Foundations of Module Theory

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Algebraic Methods in Topology

Day 3 Preface

Take-Home Points (from Days 1 and 2)

1. Rings and modules have many applications.
2. Modules give a unified way to study vector spaces, abelian groups, and other constructions.

1. Understand a ring by understanding its modules.
2. This is like the Sylow Theorems, but for rings.
3. This is representation theory for rings.
4. Elementary rings have only elementary modules and conversely.

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Take-Home Points (from Day 3)

1. In general, modules are not nice like they are over a principal ideal domain.
2. There are other niceness conditions one can see via modules.

Day 3 Outline

1. Niceness
2. Presentations of Modules
3. Exact Sequences
4. Hilbert's Syzygy Theorem
5. Conclusion

3.1. Niceness

Assumption

Let $R \neq 0$ be a ring, i.e., a commutative ring with identity.
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Answer

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Examples

1. R is a field iff every R-module is free.
2. R is a principal ideal domain iff every submodule of every free module is free.

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It measures how far M is from being free.

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This is one way to prove the Fundamental Theorem for Finitely Generated Modules over a principal ideal domain.

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Exact sequences provide a technology for tracking the process.

They have many other application.

3.3. Exact Sequences, cont.

Definition (Let $L \xrightarrow{f} M \xrightarrow{g} N$ be R-module homomorphisms.)

The given sequence is **exact** if $\operatorname{Im}(f) = \operatorname{Ker}(g)$.

3.3. Exact Sequences, cont.

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Example

1. For $r \in R$ we have exact sequences

$$\langle r \rangle \xrightarrow{i} R \xrightarrow{\tau} R/\langle r \rangle \qquad \text{and} \qquad R \xrightarrow{r\cdot} R \xrightarrow{\tau} R/\langle r \rangle$$

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2. For any ideal $I \subset R$, we have an exact sequence

$$I \xrightarrow{i} R \xrightarrow{\tau} R/I \qquad \operatorname{Im}(i) = I = \operatorname{Ker}(\tau)$$

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Proposition

1. *A sequence $A \xrightarrow{\alpha} B \to 0$ is exact iff α is onto.*

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Proof.

(1) $\mathrm{Ker}(B \to 0) = B$ and α is onto iff $\mathrm{Im}(\alpha) = B$.

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(1) $\mathrm{Ker}(B \to 0) = B$ and α is onto iff $\mathrm{Im}(\alpha) = B$.
(2) $\mathrm{Im}(0 \to A) = 0$ and α is 1-1 iff $\mathrm{Ker}(\alpha) = 0$. □

3.3. Exact Sequences, cont.

Definition

A longer sequence $\cdots \xrightarrow{f_{i+2}} M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \xrightarrow{f_{i-1}} \cdots$ is exact if $\operatorname{Im}(f_{j+1}) = \operatorname{Ker}(f_j)$ for all j,

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By the previous proposition. □

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Example (Let $\phi\colon M \to N$ be an R-module homomorphism.)

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3. We have exact sequences (see next proposition)

$$\begin{array}{ccccccccc} 0 \to \operatorname{Ker}(\phi) & \xrightarrow{i} & M & & \xrightarrow{\phi} & & N & \xrightarrow{\tau} & N/\operatorname{Im}(\phi) \to 0 \\ & & & \searrow^{\phi'} & & \nearrow_{j} & & & \\ & & & & \operatorname{Im}(\phi) & & & & \\ & & & \nearrow & & \searrow & & & \\ & & 0 & & & & 0 & & \end{array}$$

3.3. Exact Sequences, cont.

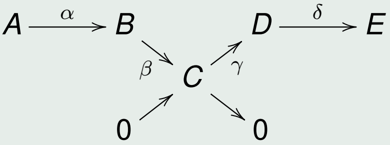
Proposition

Given exact sequences

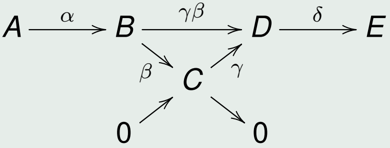
$$\begin{array}{ccccccccc}
A & \xrightarrow{\alpha} & B & & & & D & \xrightarrow{\delta} & E \\
& & & \overset{\beta}{\searrow} & & \overset{\gamma}{\nearrow} & & & \\
& & & & C & & & & \\
& & & \nearrow & & \searrow & & & \\
& & 0 & & & & 0 & &
\end{array}$$

Proposition

Given exact sequences



the following spliced sequence is also exact.



$\operatorname{Ker}(\gamma\beta) = \operatorname{Ker}(\beta)$ because γ is 1-1.
$= \operatorname{Im}(\alpha)$

$\operatorname{Im}(\gamma\beta) = \operatorname{Im}(\gamma) = \operatorname{Ker}(\delta)$ because β is onto

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Proof.

Diagram chase - if time. □

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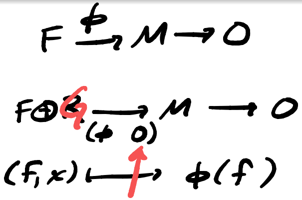

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Definition (Let M be an R-module.)

M is approximately free if there is an epimorphism $\tau\colon F \to M$ such that F and $\mathrm{Ker}(\tau)$ are both free.

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1. The $\mathbb{K}[X, Y]$-module $\mathbb{K}[X, Y]/\langle X, Y\rangle$ is not approx. free.

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2. Similarly for $\mathbb{Z}[X]/\langle 2, X\rangle$ over $\mathbb{Z}[X]$.

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Proposition (Let M be an R-module.)

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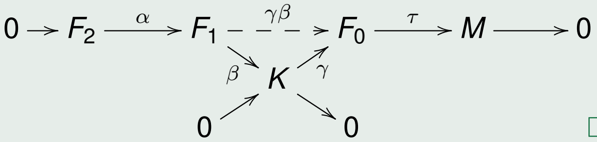
Proof. Slice and splice.



□

3.4. Hilbert's Syzygy Theorem, cont.

Definition (Let M be an R-module.)

M is approximately approximately free if there is an epimorphism $\tau\colon F \to M$ with F free and $\mathrm{Ker}(\tau)$ approximately free.

Proposition (Let M be an R-module.)

M is approximately approximately free iff there is an exact sequence $0 \to F_2 \to F_1 \to F_0 \to M \to 0$ with F_0, F_1, F_2 free.

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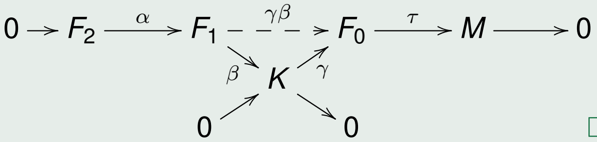

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Question

When is every R-module approximately approximately free?

3.4. Hilbert's Syzygy Theorem, cont.

Theorem (Hilbert, Quillen, Suslin)

1. *Every finitely generated* $\mathbb{K}[X, Y]$*-module is approximately approximately free.*

3.4. Hilbert's Syzygy Theorem, cont.

Theorem (Hilbert, Quillen, Suslin)

1. *Every finitely generated* $\mathbb{K}[X, Y]$*-module is approximately approximately free.*
2. *If* A *is a principal ideal domain, then every finitely generated* $A[X]$*-module is approximately approximately free.*

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1. *Every finitely generated* $\mathbb{K}[X, Y]$*-module is approximately approximately free.*
2. *If A is a principal ideal domain, then every finitely generated* $A[X]$*-module is approximately approximately free.*

In each case, can choose $0 \to R^{b_2} \to R^{b_1} \to R^{b_0} \to M \to 0$.

3.4. Hilbert's Syzygy Theorem, cont.

Theorem (Hilbert, Quillen, Suslin)

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Example

1. Over $R = \mathbb{K}[X, Y]$ the module $R/\langle X, Y\rangle$ is approximately approximately free.

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Theorem (Hilbert, Quillen, Suslin)

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Example

1. Over $R = \mathbb{K}[X, Y]$ the module $R/\langle X, Y\rangle$ is approximately approximately free.

$(x \quad y)\begin{pmatrix} -y \\ x \end{pmatrix} = (-xy + yx) = 0$

commutative!

$$0 \to R \xrightarrow{\begin{pmatrix} -Y \\ X \end{pmatrix}} R^2 \xrightarrow{(X \quad Y)} R \xrightarrow{\tau} R/\langle X, Y\rangle \to 0$$

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2. Similarly for $\mathbb{Z}[X]/\langle 2, X\rangle$ over $\mathbb{Z}[X]$.

Example

1. Over $R = \mathbb{K}[X, Y, Z]$ the module $R/\langle X, Y, Z \rangle$ is not approximately approximately free.

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Question

What can be said for these rings?

3.4. Hilbert's Syzygy Theorem, cont.

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Question

What can be said for these rings?

Theorem (Hilbert, Quillen, Suslin)

Let A be a principal ideal domain. For every finitely generated module M over $\mathbb{K}[X_1, \ldots, X_n]$ *or* $A[X_1, \ldots, X_{n-1}]$ *there is an exact sequence* $0 \to R^{b_n} \to \cdots \to R^{b_1} \to R^{b_0} \to M \to 0$.

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This conclusion fails for most other rings.

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This conclusion fails for most other rings.

Example

Let $R = A[X]/\langle X^2 \rangle \ni \overline{X} = x$.

$$\cdots R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{\tau} R/\langle x \rangle \to 0.$$

Similarly over $A[X, Y]/\langle XY \rangle$ and $\mathbb{Z}/\langle 4 \rangle$ etc.

Day 3 Outline

1. Niceness
2. Presentations of Modules
3. Exact Sequences
4. Hilbert's Syzygy Theorem
5. Conclusion

Take-Home Points

1. Rings and modules have many applications.
2. Modules give a unified way to study vector spaces, abelian groups, and other constructions.

1. Understand a ring by understanding its modules.
2. This is like the Sylow Theorems, but for rings.
3. This is representation theory for rings.
4. Elementary rings have only elementary modules and conversely.

1. In general, modules are not nice like they are over a principal ideal domain.
2. There are other niceness conditions one can see via modules.

Exercise (1)

Let $R = \mathbb{K}[X, Y]$. Prove that the following sequence is exact.

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Let $R = \mathbb{K}[X, Y]$. Prove that the following sequence is exact.

$r \mapsto \begin{pmatrix} -y \\ x \end{pmatrix} r = \begin{pmatrix} -yr \\ xr \end{pmatrix}$

$$0 \to R \xrightarrow{\begin{pmatrix} -Y \\ X \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} X & Y \end{pmatrix}} R \xrightarrow{\tau} R/\langle X, Y\rangle \to 0$$

$\psi \begin{pmatrix} a \\ b \end{pmatrix} \mapsto (x \ y)\begin{pmatrix} a \\ b \end{pmatrix} = xa + yb$

Exercise (2)

Let $R = A[X]/\langle X^2\rangle \ni \overline{X} = x$. Prove that the sequence is exact:

$$\cdots R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{\tau} R/\langle x\rangle \to 0.$$

$\bar{r} \mapsto x\bar{r}$

Exercise (3 - Hint: First Isomorphism Theorem $\times 2$)

1. Consider an exact sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$. Prove that $C \cong B/\operatorname{Im}(\alpha)$. $C = \operatorname{Im}\beta \cong B/\operatorname{Ker}\beta = B/\operatorname{Im}\alpha$
2. Consider an exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$. Prove that $A \cong \operatorname{Ker}(\beta)$. $\operatorname{Ker}\beta = \operatorname{Im}\alpha \cong A/\operatorname{Ker}\alpha = A/0 \cong A$

3.5. DCC. P. Vámos. The dual of the notion of "finitely generated". J. London Math. Soc. 43 (1968), 643–646.

Definition

An R-module M is finitely embedded if $E(M) \cong \bigoplus_{i=1}^{n} E(R/\mathfrak{m}_i)$ for (not necessarily distinct) maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \subseteq R$.

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Theorem (Vámos)

An R-module M is artinian (i.e., satisfies DCC on submodules) if and only if M/K is finitely embedded for all submodules K.

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2. *every finitely generated R-module is finitely embedded.*
3. *R/I is finitely embedded for all ideals I.*
4. *R is finitely embedded.*

dual of approximately free? approximately injective

$$0 \to M \to I \to J \to 0$$

$$0 \to M \to I^0 \to I^1 \to I^2 \to 0$$

$\mathbb{K}[X_1, \dots, X_n]$ or $A[x_1, \dots, x_{n-1}]$

$$0 \to M \to I^0 \to I^1 \to \cdots \to I^n \to 0$$

Auslander - Buchsbaum, Serre