Homotopy-theory techniques in commutative algebra

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Outline

Set-Up for the Talk

Throughout this talk, let R be a commutative ring with identity. Examples include:

- The rings of integers $\mathbb{Z}$ and p-adic integers $\mathbb{Z}_p$
- A field k like $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}/p\mathbb{Z}$
- Polynomial rings $A[x_1, \ldots, x_n]$ with coefficients in a commutative ring A with identity
- Rings of formal power series $A[\![x_1, \ldots, x_n]\!]$
- Quotient rings $A[x_1, \ldots, x_n]/I$ and $A[\![x_1, \ldots, x_n]\!]/J$

R-modules are objects that can be "acted upon" by the ring R. Modules are like vector spaces, but more interesting.

Examples of Modules

- If k is a field, then M is a k-module if and only if it is a k-vector space.
- M is a $\mathbb{Z}$-module if and only if it is an abelian group.
- If I is an ideal of R, then I is an R-module and so is the quotient R/I.

In a sense, modules unify the notions of "vector space," "abelian group," "ideal," and "quotient by an ideal."

- Consider the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_n]$. If H is a Hilbert space and $T_1, \ldots, T_n$ are pairwise commuting operators on H, then H is an R-module

$$f\xi = f(T_1, \ldots, T_n)\xi.$$

"Modules are like vector spaces, but more interesting."

Justification of "more interesting"

An R-module is free if it has a basis. If F is free and has a finite basis $e_1, \ldots, e_r$ then $F \cong R^r$ and r is the rank of F.

Most R-modules are not free.

Example. If $n \geq 2$, then the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ does not have a basis since $nb = 0$ for all $b \in \mathbb{Z}/n\mathbb{Z}$: linear independence fails.

More generally, if I is an ideal of R and $(0) \subsetneq I \subsetneq R$, then the R-module R/I does not have a basis.

In fact, R is a field if and only if every R-module has a basis.

This is a good thing! The complexity of the modules should mirror the complexity of the ring.

Finitely Generated Modules over a PID

Let R be a PID and M be a finitely generated R-module.

- M is finitely generated, so there is a surjection $\tau_0 \colon R^r \twoheadrightarrow M$.
- $\operatorname{Ker}(\tau_0)$ is a submodule of the free module R^r.
- Since R is a PID, there is an isomorphism $\operatorname{Ker}(\tau_0) \cong R^s$.
- Hence, there is an exact sequence

  $$0 \to R^s \xrightarrow{\iota} R^r \xrightarrow{\tau_0} M \to 0$$

  meaning that the kernel of each map equals the image of the preceding map:
  ι is injective, τ_0 is surjective, and $\operatorname{Ker}(\tau_0) = \operatorname{Im}(\iota)$.
- The map $\iota \colon R^s \to R^t$ is given by a matrix A. Linear algebra!

Resolutions: Use Linear Algebra to Study Modules

R is Noetherian and M is a finitely generated R-module.
For example, $R = A[x_1, \ldots, x_n]/I$ or $R = A[\![x_1, \ldots, x_n]\!]/J$ where $A = k$ or $A = \mathbb{Z}$ or $A = \mathbb{Z}_p$

- There exists a surjection $\tau_0 \colon R^{r_0} \twoheadrightarrow M$.
- R is Noetherian, so $\operatorname{Ker}(\tau_0)$ is finitely generated.
- If $\operatorname{Ker}(\tau_0)$ is free, then stop.
- If $\operatorname{Ker}(\tau_0)$ is not free, then repeat.
- There exists a surjection $\tau_1 \colon R^{r_1} \twoheadrightarrow \operatorname{Ker}(\tau_0)$.
- The composition $R^{r_1} \xrightarrow{\tau_1} \operatorname{Ker}(\tau_0) \to R^{r_0}$ is given by a matrix.
- Repeating the process yields a free resolution of M

$$\cdots \xrightarrow{A_3} R^{r_2} \xrightarrow{A_2} R^{r_1} \xrightarrow{A_1} R^{r_0} \xrightarrow{\tau_0} M \to 0$$

which is an exact sequence that may or may not terminate.

A Computation of a Resolution

Example. $R = A[x, y]$ and $I = (x, y)R$ and $M = R/I$.

- $\tau_0 \colon R \to R/I$ is the canonical surjection, and $\mathrm{Ker}(\tau_0) = I$.
- A surjection $\tau_1 \colon R^2 \to I$ is given by

$$\tau_1\left(\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]\right) = f \cdot x + g \cdot y.$$

- The relation $yx - xy = 0$ implies

$$\tau_1\left(\left[\begin{smallmatrix} y \\ -x \end{smallmatrix}\right]\right) = yx - xy = 0 \implies \left[\begin{smallmatrix} y \\ -x \end{smallmatrix}\right] \in \mathrm{Ker}(\tau_1).$$

- For all $f \in R$ we have $f\left[\begin{smallmatrix} y \\ -x \end{smallmatrix}\right] \in \mathrm{Ker}(\tau_1)$, and furthermore

$$\mathrm{Ker}(\tau_1) = R\left[\begin{smallmatrix} y \\ -x \end{smallmatrix}\right] \cong R.$$

- So $\mathrm{Ker}(\tau_1)$ is free! This gives a free resolution

$$0 \to R \xrightarrow{\left[\begin{smallmatrix} y \\ -x \end{smallmatrix}\right]} R^2 \xrightarrow{[x\ y]} R \xrightarrow{\tau_0} R/I \to 0$$

Example: Resolutions of R/I

$R = A[x_1, \ldots, x_n]$ or $R = A[\![x_1, \ldots, x_n]\!]$ and $I = (x_1, \ldots, x_n)R$.

$n = 1$:

$$0 \to R^1 \xrightarrow{[\, x_1 \,]} R^1 \xrightarrow{\tau_0} R/I \to 0$$

$n = 2$:

$$0 \to R^1 \xrightarrow{\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}} R^2 \xrightarrow{[\, x_1 \;\; x_2 \,]} R^1 \xrightarrow{\tau_0} R/I \to 0$$

$n = 3$:

$$0 \to R^1 \xrightarrow{\begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \\ 0 & -x_1 & -x_2 \end{bmatrix}} R^3 \xrightarrow{[\, x_1 \;\; x_2 \;\; x_3 \,]} R^1 \xrightarrow{\tau_0} R/I \to 0$$

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Binomial coefficients!

Exterior Powers

How the Binomial Coefficients Arise

Fix an integer $n \geq 1$ and consider the free module R^n.

- Let $e_1, \ldots, e_n \in R^n$ be the standard basis.
- For each integer d the dth exterior power of R^n is the free R-module $\wedge^d R^n \cong R^{\binom{n}{d}}$ with basis

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d} \mid 1 \leq i_1 < i_2 < \cdots < i_d \leq n\}.$$

$\wedge^d R^n$	$d = 4$	$d = 3$	$d = 2$	$d = 1$	$d = 0$	$d = -1$
$n = 1$			0	R^1	R^1	0
$n = 2$		0	R^1	R^2	R^1	0
$n = 3$	0	R^1	R^3	R^3	R^1	0
basis		$\{e_{i_1} \wedge e_{i_2} \wedge e_{i_3}\}$	$\{e_{i_1} \wedge e_{i_2}\}$	$\{e_i\}$	$\{1\}$	

Let $x_1, \ldots, x_n \in R$ and set $I = (x_1, \ldots, x_n)R$.

The Augmented Koszul Complex

Define homomorphisms $\partial_d^K\colon \wedge^d R^n \to \wedge^{d-1} R^n$ for $d = 1, \ldots, n$.

$$0 \longrightarrow \underset{\deg n}{R^{\binom{n}{n}}} \xrightarrow{\partial_n^K} \underset{\deg n-1}{R^{\binom{n}{n-1}}} \xrightarrow{\partial_{n-1}^K} \cdots \xrightarrow{\partial_2^K} \underset{\deg 1}{R^{\binom{n}{1}}} \xrightarrow{\partial_1^K} \underset{\deg 0}{R^{\binom{n}{0}}} \xrightarrow{\tau_0} \underset{\deg -1}{R/I} \longrightarrow 0$$

$$\partial_1^K(e_i) = x_i \qquad R^n \to R$$

$$\partial_2^K(e_{i_1} \wedge e_{i_2}) = x_{i_1} e_{i_2} - x_{i_2} e_{i_1} \qquad R^{\binom{n}{2}} \to R^n$$

$$\partial_d^K(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d}) = \sum_{j=1}^{d} (-1)^{j+1} x_{i_j} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_d}$$

- $\partial_{d-1}^K \circ \partial_d^K = 0$ for $d = 2, \ldots, n$ so this is a chain complex.
- $\operatorname{Ker}(\tau_0) = I = \operatorname{Im}(\partial_1^K)$
- If $R = A[x_1, \ldots, x_n]$, this is the free resolution of R/I.

The Wedge Product

The Koszul complex $K^R = K^R(x_1, \ldots, x_n)$ is the chain complex

$$0 \xrightarrow{\partial^K_{n+1}} R^{\binom{n}{n}} \xrightarrow{\partial^K_n} R^{\binom{n}{n-1}} \xrightarrow{\partial^K_{n-1}} \cdots \xrightarrow{\partial^K_2} R^{\binom{n}{1}} \xrightarrow{\partial^K_1} R^{\binom{n}{0}} \xrightarrow{\partial^K_0} 0$$

The wedge product provides a product on the Koszul complex.

$$(\wedge^d R^n) \times (\wedge^{d'} R^n) \to \wedge^{d+d'} R^n \qquad (v, w) \mapsto v \wedge w =: vw$$

This gives rise to elements of the form

$$(e_{i_1} \wedge \cdots \wedge e_{i_d})(e_{j_1} \wedge \cdots \wedge e_{j_{d'}}) = e_{i_1} \wedge \cdots \wedge e_{i_d} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{d'}}$$

which are not necessarily basis elements because the subscripts may not be in strictly ascending order.
Use the following two relations:

$$e_{i_1} \wedge \cdots \wedge e_{i_j} \wedge e_{i_{j+1}} \wedge \cdots \wedge e_{i_{d+d'}} = -e_{i_1} \wedge \cdots \wedge e_{i_{j+1}} \wedge e_{i_j} \wedge \cdots \wedge e_{i_{d+d'}}$$

$$e_{i_1} \wedge \cdots \wedge e_{i_j} \wedge e_{i_j} \wedge \cdots \wedge e_{i_{d+d'}} = 0$$

Differential Graded Algebras

The Homotopy-Theoretic Methods

Basic properties. For $v \in \wedge^d R^n$ and $w \in \wedge^{d'} R^n$

- $wv = (-1)^{dd'} vw$
- $v^2 = 0$ when d is odd
- Leibniz Rule: $\partial^K_{d+d'}(vw) = \partial^K_d(v)w - (-1)^d v\partial^K_{d'}(w)$

This gives the Koszul complex the structure of a differential graded commutative algebra or DG algebra for short.

- The ring R is a DG algebra concentrated in degree 0.
- The natural map $R \to K^R$ is a DG algebra homomorphism.

$$\begin{array}{ccccccccccccc}
R & & & & & & & 0 & \longrightarrow & R & \longrightarrow & 0 \\
\downarrow \iota & & & & & & & \downarrow & & \downarrow \cong & & \downarrow \\
K^R & & 0 & \longrightarrow & R^{\binom{n}{n}} & \longrightarrow & \cdots \longrightarrow & R^{\binom{n}{1}} & \longrightarrow & R^{\binom{n}{0}} & \longrightarrow & 0
\end{array}$$

Differential Graded Algebras (cont.)

Compatibility with Ring Homomorphisms

- A ring homomorphism $\varphi\colon R \to S$ induces a DG algebra homomorphism $K^R(x_1,\dots,x_n) \to K^S(\varphi(x_1),\dots,\varphi(x_n))$

$$\begin{array}{ccccccccccccc}
K^R & & 0 & \longrightarrow & R^{\binom{n}{n}} & \longrightarrow & \cdots & \longrightarrow & R^{\binom{n}{1}} & \longrightarrow & R^{\binom{n}{0}} & \longrightarrow & 0 \\
\Big\downarrow K^\varphi & & & & \Big\downarrow \varphi^{\binom{n}{n}} & & & & \Big\downarrow \varphi^{\binom{n}{1}} & & \Big\downarrow \varphi^{\binom{n}{0}} & & \\
K^S & & 0 & \longrightarrow & S^{\binom{n}{n}} & \longrightarrow & \cdots & \longrightarrow & S^{\binom{n}{1}} & \longrightarrow & S^{\binom{n}{0}} & \longrightarrow & 0
\end{array}$$

- The maps ι and K^φ make the following diagram commute.

$$\begin{array}{ccc}
R & \xrightarrow{\iota} & K^R \\
\varphi \Big\downarrow & & \Big\downarrow K^\varphi \\
S & \xrightarrow{\iota} & K^S
\end{array}$$

The Homothety Homomorphism

Let C be a chain complex over R.

- The *i*th homology module of C is the R-module $\mathrm{H}_i(C) = \mathrm{Ker}(\partial_i^C)/\operatorname{Im}(\partial_{i+1}^C)$.
- The endomorphism complex of C is denoted $\mathrm{End}(C)$.
- For each $r \in R$ there is a commutative diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial_{j+1}^C} & C_j & \xrightarrow{\partial_j^C} & C_{j-1} & \xrightarrow{\partial_{j-1}^C} & \cdots \\ & & \downarrow{r\cdot} & & \downarrow{r\cdot} & & \\ \cdots & \xrightarrow{\partial_{j+1}^C} & C_j & \xrightarrow{\partial_j^C} & C_{j-1} & \xrightarrow{\partial_{j-1}^C} & \cdots \end{array}$$

  The homothety $C \xrightarrow{r\cdot} C$ is in $\mathrm{Ker}\left(\partial_0^{\mathrm{End}(C)}\right) \subseteq \mathrm{End}(C)_0$.
- The map $R \to \mathrm{H}_0(\mathrm{End}(C))$ given by $r \mapsto (C \xrightarrow{r\cdot} C)$ is an R-module homomorphism.

Semidualizing Complexes

A chain complex C over R is semidualizing if

- each C_i is a free R-module of finite rank,
- $C_i = 0$ for each $i < 0$,
- $\mathrm{H}_i(C) = 0$ for $i \gg 0$,
- $R \cong \mathrm{H}_0(\mathrm{End}(C))$, and
- $\mathrm{H}_i(\mathrm{End}(C)) = 0$ for each $i \neq 0$.

Example. R is semidualizing.

Example. A dualizing complex is semidualizing.

Semidualizing complexes arise in the study of the homological algebra of ring homomorphisms, e.g., in the composition question for local ring homomorphisms of finite G-dimension.

Much of my recent research has been devoted to the analysis of the set of semidualizing complexes.

The Completion of a Local Ring

R is a local noetherian ring with maximal ideal $\mathfrak{m}$. For $r, s \in R$

$$\operatorname{ord}(r) = \sup\{n \geq 0 \mid r \in \mathfrak{m}^n\} \qquad \operatorname{dist}(r,s) = 2^{-\operatorname{ord}(r-s)}$$

The function $\operatorname{dist}(-,-)$ is a metric on R. The topological completion of R is denoted $\widehat{R}$.

$\widehat{R}$ is a noetherian local ring equipped with a canonical ring homomorphism $\varphi_R \colon R \to \widehat{R}$ and maximal ideal $\varphi_R(\mathfrak{m})\widehat{R}$.

Example. If $R = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}/(f_1, \ldots, f_m)$, then $\widehat{R} \cong k[\![x_1, \ldots, x_n]\!]/(f_1, \ldots, f_m)$.

If $\mathfrak{m} = (x_1, \ldots, x_n)R$, then the induced map on Koszul complexes $K^R \to K^{\widehat{R}}$ is a homology isomorphism.

Ascent of Semidualizing Complexes

A semidualizing complex over R has the following shape.

$$C = \qquad \cdots \xrightarrow{\partial_3^C} R^{\beta_2} \xrightarrow{\partial_2^C} R^{\beta_1} \xrightarrow{\partial_1^C} R^{\beta_0} \to 0$$

The completion of C is a semidualizing complex over $\widehat{R}$.

$$\widehat{C} = C \otimes_R \widehat{R} = \qquad \cdots \xrightarrow{\partial_3^C} \widehat{R}^{\beta_2} \xrightarrow{\partial_2^C} \widehat{R}^{\beta_1} \xrightarrow{\partial_1^C} \widehat{R}^{\beta_0} \to 0$$

Problem. Provide conditions on R such that every semi-dualizing complex over $\widehat{R}$ is isomorphic to one of the form $\widehat{C}$.

R has the approximation property when, for every finite system of polynomial equations $S = \{f_i(X_1, \ldots, X_N) = 0\}_{i=1}^{t}$, if S has a solution in $\widehat{R}$, then it has a solution in R.

A Descent Theorem for Semidualizing Complexes

Theorem. (L.W.Christensen-SSW, 2006) *If R has the approximation property, then every semidualizing complex over $\widehat{R}$ is isomorphic to $\widehat{C}$ for some semidualizing complex C over R.*

Sketch of Proof. Let B be a semidualizing complex over $\widehat{R}$.
Let $x_1, \ldots, x_n \in R$ be a minimal generating set for $\mathfrak{m}$.
Set $K^R = K^R(x_1, \ldots, x_n)$ and $K^{\widehat{R}} = K^{\widehat{R}}(\varphi_R(x_1), \ldots, \varphi_R(x_n))$.
The map $\varphi_R \colon R \to \widehat{R}$ provides a commutative diagram

$$
\begin{array}{ccccccc}
C & \longrightarrow & & & \widehat{C} & \cong & B \\
\downarrow & R & \longrightarrow & \widehat{R} & \downarrow & & \downarrow \\
\downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\
\downarrow & K^R & \longrightarrow & K^{\widehat{R}} & \widehat{C} \otimes_{\widehat{R}} K^{\widehat{R}} & \cong & \downarrow \\
A & \longrightarrow & & & A \otimes_{K^R} K^{\widehat{R}} & \cong & B \otimes_{\widehat{R}} K^{\widehat{R}}
\end{array}
$$

QED

Summary

- Free resolutions allow us to use linear algebra to study modules that are not free.
- The DG algebra structure on the Koszul complex allows us to solve certain problems about commutative rings by leaving the realm of rings.
- Outlook
  - **Question.** Is the set of isomorphism classes of semidualizing complexes over R a finite set?
  - The proof of a special case suggests that one needs to build a deformation theory for DG algebras.

Tak!

The Augmented Koszul Complex May Not Be Exact.

Example. Set $R = k[X_1, X_2]/(X_1X_2)$ and let $x_i = \overline{X_i}$ for $i = 1, 2$. The relation $x_1x_2 = 0$ in R makes the augmented Koszul complex non-exact in degree 1.

$$0 \to R^1 \xrightarrow{\partial_2^K = \left[\begin{smallmatrix} x_2 \\ -x_1 \end{smallmatrix}\right]} R^2 \xrightarrow{\partial_1^K = [\, x_1 \;\; x_2 \,]} R^1 \xrightarrow{\tau_0} R/I \to 0$$

The vectors $\left[\begin{smallmatrix} x_2 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ x_1 \end{smallmatrix}\right]$ are in $\mathrm{Ker}(\partial_1^K)$ because

$$[\, x_1 \;\; x_2 \,]\left[\begin{smallmatrix} x_2 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \quad \text{and} \quad [\, x_1 \;\; x_2 \,]\left[\begin{smallmatrix} 0 \\ x_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$$

However, $\left[\begin{smallmatrix} x_2 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ x_1 \end{smallmatrix}\right] \notin \mathrm{Im}(\partial_2^K)$.

Again, this is good! The complexity of the Koszul complex should mirror the complexity of the sequence $x_1, \ldots, x_n$.

The Endomorphism Complex

Let C be a chain complex over R.
The endomorphism complex of C is the chain complex $\mathrm{End}(C)$:

modules $$\mathrm{End}(C)_i = \prod_{j\in\mathbb{Z}} \mathrm{Hom}_R(C_j, C_{i+j}) \ni (\psi_j)_{j\in\mathbb{Z}}$$

elements
$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\partial^C_{j+1}} & C_j & \xrightarrow{\partial^C_j} & C_{j-1} & \xrightarrow{\partial^C_{j-1}} & \cdots \\
 & & \downarrow \psi_j & & \downarrow \psi_{j-1} & & \\
\cdots & \xrightarrow{\partial^C_{j+i+1}} & C_{j+i} & \xrightarrow{\partial^C_{j+i}} & C_{j+i-1} & \xrightarrow{\partial^C_{j+i-1}} & \cdots
\end{array}$$

differentials $$\partial_i^{\mathrm{End}(C)} \colon \prod_{j\in\mathbb{Z}} \mathrm{Hom}_R(C_j, C_{i+j}) \to \prod_{j\in\mathbb{Z}} \mathrm{Hom}_R(C_j, C_{i+j-1})$$

$$\partial_i^{\mathrm{End}(C)}((\psi_j)_{j\in\mathbb{Z}}) = (\partial^C_{i+j} \circ \psi_j - (-1)^i \psi_{j-1} \circ \partial^C_j)_{j\in\mathbb{Z}}$$