Complete Intersection Dimensions and Foxby Classes

Sean Sather-Wagstaff

North Dakota State University

DePaul University
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Ascent and descent of the Gorenstein property

Throughout this talk, $\varphi\colon R \to S$ and $\sigma\colon S \to T$ are local ring homomorphisms: $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ and $(T, \mathfrak{r})$ are local rings and $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$ and $\sigma(\mathfrak{n}) \subseteq \mathfrak{r}$.

Theorem. *Assume that φ is flat and set $\overline{S} = S/\mathfrak{m}S$. The ring S is Gorenstein if and only if R and $\overline{S}$ are Gorenstein. Moreover, there is an equality of Bass series $I_S^S(t) = I_R^R(t) I_{\overline{S}}^{\overline{S}}(t)$.*

What restrictions on φ are really needed for such a result?

Definition. A finitely generated R-module G is *totally reflexive* if
(1) $\operatorname{Ext}_R^{\geqslant 1}(G, R) = 0 = \operatorname{Ext}_R^{\geqslant 1}(\operatorname{Hom}_R(G, R), R)$, and
(2) $G \xrightarrow{\cong} \operatorname{Hom}_R(\operatorname{Hom}_R(G, R), R)$.

The *G-dimension* of a finitely generated R-module M is

$$\operatorname{G-dim}_R(M) = \inf \left\{ n \geqslant 0 \;\middle|\; \begin{array}{c} \exists\ 0 \to G_n \to \cdots \to G_0 \to M \to 0 \\ \text{exact, each } G_i \text{ totally reflexive} \end{array} \right\}$$

Ring homomorphisms of finite G-dimension

Definition. The *semi-completion* of φ, denoted $\grave{\varphi}$, is the composition $R \xrightarrow{\varphi} S \to \widehat{S}$.

A *Cohen factorization* of $\grave{\varphi}$ is a diagram of local ring homomorphisms $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$, where $\grave{\varphi} = \varphi'\dot{\varphi}$, with $\dot{\varphi}$ flat, the closed fibre $R'/\mathfrak{m}R'$ regular, R' complete, and φ' surjective.

Write $\mathrm{G}\text{-}\dim(\varphi) < \infty$ if $\mathrm{G}\text{-}\dim_{R'}(\widehat{S}) < \infty$ for some (equivalently, for every) Cohen factorization $R \to R' \to \widehat{S}$ of $\grave{\varphi}$.

Theorem. (Avramov-Foxby, 1997) *Assume* $\mathrm{G}\text{-}\dim(\varphi) < \infty$*.*

There is a formal Laurent series $I_\varphi(t)$ with nonnegative integer coefficients such that $I_S^S(t) = I_R^R(t)I_\varphi(t)$.

The ring S is Gorenstein if and only if R is Gorenstein and $I_\varphi(t)$ is a monomial.

Composition of homomorphisms of finite G-dimension

Question. If $\mathrm{G}\text{-}\dim(\varphi), \mathrm{G}\text{-}\dim(\sigma) < \infty$, then $\mathrm{G}\text{-}\dim(\sigma\varphi) < \infty$?

(Avramov-Foxby) If $\mathrm{fd}(\sigma) < \infty$, then yes.

Theorem. (SSW, 2007) *If* $\mathrm{G}\text{-}\dim(\varphi) < \infty$ *and* $\mathrm{CI}\text{-}\dim(\sigma) < \infty$, *then* $\mathrm{G}\text{-}\dim(\sigma\varphi) < \infty$.

Definition. A *quasi-deformation* of R is a diagram of local ring homomorphisms $R \xrightarrow{\varphi} R' \xleftarrow{\tau} Q$ such that φ is flat, and τ is surjective with kernel generated by a Q-regular sequence.

The *complete intersection dimension* of a finitely generated R-modules M is

$$\mathrm{CI}\text{-}\dim_R(M) := \inf \left\{ \begin{matrix} \mathrm{pd}_Q(R' \otimes_R M) \\ -\mathrm{pd}_Q(R') \end{matrix} \middle| \begin{matrix} R \to R' \leftarrow Q \text{ is a} \\ \text{quasi-deformation} \end{matrix} \right\}$$

and $\mathrm{CI}\text{-}\dim(\varphi) < \infty$ if $\mathrm{CI}\text{-}\dim_{R'}(\widehat{S}) < \infty$ for some Cohen factorization $R \to R' \to \widehat{S}$ of $\grave{\varphi}$.

Question. Is this independent of the Cohen factorization?

Foxby classes

Definition. A finitely generated R-module C is *semidualizing* if $\operatorname{Ext}_R^{\geqslant 1}(C,C)=0$ and $R \xrightarrow{\cong} \operatorname{Hom}_R(C,C)$.

Example. R^1 is a semidualizing R-module.

D is a dualizing R-module if and only if D is semidualizing and $\operatorname{id}_R(D)<\infty$.

Assume (1) $\mathrm{G}\text{-}\dim(\varphi)<\infty$ and S is Cohen-Macaulay, or (2) φ is flat and $S/\mathfrak{m}S$ is Cohen-Macaulay. Then a relative dualizing module $D^{\grave{\varphi}}$ is a semidualizing $\widehat{S}$-module and $I_\varphi(t)=P^{\widehat{S}}_{D^{\grave{\varphi}}}(t)$.

Definition. Let C be a semidualizing R-module. An R-module M is in the *Auslander class* $\mathrm{A}_C(R)$ if $M \xrightarrow{\cong} \operatorname{Hom}_R(C, C\otimes_R M)$ and $\operatorname{Tor}^R_{\geqslant 1}(C,M)=0=\operatorname{Ext}_R^{\geqslant 1}(C, C\otimes_R M)$.

Example. If $\operatorname{fd}_R(M)<\infty$, then $M\in \mathrm{A}_C(R)$.

If R is Cohen-Macaulay with a dualizing module D, then $\mathrm{G}\text{-}\dim_R(M)<\infty$ if and only if $M\in\mathrm{A}_D(R)$.

Foxby classes (cont.)

Definition. A homologically finite R-complex C is *semidualizing* if $R \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(C, C)$.

Example. Each semidualizing R-module is a semidualizing R-complex concentrated in degree 0.

D is a dualizing R-complex if and only if D is semidualizing and $\mathrm{id}_R(D) < \infty$.

If $\mathrm{G}\text{-}\dim(\varphi) < \infty$, then a relative dualizing complex $D^{\grave{\varphi}}$ is a semidualizing $\widehat{S}$-complex and $I_\varphi(t) = P^{\widehat{S}}_{D^{\grave{\varphi}}}(t)$.

Definition. The *Auslander class* with respect to C is the full subcategory $\mathrm{A}_C(R) \subseteq \mathrm{D_b}(R)$ consisting of the complexes M such that $C \otimes^{\mathbf{L}}_R M \in \mathrm{D_b}(R)$ and $M \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R M)$.

Example. If $\mathrm{fd}_R(M) < \infty$, then $M \in \mathrm{A}_C(R)$.

If R is has a dualizing complex D, then $\mathrm{G}\text{-}\dim_R(M) < \infty$ if and only if $M \in \mathrm{A}_D(R)$.

Theorem. (SSW, 2007) *If* $\operatorname{CI-dim}_R(M) < \infty$, *then* $M \in \mathsf{A}_C(R)$ *for each semidualizing* R*-complex* C.

Proof. As $\operatorname{CI-dim}_R(M) < \infty$, there exists a quasi-deformation $R \xrightarrow{\varphi} R' \xleftarrow{\tau} Q$ such that Q is complete and $\operatorname{pd}_Q(R' \otimes_R M) < \infty$.

φ is flat, so $R' \otimes_R C$ is a semidualizing R'-complex.

Q is complete, and τ is surjective with kernel generated by a Q-sequence. So $\operatorname{Ext}^{\geqslant 1}_{R'}(R' \otimes_R C, R' \otimes_R C) = 0$ implies that there is a semidualizing Q-complex B such that $R' \otimes_R C \simeq R' \otimes^{\mathbf{L}}_Q B$.

$\operatorname{pd}_Q(R' \otimes_R M) < \infty$ implies $R' \otimes_R M \in \mathsf{A}_B(Q)$.

This implies $R' \otimes_R M \in \mathsf{A}_{R' \otimes^{\mathbf{L}}_Q B}(R') = \mathsf{A}_{R' \otimes_R C}(R')$.

This implies $M \in \mathsf{A}_C(R)$. □

Structure of quasi-deformations

If $\text{CI-dim}_R(M) < \infty$, then there exists a quasi-deformation $R \to R' \leftarrow Q$ such that $\text{pd}_Q(R' \otimes_R M) < \infty$.

One may assume that $R'/\mathfrak{m}R'$ is artinian (so, Cohen-Macaulay) and that Q is complete.

Theorem.(SSW, 2007) *If* $\text{CI-dim}_R(M) < \infty$*, then there exists a quasi-deformation* $R \to R'' \leftarrow Q'$ *such that* Q' *is complete,* $R''/\mathfrak{m}R''$ *is artinian and Gorenstein, and* $\text{pd}_{Q'}(R'' \otimes_R M) < \infty$*.*

Proof. There is a quasi-deformation $R \xrightarrow{\varphi} R' \xleftarrow{\tau} Q$ where Q is complete, $R'/\mathfrak{m}R'$ is Cohen-Macaulay, and $\text{pd}_Q(R' \otimes_R M) < \infty$.

R' is complete and φ is flat with Cohen-Macaulay closed fibre, so φ has a relative dualizing module D^φ.

Q is complete and τ is surjective with kernel generated by a Q-sequence, so there is a semidualizing Q-module B such that $R' \otimes^{\mathbf{L}}_Q B \simeq D^\varphi$.

Use the trivial extensions $R'' = R' \ltimes D^\varphi$ and $Q' = Q \ltimes B$. □