ are pairwise disjoint. The left hand side simplifies to

$$\begin{aligned}(e_\Lambda e_\Gamma)e_\Omega &= \left(\operatorname{sgn}(\Lambda,\Gamma)\frac{f_\Lambda f_\Gamma}{f_{\Lambda\cup\Gamma}}e_{\Lambda\cup\Gamma}\right)e_\Omega \\ &= \operatorname{sgn}(\Lambda,\Gamma)\operatorname{sgn}(\Lambda\cup\Gamma,\Omega)\frac{f_\Lambda f_\Gamma}{f_{\Lambda\cup\Gamma}}\frac{f_{\Lambda\cup\Gamma}f_\Omega}{f_{(\Lambda\cup\Gamma)\cup\Omega}}e_{(\Lambda\cup\Gamma)\cup\Omega} \\ &= \operatorname{sgn}(\Lambda,\Gamma)\operatorname{sgn}(\Lambda\cup\Gamma,\Omega)\frac{f_\Lambda f_\Gamma f_\Omega}{f_{\Lambda\cup\Gamma\cup\Omega}}e_{\Lambda\cup\Gamma\cup\Omega}.\end{aligned}$$

The right hand side simplifies to

$$\begin{aligned}e_\Lambda(e_\Gamma e_\Omega) &= e_\Lambda\left(\operatorname{sgn}(\Gamma,\Omega)\frac{f_\Gamma f_\Omega}{f_{\Gamma\cup\Omega}}e_{\Gamma\cup\Omega}\right) \\ &= \operatorname{sgn}(\Gamma,\Omega)\operatorname{sgn}(\Lambda,\Gamma\cup\Omega)\frac{f_\Lambda f_{\Gamma\cup\Omega}}{f_{\Lambda\cup(\Gamma\cup\Omega)}}\frac{f_\Gamma f_\Omega}{f_{\Gamma\cup\Omega}}e_{\Lambda\cup(\Gamma\cup\Omega)} \\ &= \operatorname{sgn}(\Gamma,\Omega)\operatorname{sgn}(\Lambda,\Gamma\cup\Omega)\frac{f_\Lambda f_\Gamma f_\Omega}{f_{\Lambda\cup\Gamma\cup\Omega}}e_{\Lambda\cup\Gamma\cup\Omega}.\end{aligned}$$

Notice that the monomial coefficients agree and the signs agree using the same proof as for the Koszul complex in Theorem III.A.14. ☐

Theorem III.A.20. *Multiplication on T is graded commutatitive.*

Proof. First, $e_\Lambda^2 = 0$ for all $\Lambda \neq \emptyset$. We want to show that the following equation holds for all $\Lambda, \Gamma \subseteq [n]$:

$$e_\Lambda e_\Gamma \stackrel{?}{=} (-1)^{|\Lambda||\Gamma|}e_\Gamma e_\Lambda.$$

This is automatic if $\Lambda\cap\Gamma \neq \emptyset$, so assume $\Lambda\cap\Gamma = \emptyset$. The signs agree using the same proof as for the Koszul complex in Theorem III.A.15 and the monomial coefficients are

$$\frac{f_\Gamma f_\Lambda}{f_{\Gamma\cup\Lambda}} = \frac{f_\Lambda f_\Gamma}{f_{\Lambda\cup\Gamma}}.$$

☐

Theorem III.A.21. *The Taylor resolution $T = T^R(\mathbf{f})$ is a DG algebra.*

Proof. We want to show that the Leibniz rule is satisfied on basis vectors e_Λ, e_Γ. We have three cases to consider, as in the proof of Theorem III.A.16.

Case 1: Suppose $|\Lambda\cap\Gamma| \geq 2$. This case follows the same process as in the proof of Theorem III.A.16.

Case 2: Suppose $|\Lambda\cap\Gamma| = 1$, say $\lambda_{p_0} = \gamma_{q_0} \in \Lambda\cap\Gamma$. The left hand side of the Leibniz rule is 0, so we want to show the right hand side is also 0. For $\lambda \neq \lambda_{p_0}$, $\lambda_{p_0} \in (\Lambda\setminus\{\lambda\})\cap\Gamma$, so $e_{\Lambda\setminus\{\lambda\}}e_\Gamma = 0$. Similarly, for $\gamma \neq \gamma_{q_0}$, $e_\Lambda e_{\Gamma\setminus\{\gamma\}} = 0$. So the right hand side of the Leibniz rule reduces to

$$\begin{aligned}&(-1)^{\phi(\lambda_{p_0},\Lambda)}\frac{f_\Lambda}{f_{\Lambda\setminus\{\lambda_{p_0}\}}}e_{\Lambda\setminus\{\lambda_{p_0}\}}e_\Gamma + (-1)^{|\Lambda|+\phi(\gamma_{q_0},\Gamma)}\frac{f_\Gamma}{f_{\Gamma\setminus\{\gamma_{p_0}\}}}e_\Lambda e_{\Gamma\setminus\{\gamma_{q_0}\}} \\ &\qquad = \pm\frac{f_\Lambda}{f_{\Lambda\setminus\{\lambda_{p_0}\}}}\frac{f_{\Lambda\setminus\{\lambda_{p_0}\}}f_\Gamma}{f_{(\Lambda\setminus\{\lambda_{p_0}\})\cup\Gamma}}e_{(\Lambda\setminus\{\lambda_{p_0}\})\cup\Gamma} \mp \frac{f_\Gamma}{f_{\Gamma\setminus\{\gamma_{q_0}\}}}\frac{f_\Lambda f_{\Gamma\setminus\{\gamma_{q_0}\}}}{f_{\Lambda\cup(\Gamma\setminus\{\gamma_{q_0}\})}}e_{\Lambda\cup(\Gamma\setminus\{\gamma_{q_0}\}} \\ &\qquad = \pm\frac{f_\Lambda f_\Gamma}{f_{\Lambda\cup\Gamma}}e_{\Lambda\cup\Gamma} \mp \frac{f_\Gamma f_\Lambda}{f_{\Lambda\cup\Gamma}}e_{\Lambda\cup\Gamma} = 0\end{aligned}$$

where the signs in the terms are opposites as in the proof of Theorem III.A.16.

Case 3: Suppose $\Lambda\cap\Gamma = \emptyset$. As in Case 2, this is similar to Case 3 in the proof of Theorem III.A.16. ☐

CHAPTER III.B

General Construction of DG Algebra Resolutions

Throughout this chapter, assume R is a noetherian commutative ring with identity and $I \leq R$ is an ideal and $\overline{R} = R/I$.

Definition III.B.1. A <u>DG algebra resolution</u> of $\overline{R}$ over R is a free resolution A of $\overline{R}$ over R such that A is a DG R-algebra.

Example III.B.2. (a) If $\mathbf{f} \in R$ is a weakly R-regular sequence, then the Koszul complex $K^R(\mathbf{f})$ is a DG algebra resolution over R of $R/\langle \mathbf{f}\rangle$.

(b) If $R = k[X_1, \ldots, X_d]$ is a polynomial ring and $\mathbf{f} \in [|R|]$, then the Taylor resolution $T^R(\mathbf{f})$ is a DG algebra resolution of $R/\langle \mathbf{f}\rangle$.

Goals. (1) There exists a DG algebra resolution of $\overline{R}$ over R.
(2) If $R = k[X_1, \ldots, X_d]$, then $\overline{R}$ has a bounded DG algebra resolution over R.

Strategies. (1) Start with $K = K^R(\mathbf{f})$ where $I = \langle \mathbf{f}\rangle$. Then K is a DG algebra and $H_0(K) = R/I$, but this is not generally a resolution, because usually $H_1(K) \neq 0$. Then we will reduce the homology degree-by-degree in a manner similar to the algorithm kernel-surject-kernel-surject-....
(2) For the special case when $R = k[X_1, \ldots, X_d]$, Hilbert's Syzygy Theorem implies we can truncate to get a bounded resolution. We then need to show that this truncation is a DG algebra.

Discussion III.B.3. For strategy (1), how does one reduce homology? Start with any DG algebra A (some approximation of a resolution of $\overline{R}$). Let $z \in Z_i(A)$ such that $0 \neq \overline{z} \in H_i(A)$. Build a new DG algebra $A[y]$ such that A is a DG subalgebra of $A[y]$ and y is a variable and

$$H_j(A[y]) = \begin{cases} H_j(A) & \forall j < i \\ \dfrac{H_i(A)}{S} & j = i, \end{cases}$$

where in the latter case $S \subseteq H_i(A)$ is a submodule such that $\overline{z} \in S$. So the homology of $A[y]$ in degree less than i is the same as the homology of A, but the homology of $A[y]$ in degree i requires one fewer generator (assuming $\overline{z}$ is a generator of $H_i(A)$). Then start with $A = K^R(\mathbf{f})$ where $I = \langle \mathbf{f}\rangle$. Let $z_{1,1}, \ldots, z_{1,m} \in Z_1(A)$ such that $H_1(A) = \langle \overline{z_{1,1}}, \ldots, \overline{z_{1,m}}\rangle$. Construct $A^{(1)} = A[y_{1,1}, \ldots, y_{1,m}]$ such that $H_0(A^{(1)}) = \overline{R}$ and

$$H_1(A^{(1)}) = H_1(A)/\langle \overline{z_{1,1}}, \ldots, \overline{z_{1,m}}\rangle = 0.$$

We then repeat this process for $H_\ell(A^{(1)})$ for $\ell \geq 2$ in order to construct a DG algebra resolution of $\overline{R}$. Most of the remainder of this chapter is devoted to filling in the details of this argument.

Definition III.B.4. Let A be a DG algebra over R and let $z \in Z_i(A)$ such that $i \geq 0$ is even. Let y be a symbol and define the degree of y to be $|y| = i+1$ (odd). Define $A[y]$ as follows. The R-modules in the resolution are given by

$$A[y]_n = A_n \oplus A_{n-(i+1)}y = \left\{\alpha + ay \;\middle|\; \alpha \in A_n \text{ and } a \in A_{n-(i+1)}\right\}$$

(so $A[y]_n$ is an R module) and the differentials in the resolution are given by

$$\partial_n^{A[y]}(\alpha + ay) = \partial_n^A(\alpha) + \partial_{n-(i+1)}^A(a)y + (-1)^{|a|}az,$$

so they are R-linear, satisfy the Leibniz rule for elements of the form ay with $\partial_{i+1}^{A[y]}(y) = z$. We define multiplication as follows. Since $|y|$ is odd (i.e,, $|y| \equiv 1 \pmod 2$), we set $y^2 = 0$ and

$$yb = (-1)^{|y||b|}by = (-1)^{|b|}by.$$

We also set

$$\begin{aligned}(\alpha + ay)(\beta + by) &= \alpha\beta + \alpha by + ay\beta + \underbrace{ayby}_{=0\ \because\ y^2=0} \\ &= \alpha\beta + (\alpha b + (-1)^{|\beta|} a\beta)y.\end{aligned}$$

These definitions merit reality checks. For the differential, since $\alpha + ay \in A[y]_n$, we know $\alpha \in A_n$ and $a \in A_{n-(i+1)}$. Also note

$$\partial_n^A(\alpha) \in A_{n-1} \subseteq A_{n-1} \oplus A_{n-1-(i+1)}y = A[y]_{n-1}$$

and

$$\underbrace{\partial^A_{n-(i+1)}(a)}_{\in A_{n-(i+1)-1}} \cdot \underbrace{y}_{\in A[y]_{i+1}} \overset{\therefore}{\in} A[y]_{n-1}.$$

Since $a \in A_{n-(i+1)}$ and $z \in A_i$, we also have $az \in A_{n-1} \subseteq A[y]_{n-1}$, so the differential lands as we would like. Similarly, multiplication defined $A[y]_n \times A[y]_m \to A[y]_{m+n}$ lands well also.

Theorem III.B.5. *Using the notation of Definition III.B.4 we have the following.*

(a) *$A[y]$ is a DG algebra.*
(b) *If A_n is free for all n, then $A[y]_n$ is free for all n.*
(c) *$A[y]_n = A_n$ and $\partial_n^{A[y]} = \partial_n^A$ for all $n \leq i$ and therefore $H_n(A[y]) = H_n(A)$ for all $n < i$.*
(d) *$A \subseteq A[y]$ is a DG subalgebra.*
(e) *$H_i(A[y]) \cong H_i(A)/S$ where $S \subseteq H_i(A)$ is a submodule such that $\overline{z} \in S$.*
(f) *If $w \in Z_i(A)$, then $w \in Z_i(A[y])$, i.e., $Z_i(A) \subseteq Z_i(A[y])$.*

Proof. (a) This part is tedious, but routine. For instance, $A[y]$ is a complex since using the Leibniz rule in A we see that

$$\begin{aligned}\partial(\partial(\alpha + ay)) &= \partial\left(\partial(\alpha) + \partial(a)y + (-1)^{|a|}az\right) \\ &= \underbrace{\partial(\partial(\alpha))}_{=0} + \partial(\partial(a)y) + (-1)^{|a|}\partial(az) \\ &= \underbrace{\partial(\partial(a))}_{=0} y + (-1)^{|a|-1}\partial(a)\underbrace{\partial(y)}_{=z} + (-1)^{|a|}[\partial(a)z + (-1)^{|a|}a\underbrace{\partial(z)}_{=0}] \\ &= 0.\end{aligned}$$

As another for instance, one considers

$$1_{A[y]} = 1_A + 0_A y \in A[y]_0$$

and checks that this is the multiplicative identity.

(b) This holds since the direct sum of free modules is free.

(c) If $n \leq i$, then

$$n - (i+1) = n - i - 1 < n - i \leq 0$$

and therefore

$$A[y]_n = A_n \oplus \underbrace{A_{n-(i+1)}y}_{=0} = A_n \oplus 0 = A_n.$$

Hence we have $\partial(\alpha + 0y) = \partial(\alpha)$ and therefore $\partial_n^{A[y]} = \partial_n^A$ (for these n). Thus we have

$$\begin{tikzcd} A[y] = & \cdots \arrow[r] & A[y]_{i+1} \arrow[r] & A[y]_i \arrow[r, "\partial_i^{A[y]}", "\partial_i^A"'] \arrow[d, equal] & A[y]_{i-1} \arrow[r, "\partial_{i-1}^{A[y]}", "\partial_{i-1}^A"'] \arrow[d, equal] & \cdots \\ & & & A_i & A_{i-1} & \end{tikzcd}$$

and therefore $H_n(A[y]) = H_n(A)$ for $n < i$.

(d) $\partial(\alpha + 0y) = \partial(\alpha)$ and $(\alpha + 0y)(\beta + 0y) = \alpha\beta$ etc.

(e) Since the inclusion $A \subseteq A[y]$ is a chain map by part (d), there exists a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\subseteq"] & A[y] \arrow[r, "\pi"] & A[y]/A \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & & & y \arrow[dd, bend left, mapsto] & & \\
0 \arrow[r] & A_{i+1} \arrow[r] \arrow[d] & A_{i+1} \oplus A_0 y \arrow[rr] \arrow[d] & & A_0 y \arrow[r] \arrow[d] & 0 \\
 & & & z & & \\
0 \arrow[r] & A_i \arrow[r] \arrow[d] & A_i \arrow[rr] \arrow[d] & & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r] \arrow[d] & A_{i-1} \arrow[rr] \arrow[d] & & 0 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & & \vdots &
\end{tikzcd}$$

Then the long exact sequence in homology yields

$$\begin{tikzcd}
\cdots \arrow[r] & H_{i+1}\left(A[y]/A\right) \arrow[r, "\eth"] \arrow[d, equal] & H_i(A) \arrow[r] & H_i(A[y]) \arrow[r] \arrow[d, phantom, "\wr\Vert"] & 0. \\
 & \dfrac{A_0 y}{\operatorname{Im}(\partial)} \ni \overline{y} & & \dfrac{H_i(A)}{\operatorname{Im}(\eth)} &
\end{tikzcd}$$

We set $S = \operatorname{Im}(\eth)$ and a diagram chase shows that $\overline{z} = \eth(\overline{y}) \in S$ by definition of $\eth$.

(f) We simply observe that

$$\partial^{A[y]}(w) = \partial^{A[y]}(w + 0y) = \partial^{A}(w) = 0.$$

$\square$

Note III.B.6. Use the notation from Definition III.B.4.

(a) First, observe that $\Sigma^i A \xrightarrow{z} A$ is a chain map. Then as an R-complex, we get

$$A[y] \cong \operatorname{Cone}(\Sigma^i A \xrightarrow{z} A).$$

However, $A[y]$ has an extra DG algebra structure that the mapping cone does not convey.

(b) Second, we make two observations about the Leibniz rule. First, use graded commutativity on the Leibniz rule to rewrite the second term as follows:

$$\begin{aligned}
\partial(ab) &= \partial(a)b + (-1)^{|a|} a\partial(b) \\
&= \partial(a)b + (-1)^{|a|+|a||\partial(b)|} \partial(b)a \\
&= \partial(a)b + (-1)^{|a||b|} \partial(b)a
\end{aligned}$$

Then we generalize the Leibniz rule to a product of m terms inductively as follows:

$$\begin{aligned}
\partial(a_1 \cdots a_m) &= \sum_{j=1}^{m} (-1)^{\sum_{t=1}^{j-1} |a_t|} a_1 \cdots a_{j-1} \partial(a_j) a_{j+1} \cdots a_m \\
&= \sum_{j=1}^{m} (-1)^{\left(\sum_{t=1}^{j-1} |a_t|\right)|a_j|} \partial(a_j) a_1 \cdots a_{j-1} a_{j+1} \cdots a_m.
\end{aligned}$$

Now we extend our definition for multiple elements of $Z_i(A)$ where i is fixed and even.

Definition III.B.7. Let A be a DG algebra and $z_1, \ldots, z_m \in Z_i(A)$ such that $i \geq 0$ is even. Define

$$A[\mathbf{y}] = A[y_1, \ldots, y_m] = A[y_1, \ldots, y_{m-1}][y_m]$$

and $\partial(y_j) = z_j$ for all $j = 1, \ldots, m$. This is inductively well-defined because $z_m \in Z_i(A) \subseteq Z_i(A[y_1, \ldots, y_{m-1}])$ by Theorem III.B.5(f).

Theorem III.B.8. *Using the notation of Definition III.B.7 we have the following.*

(a) *$A[\mathbf{y}]$ is a DG algebra.*
(b) *If A_n is free for all n, then $A[\mathbf{y}]_n$ is free for all n.*
(c) *$A[\mathbf{y}]_n = A_n$ and $\partial_n^{A[\mathbf{y}]} = \partial_n^A$ for all $n \leq i$ and therefore $H_n(A[\mathbf{y}]) = H_n(A)$ for all $n < i$.*
(d) *$A \subseteq A[\mathbf{y}]$ is a DG subalgebra.*
(e) *$H_i(A[\mathbf{y}]) \cong H_i(A)/S$ where $S \subseteq H_i(A)$ is a submodule such that $\overline{z}_1, \ldots, \overline{z}_m \in S$. In particular, if $H_i(A) = \langle \overline{z}_1, \ldots, \overline{z}_m \rangle$, then $H_i(A[\mathbf{y}]) = 0$.*
(f) *If $w \in Z_i(A)$, then $w \in Z_i(A[\mathbf{y}])$, i.e., $Z_i(A) \subseteq Z_i(A[\mathbf{y}])$.*

Proof. Induct on m. □

Note III.B.9. Use the notation from Definition III.B.7. The elements in $A[\mathbf{y}]$ are finite sums of terms of the form $ay_{p_1} \cdots y_{p_\ell}$, where $a \in A$ and $1 \leq p_1 < \cdots < p_\ell \leq m$. Also,

$$|ay_{p_1} \cdots y_{p_\ell}| = |a| + |y_{p_1}| + \cdots + |y_{p_\ell}| = |a| + \ell(i+1).$$

and by Note III.B.6(b) the differential applied to such terms yields

$$\begin{aligned}
\partial(ay_{p_1} \cdots y_{p_\ell}) &= \partial(a) y_{p_1} \cdots y_{p_\ell} + (-1)^{|a|} a \partial(y_{p_1} \cdots y_{p_\ell}) \\
&= \partial(a) y_{p_1} \cdots y_{p_\ell} + (-1)^{|a|} a \sum_{j=1}^{\ell} (-1)^{|y_{p_j}| \sum_{t=1}^{j-1} |y_{p_t}|} \partial(y_{p_j}) y_{p_1} \cdots y_{p_{j-1}} y_{p_{j+1}} \cdots y_{p_\ell} \\
&= \partial(a) y_{p_1} \cdots y_{p_\ell} + (-1)^{|a|} a \sum_{j=1}^{\ell} (-1)^{j-1} \partial(y_{p_j}) y_{p_1} \cdots y_{p_{j-1}} y_{p_{j+1}} \cdots y_{p_\ell}
\end{aligned}$$

where the last line comes about because $|y_{p_t}| = i+1$ is odd, i.e., $|y_{p_t}| \equiv 1 \pmod 2$. Notice that these operations are similar to those on the Koszul complex. Furthermore, multiplication of two such terms is given by

$$(ay_{p_1} \cdots y_{p_\ell})(by_{q_1} \cdots y_{q_k}) = \begin{cases} 0 & \text{if } p_r = q_s \text{ for some } r, s \\ (-1)^{|b|\ell} \sigma(p_1, \ldots, p_\ell, q_1, \ldots, q_k) ab y_{t_1} \cdots y_{t_{\ell+k}} & \text{otherwise,} \end{cases}$$

where $t_1 < \cdots < t_{\ell+k}$ and $\{t_1, \ldots, t_{\ell+k}\} = \{p_1, \ldots, p_\ell, q_1, \ldots, q_k\}$.

Now we move to the case where i is odd.

Definition III.B.10. Let A be a DG algebra and $z \in Z_i(A)$ such that $i > 0$ is odd. Let y be a symbol so that $|y| = i+1$ is even. Define $A[y]$ so that the R-modules are as follows:

$$A[y]_n = A_n \oplus A_{n-(i+1)} y \oplus A_{n-2(i+1)} y^2 \oplus \cdots.$$

This sum is finite because $n - j(i+1) < 0$ for $j \gg 0$, so $A_{n-j(i+1)} = 0$ for all $j \gg 0$. Furthermore, elements in $A[y]_n$ look like $a_0 + a_1 y + a_2 y^2 + \cdots$ for $a_j \in A_{n-j(i+1)}$. Let $\partial(y) = z$, then

$$\partial(y^2) = \partial(y \cdot y) = \partial(y) y + \underbrace{(-1)^{|y|}}_{=1} y \partial(y) = 2\partial(y) y = 2zy.$$

Inductively, we have $\partial(y^j) = jzy^{j-1}$ for $j \geq 0$. Notice that this looks like the power rule for derivatives. Then we can define the differential on $A[y]_n$ as

$$\partial\left(\sum_j a_j y^j\right) = \sum_j \left(\partial(a_j) y^j + (-1)^{|a_j|} a_j \cdot jzy^{j-1}\right).$$

Multiplication is similar to the multiplication in Definition III.B.4, but now there are more terms. We define multiplication as

$$\left(\sum_j a_j y^j\right)\left(\sum_k b_k y^k\right) = \sum_{j,k} a_j b_k y^{j+k},$$

where we can swap the order of y^j and b_k because $\left|y^j\right| = j\,|y|$ is even.

Theorem III.B.11. *Using the notation of Definition III.B.10 we have the following.*

(a) *$A[y]$ is a DG algebra.*
(b) *If A_n is free for all n, then $A[y]_n$ is free for all n.*
(c) *$A[y]_n = A_n$ and $\partial_n^{A[y]} = \partial_n^A$ for all $n \leq i$ and therefore $H_n(A[y]) = H_n(A)$ for all $n < i$.*
(d) *$A \subseteq A[y]$ is a DG subalgebra.*
(e) *$H_i(A[y]) \cong H_i(A)/S$ where $S \subseteq H_i(A)$ is a submodule such that $\overline{z} \in S$.*
(f) *If $w \in Z_i(A)$, then $w \in Z_i(A[y])$, i.e., $Z_i(A) \subseteq Z_i(A[y])$.*

Proof. Argue as in the proof of Theorem III.B.5. □

Definition III.B.12. Let A be a DG algebra and $z_1, \ldots, z_m \in Z_i(A)$ such that $i > 0$ is odd. Define

$$A[\mathbf{y}] = A[y_1, \ldots, y_m] = A[y_1, \ldots, y_{m-1}][y_m]$$

and $\partial(y_j) = z_j$ for all $j = 1, \ldots, m$. This is inductively well-defined because $z_m \in Z_i(A) \subseteq Z_i(A[y_1, \ldots, y_{m-1}])$ by Theorem III.B.11(f).

Theorem III.B.13. *Using the notation of Definition III.B.12 we have the following.*

(a) *$A[\mathbf{y}]$ is a DG algebra.*
(b) *If A_n is free for all n, then $A[\mathbf{y}]_n$ is free for all n.*
(c) *$A[\mathbf{y}]_n = A_n$ and $\partial_n^{A[\mathbf{y}]} = \partial_n^A$ for all $n \leq i$ and therefore $H_n(A[\mathbf{y}]) = H_n(A)$ for all $n < i$.*
(d) *$A \subseteq A[\mathbf{y}]$ is a DG subalgebra.*
(e) *$H_i(A[\mathbf{y}]) \cong H_i(A)/S$ where $S \subseteq H_i(A)$ is a submodule such that $\overline{z}_1, \ldots, \overline{z}_m \in S$. In particular, if $H_i(A) = \langle \overline{z}_1, \ldots, \overline{z}_m \rangle$, then $H_i(A[\mathbf{y}]) = 0$.*
(f) *If $w \in Z_i(A)$, then $w \in Z_i(A[\mathbf{y}])$, i.e., $Z_i(A) \subseteq Z_i(A[\mathbf{y}])$.*

Proof. Induct on m. □

We can now prove our first goal of the section.

Theorem III.B.14. *There exists a DG algebra resolution of $\overline{R}$ over R.*

Proof. Recall that $I = \overline{\mathbf{f}}$ and $\overline{R} = R/I$. We construct an ascending chain of DG algebras that approximate the desired resolution. Let

$$A^{(0)} = K^R(\mathbf{f}).$$

Because of our noetherian assumption, there exist $z_{1,1}, \ldots, z_{1,m_1} \in Z_1(A^{(0)})$ so that $H_1(A^{(0)}) = \langle \overline{z}_{1,1}, \ldots, \overline{z}_{1,m_1} \rangle$. Then define

$$A^{(1)} = A^{(0)}[y_{1,1}, \ldots, y_{1,m_1}],$$

where $\partial(y_{1,j}) = z_{1,j}$ for all $j = 1, \ldots, m_1$ and $|y_{1,j}| = 2$. We must have $H_0(A^{(1)}) = \overline{R}$ and $H_1(A^{(1)}) = 0$ by Theorem III.B.13(e). Furthermore, the $A_j^{(1)}$ are finitely generated and free for all j. Therefore, there exist $z_{2,1}, \ldots, z_{2,m_2} \in Z_2(A^{(1)})$ so that $H_2(A^{(1)}) = \langle \overline{z}_{2,1}, \ldots, \overline{z}_{2,m_2} \rangle$. Then define

$$A^{(2)} = A^{(1)}[y_{2,1}, \ldots, y_{2,m_2}],$$

where $\partial(y_{2,j}) = z_{2,j}$ for all $j = 1, \ldots, m_2$ and $|y_{2,j}| = 3$. Continuing in this fashion, we find that $A^{(h)}$ for $h \geq 1$ is a free DG algebra that satisfies $H_0(A^{(h)}) = \overline{R}$ and $H_j(A^{(h)}) = 0$ for all $j = 1, \ldots, h$ and $A_j^{(h)} = A_j^{(h-1)}$ for all $j \leq h-1$. Also, we have an ascending chain of DG algebras

$$A^{(0)} \subseteq A^{(1)} \subseteq A^{(2)} \subseteq \cdots \subseteq A^{(h)} \subseteq \cdots.$$

We claim that $A = \bigcup_{h=0}^{\infty} A^{(h)}$ is well-defined and is a DG algebra resolution of $\overline{R}$ over R. For all $j \geq 0$, we have

$$A_j^{(0)} \subseteq \cdots \subseteq A_j^{(j)} = A_j^{(j+1)} = \cdots,$$

so $A_j = A_j^{(j)}$ is the stable value in the ascending chain. To check the differential, consider the following commutative diagram:

$$\begin{tikzcd}
A_j^{(0)} \arrow[d, "\partial_j^{A^{(0)}}"] & \subseteq & \cdots & \subseteq & A_j^{(j)} \arrow[d, "\partial_j^{A^{(j)}}"] & = & A_j^{(j+1)} \arrow[d, "\partial_j^{A^{(j+1)}}"] & = & A_j^{(j+2)} \arrow[d, "\partial_j^{A^{(j+2)}}"] & = & \cdots \\
A_{j-1}^{(0)} & \subseteq & \cdots & = & A_{j-1}^{(j)} & = & A_{j-1}^{(j+1)} & = & A_{j-1}^{(j+2)} & = & \cdots .
\end{tikzcd}$$

Then $\partial_j^A = \partial_j^{A^{(j)}}$ is the differential of the stable value in the ascending chain. Furthermore, A_j is free and finitely generated because $A_j^{(j)}$ is free and finitely generated. To check multiplication, let $a \in A_j$ and $b \in A_k$ and $\ell \geq j, k$. Then $a = A_j = A_j^{(j)} = A_j^{(\ell)}$ and $b = A_k = A_k^{(k)} = A_k^{(\ell)}$, so the multiplication ab makes sense in $A^{(\ell)}$. Furthermore, this is independent of the choice of ℓ because $A^{(1)} \subseteq A^{(2)} \subseteq \cdots$ are subalgebras. The axioms for a DG algebra are inherited from $A^{(h)}$, so A is a DG algebra. Finally,

$$H_j(A) = H_j(A^{(j+1)}) = H_j(A^{(j)}) = 0$$

for all $j \geq 1$, so A is a DG algebra resolution. $\square$

THEOREM III.B.15. *Let $R = k[X_1, \ldots, X_d]$ be a polynomial ring over a field, let $I \leq R$ be an ideal, and set $\overline{R} = R/I$. Then there exists a DG algebra resolution $\overline{A}$ of $\overline{R}$ over R such that $\overline{A}_i = 0$ for all $i > d$.*

PROOF. Hilbert's Syzygy Theorem implies $\overline{R}$ has a free resolution F over R such that $F_i = 0$ for all $i > d$. Theorem III.B.14 implies there exists a DG algebra resolution A of $\overline{R}$ over R. Then Schanuel's Lemma implies that

$$B_{d-1}^A = Z_{d-1}^A = \operatorname{Ker}\left(\partial_{d-1}^A\right)$$

is projective. A result of Serre implies $\operatorname{Ker}\left(\partial_{d-1}^A\right)$ is free. The sequence

$$\overline{A} = \qquad 0 \longrightarrow B_{d-1}^A \longrightarrow A_{d-1} \longrightarrow \cdots \longrightarrow A_0 \longrightarrow 0$$

is exact in all degrees except in degree 0 and consists of free modules. Therefore it is a free resolution of $\overline{R}$ over R and thus we need only show $\overline{A}$ has a DG algebra structure.

We observe

$$B_{d-1}^A = \operatorname{Im}\left(\partial_d^A\right) \cong \frac{A_d}{\operatorname{Ker}\left(\partial_d^A\right)}$$

and produce the following commutative diagram.

$$\begin{tikzcd}
\mathcal{I} = \arrow[d, hook] & \cdots \arrow[r, "\partial_{d+3}^A"] & A_{d+2} \arrow[r, "\partial_{d+2}^A"] \arrow[d, "\mathrm{id}"'] & A_{d+1} \arrow[r, "\partial_{d+1}^A"] \arrow[d, "\mathrm{id}"'] & \operatorname{Ker}\left(\partial_d^A\right) \arrow[r] \arrow[d, "\subseteq"'] & 0 & \\
A = & \cdots \arrow[r, "\partial_{d+3}^A"'] & A_{d+2} \arrow[r, "\partial_{d+2}^A"'] & A_{d+1} \arrow[r, "\partial_{d+1}^A"'] & A_d \arrow[r, "\partial_d^A"'] & A_{d-1} \arrow[r] & \cdots
\end{tikzcd}$$

$\mathcal{I} \subseteq A$ is a subcomplex such that $\overline{A} \cong A/\mathcal{I}$. As in the proof of Theorem III.A.6, it suffices to show that $\mathcal{I}$ is a DG ideal of A, i.e., is a subcomplex that absorbs multiplication by elements of A, i.e., it suffices to show that for all $a \in A$ and for all $x \in \mathcal{I}$ we have $ax \in \mathcal{I}$.

Assume without loss of generality that x is nonzero. Then $|x| \geq d$. If $|ax| > d$, then we are done since in this case $ax \in A_{|ax|} = \mathcal{I}_{|ax|}$. Therefore again without loss of generality assume that $|ax| = d$, i.e., $|x| = d$ and $|a| = 0$. This implies $x \in Z_d^A$ and $a \in R$, which then implies $ax \in Z_d^A = \mathcal{I}_d$. $\square$

Note if $\Delta = \operatorname{depth}(\overline{R})$, then Theorem II.C.16 implies $p := \operatorname{pd}_R(\overline{R}) = d - \Delta$. The same proof as for Theorem III.B.15 yields a DG algebra resolution A' of $\overline{R}$ such that $A'_i = 0$ for all $i > d - \Delta$.

CHAPTER III.C

Applications

The Tor Algebra

Assume R is a noetherian commutative ring with identity and $I, J \leq R$ are ideals. Recall that for R-modules M and N, if we let P be a projective resolution of M over R, then we have

$$\operatorname{Tor}_i^R(M,N) = H_i(P \otimes_R N).$$

If M is finitely generated then P is of the form

$$P = \qquad \cdots \xrightarrow{\partial_3} R^{\beta_2} \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \longrightarrow 0$$

and we tensor to obtain

$$P \otimes_R N = \qquad \cdots \xrightarrow{\partial_3} N^{\beta_2} \xrightarrow{\partial_2} N^{\beta_1} \xrightarrow{\partial_1} N^{\beta_0} \longrightarrow 0.$$

We use the same matrices in P and $P \otimes_R N$ for the differential.

Remark III.C.1. Properties of Tor to be proved in the spring. We have

$$\operatorname{Tor}_i^R(M,N) = H_i(P \otimes_R N) \cong H_i(P \otimes_R Q) \cong H_i(M \otimes_R Q)$$

where Q is a projective resolution of N. (This says that Tor is "balanced".) The differential on $P \otimes_R Q$ is defined as

$$\begin{array}{ccccc} (P \otimes_R Q)_n & = & \bigoplus_{i+j=n}(P_i \otimes_R Q_j) & \supseteq & P_i \otimes_R Q_j \ni p \otimes q \\ \Big\downarrow{\scriptstyle \partial_n^{P\otimes Q}} & & & & \\ (P \otimes_R Q)_{n-1} & = & \bigoplus_{k+1=n-1}(P_k \otimes_R Q_\ell) & \supseteq & (P_{i-1} \otimes_R Q_j) \oplus (P_i \otimes_R Q_{j-1}) \end{array}$$

where

$$\partial(p \otimes q) := \partial(p) \otimes q + (-1)^{|p|} p \otimes \partial(q).$$

One has to check that $P \otimes_R Q$ is an R-complex. We consider augmentations $P \xrightarrow[\simeq]{\tau} M$ and $Q \xrightarrow[\simeq]{\pi} N$. Then we have

$$P \otimes N \xleftarrow[P\otimes\pi]{\simeq} P \otimes_R Q \xrightarrow[\simeq]{\tau\otimes Q} M \otimes_R Q.$$

(We use $\simeq$ to denote that the induced map on homology is an isomorphism, i.e., the map is a quasiisomorphism.)

Theorem III.C.2. *Let A be a DG algebra resolution of R/I over R.*

(a) $A' := A \otimes_R (R/J)$ *is a DG algebra.*
(b) $H(A') = \oplus_n H_n(A') = \oplus_n \operatorname{Tor}_n^R(R/I, R/J)$ *is a graded commutative ring, "the Tor algebra".*

Note III.C.3. This shows that $\operatorname{Tor}_n^R(R/I, R/J)$ is not a random list of modules. They fit together with a strong structure, and so there are restrictions on what these modules can look like.

Example III.C.4. Let $R = k[X_1, \ldots, X_d]$ be a polynomial ring over a field and set $I = \langle \underline{X} \rangle$. The DG algebra resolution of $R/I \cong k$ is the Koszul complex $A = K^R(\underline{X})$ and thus

$$A' = K^R(\underline{X}) \otimes_R (R/J) \cong K^{R/J}(\underline{x})$$

where $x_i = \overline{X_i} \in R/J$, and the isomorphism is described in the following diagram.

$$\begin{tikzcd}
e_\Lambda \otimes \overline{1} \arrow[d, mapsto] \arrow[rrrr, mapsto, bend left=10] & \in & R^{\binom{d}{i}} \otimes_R (R/J) \arrow[d, "\partial_i^{K^R(\underline{X})} \otimes (R/J)"'] & (R/J)^{\binom{d}{i}} \arrow[l, "\Phi_i"', "\cong"] \arrow[d, "\partial_i^{K^{R/J}(\underline{x})}"] \arrow[dl, phantom, "\curvearrowright"] & \ni & \overline{e_\Lambda} \arrow[d, mapsto] \\
\sum_{\lambda\in\Lambda} x_\lambda e_{\Lambda\setminus\{\lambda\}} \otimes \overline{1} \arrow[rrrrr, mapsto, bend right=10] & \in & R^{\binom{d}{i-1}} \otimes_R (R/J) & (R/J)^{\binom{d}{i-1}} \arrow[l, "\cong", "\Phi_{i-1}"'] & \ni & \sum_{\lambda\in\Lambda}(-1)^{\phi(\lambda,\Lambda)} x_\lambda \overline{e_{\Lambda\setminus\{\lambda\}}}
\end{tikzcd}$$

Then Φ is an isomorphism of DG algebras, i.e., it is an isomorphism of R-complexes that respects multiplication and multiplicative identities.

$$\Phi\left(1_{K^{R/J}(\underline{x})}\right) = \Phi\left(\overline{1}\right) = 1 \otimes \overline{1} = 1_{K^R(\underline{X})\otimes_R(R/J)}$$

$$\Phi(\overline{e_\Lambda e_\Gamma}) = \Phi(\overline{e_\Lambda})\Phi(\overline{e_\Gamma})$$

(Check this.) In particular, it follows in this case that

$$\mathrm{Tor}^R(R/(\underline{X}), R/J) \cong H\left(K^{R/J}(\underline{x})\right)$$

and the graded algebra structure on the Tor algebra is the same as the algebra structure on the homology of $K^{R/J}(\underline{x})$ induced by the exterior algebra structure on $K^{R/J}(\underline{x})$.

Proof of Theorem III.C.2. (a) We use an alternate description of A' as in Example III.C.4. Set $R' = R/J$, then

$$\begin{tikzcd}
a \otimes \overline{1} \arrow[d, mapsto] \arrow[rrrr, mapsto, bend left=10] & \in & (A')_i = A_i \otimes_R R' \arrow[r, "\cong"'] \arrow[d, "\partial_i^{A\otimes R'}"'] & A_i/JA_i \arrow[d] & \ni & \overline{a} \arrow[d, mapsto] \\
\partial(a) \otimes \overline{1} \arrow[rrrrr, mapsto, bend right=10] & \in & A_{i-1} \otimes_R R' \arrow[r, "\cong"] & A_{i-1}/JA_{i-1} & \ni & \overline{\partial(a)}
\end{tikzcd}$$

is an isomorphism of R-complexes. Multiplication on the right side of the diagram is defined by

$$\begin{aligned} \frac{A_i}{JA_i} \times \frac{A_j}{JA_j} &\longrightarrow \frac{A_{i+j}}{JA_{i+j}} \\ (\overline{a}, \overline{b}) &\longmapsto \overline{ab}. \end{aligned}$$

To show this is well defined, let $\overline{a} = \overline{a'} \in \frac{A_i}{JA_i}$ and $\overline{b} = \overline{b'} \in \frac{A_j}{JA_j}$. Then $a - a' \in JA_i$ and $b - b' \in JA_j$, so $ab - a'b' \in JA_{i+j}$ and thus $\overline{ab} = \overline{a'b'}$. Therefore $\overline{a}\overline{b} = \overline{ab}$, and it is straightforward to show that the DG axioms for $A' \cong A/JA$ are inherited from A.

(b) Since A' is a DG algebra, then

$$\mathrm{Tor}^R(R/I, R/J) \cong H(A')$$

is a graded commutative ring by Theorem III.A.6 $\square$

Avramov's Hammer

For this subsection, assume $R = k[X_1, \ldots, X_d]$ is a polynomial ring and $J \leq R$ is an ideal generated by homogeneous polynomials and $R' = R/J$.

Note III.C.5. We have a general strategy in commutative algebra for proving results.

(1) Prove the result for the case when R' is a finite dimensional vector space over k.
(2) If R' is Cohen-Macaulay, then there is a maximal weakly R-regular sequence $\mathbf{f} \in R'$ which satisfies $\dim_k(R'/\langle\mathbf{f}\rangle) < \infty$. By step (1), the result holds over $R'/\langle\mathbf{f}\rangle$. Furthermore, sometimes the weakly R-regular sequence guarantees that the result then holds over R'.
(3) If R' is not Cohen-Macaulay... ¯_(ツ)_/¯

Avramov's hammer is a tool to help us deal with (3) by producing a finite dimensional DG algebra U which behaves like $R'/\langle\mathbf{f}\rangle$. We prove the result over U, then use general machinery to deduce the result for R'. The downside of Avramov's hammer is that U is a DG algebra, so we need to track more data. The payoff, however, is that we can drop the Cohen-Macaulay assumption.

Another strategy we use is to prove a result for certain finite dimensional rings, then deduce the result for certain Cohen-Macaulay rings. With Avramov's hammer, we can prove a result for certain finite dimensional DG algebras, then deduce the result for certain non-Cohen-Macaulay rings.

Problem III.C.6. Let M be a finitely generated graded R'-module (i.e., a module having a free resolution with matrices of homogeneous polynomials). If $\operatorname{Tor}_i^{R'}(M,M)=0$ for all $i \gg 0$, must $\operatorname{pd}_{R'} M$ be finite (i.e., must M have a bounded free resolution over R')?

Note III.C.7. Note that $\operatorname{Tor}_i^{R'}(M,N)=0$ for all $i \gg 0$ implies neither $\operatorname{pd}_{R'} M<\infty$ nor $\operatorname{pd}_{R'} N<\infty$.

Definition III.C.8. Let A and B be DG algebras.

(a) A chain map $\Phi : A \to B$ is a morphism of DG algebras if it respects multiplication and multiplicative identities (i.e., $\Phi(1_A)=1_B$ and $\Phi(aa')=\Phi(a)\Phi(a')$ for all $a,a' \in A$).
(b) A quasiisomorphism of DG algebras is a morphism of DG algebras that induces isomorphisms on homology in all degrees.

Example III.C.9.
(a) Let A be a DG algebra resolution of R' over R and let

$$A^+ = \left(\cdots \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\tau} R' \longrightarrow 0 \right).$$

Then τ is a quasiisomorphism of DG algebras as in the following diagram:

$$\begin{tikzcd}
A = \arrow[d, "\tau", "\simeq"'] & \cdots \arrow[r, "\partial_3"] & A_2 \arrow[r, "\partial_2"] \arrow[d] & A_1 \arrow[r, "\partial_1"] \arrow[d] & A_0 \arrow[r] \arrow[d, "\tau"] & 0 \arrow[d] \\
R' = & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & R' \arrow[r] & 0.
\end{tikzcd}$$

(b) If $\mathbf{f} \in R'$, then the following diagram is a morphism of DG alrebras:

$$\begin{tikzcd}
R' = \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R' \arrow[r] \arrow[d, "1"] & 0 \arrow[d] \\
K^{R'}(\mathbf{f}) = & \cdots \arrow[r] & (R')^{\binom{n}{2}} \arrow[r] & (R')^n \arrow[r] & R' \arrow[r] & 0.
\end{tikzcd}$$

This is not a quasiisomorphism unless $\mathbf{f}=\emptyset$. More generally, if A is a DG R-algebra, then $R' \to A_0$ defined by $\overline{r} \mapsto \overline{r} \cdot 1_A$ induces a morphism of DG algebras $R' \to A$.
(c) Consider the following alternate description of $\operatorname{Tor}^R(R/I,R/J)$. Let A be a DG algebra resolution of R/I over R and let B be a DG algebra resolution of R/J over R. Then $A \otimes_R B$ is a DG algebra where

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|}(aa') \otimes (bb').$$

Furthermore,

$$\begin{tikzcd}
A \otimes_R B \arrow[d, "A\otimes\pi", "\simeq"'] \\
A \otimes_R (R/J)
\end{tikzcd}$$

is a quasiisomorphism of DG algebras, so $\operatorname{Tor}^R(R/I,R/J)=H(A \otimes_R B)$.

Construction III.C.10 (Avramov's Hammer). Let A be a bounded DG algebra resolution of $R'=R/J$ over $R=k[X_1,\ldots,X_d]$. Then Avramov's hammer is constructed via the following chain:

$$R' \xrightarrow[\substack{\text{DG algebra}\\ \text{morphism by}\\ \text{Example III.C.9(b)}}]{} K^{R'}(\mathbf{x}) \xrightarrow[\substack{\text{by}\\ \text{Example III.C.4}}]{\simeq} K^R(\mathbf{X}) \otimes_R R' \xleftarrow[\substack{\text{by}\\ \text{Example III.C.9(c)}}]{\simeq} K^R(\mathbf{X}) \otimes_R A \xrightarrow[\substack{\text{by}\\ \text{Example III.C.9(c)}}]{\simeq} k \otimes_R A = U,$$

where

$$
\begin{array}{ll}
A = & 0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_2} \longrightarrow R^{\beta_1} \longrightarrow R \longrightarrow 0, \\[2em]
U = k \otimes_R A = & 0 \longrightarrow k^{\beta_d} \longrightarrow \cdots \longrightarrow k^{\beta_2} \longrightarrow k^{\beta_1} \longrightarrow k \longrightarrow 0,
\end{array}
$$

and $\dim_k(U) = 1 + \beta_1 + \cdots + \beta_d$ is finite.

Definition III.C.11. Let A be a DG algebra over R. A DG A-module is an R-complex Y equipped with an R-bilinear multiplication $A_i \times Y_j \to Y_{i+j}$ denoted $(a, y) \mapsto ay$ that is unital, associative, and satisfies the Liebniz rule.

Example III.C.12.
(a) First, A is a DG A-module. Moreover, $\Sigma^n A$ is a DG A-module.
(b) Let $\Phi : A \to B$ be a morphism of DG algebras. Then:
- B is a DG A-module with $ab = \Phi(a)b$.
- If Y is a DG B-module, then Y is a DG A-module with $ay = \Phi(a)y$. This is a "restriction of scalars".
- There is a notion of tensor product over A such that if Z is a DG A-module, then $B \otimes_A Z$ is a DG B-module with $b(b' \otimes z) = (bb') \otimes z$.

Definition III.C.13. Let A be a DG algebra and let L and Y be DG A-modules such that $L_i = 0$ for all $i \ll 0$.

(a) A semi-basis for L is a subset $E \subseteq L$ such that every element of L can be written uniquely as a linear combination of elements of E with coefficients from A.
(b) We call L semi-free if it has a semi-basis.
(c) A semi-free resolution of Y is a quasiisomorphism $L \xrightarrow{\simeq} Y$ such that L is semi-free and the quasi-isomorphism respects scalar multiplication.

Example III.C.14. Let $K = K^{R'}(\mathbf{x})$ and let $R' \to K$ be a morphism of DG algebras. If M is an R'-module with free resolution F over R', then $K \otimes_R M$ is a DG K-module and $K \otimes_R F$ is a semi-free DG K-module. Furthermore, the augmentation $F \xrightarrow{\simeq} M$ induces a semi-free resolution over K

$$K \otimes_R F \xrightarrow{\simeq} K \otimes_R M.$$

Definition III.C.15. Let A be a DG algebra and let X and Y be DG A-modules and let $L \xrightarrow{\simeq} X$ be a semi-free resolution. Then

$$\operatorname{Tor}_i^A(X, Y) = H_i(L \otimes_A Y).$$

Discussion III.C.16. Let M be a finitely generated graded R'-module, and construct a semi-free resolution using Construction III.C.10:

$$R' \longrightarrow K^{R'}(\mathbf{x}) \quad \cong \quad K^R(\mathbf{X}) \otimes_R R' \xleftarrow{\simeq} \underbrace{K^R(\mathbf{X}) \otimes_R A}_{=B} \xrightarrow{\simeq} k \otimes_R A = U$$

$$M \rightsquigarrow \underbrace{K \otimes_R M}_{\text{DG } K\text{-module}} \overset{\simeq}{\leftsquigarrow} \underbrace{L}_{\substack{\text{semi-free}\\ \text{resolution}\\ \text{over B}}} \overset{\simeq}{\rightsquigarrow} \underbrace{U \otimes_B L.}_{\text{DG } U\text{-module}}$$

Notice here that $K \otimes_R M$ is a DG B-module by restriction of scalars and that the final complex is a semi-free DG U-module. Then we have

$$\begin{aligned}
\operatorname{Tor}^{R'}_{\gg 0}(M,M)=0 &\iff \operatorname{Tor}^{K}_{\gg 0}(L,L)=0\\
&\iff \operatorname{Tor}^{B}_{\gg 0}(L,L)=0\\
&\iff \operatorname{Tor}^{U}_{\gg 0}(U\otimes_B L, U\otimes_B L)=0\\
&\overset{?}{\Rightarrow} \operatorname{pd}_U(U\otimes_B L)<\infty\\
&\Rightarrow \operatorname{pd}_B(L)<\infty\\
&\Rightarrow \operatorname{pd}_K(L)<\infty\\
&\Rightarrow \operatorname{pd}_{R'}(M)<\infty,
\end{aligned}$$

where the last implication is where we use the graded assumption on M. The implication labelled with a question has not been proven. The point is that this argument reduces Problem III.C.6 to a similar question over a finite dimensional DG algebra where the problem might be easier to solve.