

Foundations of Module Theory

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17–19 December 2020

Online CIMPA Pre-School El Salvador 2020

Algebraic Methods in Topology

Day 3 Preface

Take-Home Points (from Days 1 and 2)

- 1 Rings and modules have many applications.
- 2 Modules give a unified way to study vector spaces, abelian groups, and other constructions.
- 1 Understand a ring by understanding its modules.
- 2 This is like the Sylow Theorems, but for rings.
- 3 This is representation theory for rings.
- 4 Elementary rings have only elementary modules and conversely.

Take-Home Points (from Day 3)

- 1 In general, modules are not nice like they are over a principal ideal domain.
- 2 There are other niceness conditions one can see via modules.

Day 3 Outline

- 1 Niceness
- 2 Presentations of Modules
- 3 Exact Sequences
- 4 Hilbert's Syzygy Theorem
- 5 Conclusion

3.1. Niceness

Assumption

Let $R \neq 0$ be a ring, i.e., a commutative ring with identity.
Let \mathbb{K} be a field.

Question

When are all R -modules nice?

Answer

When R is particularly nice.

Examples

- 1 R is a field iff every R -module is free.
- 2 R is a principal ideal domain iff every submodule of every free module is free.

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- 2 **Presentations of Modules**
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3.2. Presentations of Modules

(Let M be an R -module.)

There is an R -module epimorphism $\tau: F \rightarrow M$ where F is free. Think of this as an approximation of M , or a model.

Given a basis B for F , we know exactly what every $v \in F$ looks like: it is uniquely written as $v = \sum_{b \in B}^{\text{finite}} r_b b$.

Then τ shows us what every element of M looks like:
$$m = \tau(v) = \tau\left(\sum_{b \in B}^{\text{finite}} r_b b\right) = \sum_{b \in B}^{\text{finite}} r_b \tau(b).$$

However, this is only an approximation of M because this representation will not generally be unique.

The error for this approximation is $\text{Ker}(\tau)$.

It measures how far M is from being free.

3.2. Presentations of Modules, cont.

(Let M be an R -module.)

Fix an R -module epimorphism $\tau: F \rightarrow M$ such that F is free.

Example (Assume that R is a field)

Then M is free, so τ can be chosen to be an isomorphism.

In this case, the approximation has no error, i.e., $\text{Ker}(\tau) = 0$.

Even if τ is not an isomorphism, $\text{Ker}(\tau) \subseteq F$ will be free, so the error can be understood.

Example (Assume that R is a principal ideal domain.)

Again $\text{Ker}(\tau)$ is free, so it can be understood.

This is one way to prove the Fundamental Theorem for Finitely Generated Modules over a principal ideal domain.

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3.3. Exact Sequences

(Let M be an R -module.)

Fix an R -module epimorphism $\tau: F \rightarrow M$ such that F is free.

If R is not a principal ideal domain, then the error $\text{Ker}(\tau)$ will usually not be free, so it will be harder to understand.

Since $\text{Ker}(\tau)$ is a module, we can approximate it.

Exact sequences provide a technology for tracking the process.

They have many other application.

3.3. Exact Sequences, cont.

Definition (Let $L \xrightarrow{f} M \xrightarrow{g} N$ be R -module homomorphisms.)

The given sequence is **exact** if $\text{Im}(f) = \text{Ker}(g)$.

Example

- ① For $r \in R$ we have exact sequences

$$\begin{array}{lcl} \langle r \rangle \xrightarrow{i} R \xrightarrow{\tau} R/\langle r \rangle & \text{and} & R \xrightarrow{r \cdot} R \xrightarrow{\tau} R/\langle r \rangle \\ \text{Im}(i) = \langle r \rangle = \text{Ker}(\tau) & \text{and} & \text{Im}(r \cdot) = \langle r \rangle = \text{Ker}(\tau) \end{array}$$

- ② For any ideal $I \subset R$, we have an exact sequence

$$I \xrightarrow{i} R \xrightarrow{\tau} R/I \quad \text{Im}(i) = I = \text{Ker}(\tau)$$

- ③ For any submodule $K \subset M$, we have an exact sequence

$$K \xrightarrow{i} M \xrightarrow{\tau} M/K \quad \text{Im}(i) = K = \text{Ker}(\tau)$$

3.3. Exact Sequences, cont.

Definition (Let $L \xrightarrow{f} M \xrightarrow{g} N$ be R -module homomorphisms.)

The given sequence is **exact** if $\text{Im}(f) = \text{Ker}(g)$.

Proposition

- 1 A sequence $A \xrightarrow{\alpha} B \rightarrow 0$ is exact iff α is onto.
- 2 A sequence $0 \rightarrow A \xrightarrow{\alpha} B$ is exact iff α is 1-1.

Proof.

- (1) $\text{Ker}(B \rightarrow 0) = B$ and α is onto iff $\text{Im}(\alpha) = B$.
- (2) $\text{Im}(0 \rightarrow A) = 0$ and α is 1-1 iff $\text{Ker}(\alpha) = 0$. □

3.3. Exact Sequences, cont.

Definition

A longer sequence $\dots \xrightarrow{f_{i+2}} M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \xrightarrow{f_{i-1}} \dots$ is **exact** if $\text{Im}(f_{j+1}) = \text{Ker}(f_j)$ for all j , i.e., iff the sequence $M_{j+1} \xrightarrow{f_{j+1}} M_j \xrightarrow{f_j} M_{j-1}$ is exact for all j .

Definition

A **short exact sequence** is an exact sequence of the form $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$.

Proposition

A sequence $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is exact iff f is 1-1, g is onto, and $\text{Im}(f) = \text{Ker}(g)$.

Proof.

By the previous proposition. □

3.3. Exact Sequences, cont.

Proposition

A sequence $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is exact iff f is 1-1, g is onto, and $\text{Im}(f) = \text{Ker}(g)$.

Example

- 1 For any ideal $I \subset R$, we have an exact sequence

$$0 \rightarrow I \xrightarrow{i} R \xrightarrow{\tau} R/I \rightarrow 0$$

- 2 For any submodule $K \subset M$, we have an exact sequence

$$0 \rightarrow K \xrightarrow{i} M \xrightarrow{\tau} M/K \rightarrow 0$$

3.3. Exact Sequences, cont.

Example (Let $\phi: M \rightarrow N$ be an R -module homomorphism.)

- 1 If ϕ is onto, then we have an exact sequence

$$0 \rightarrow \text{Ker}(\phi) \xrightarrow{i} M \xrightarrow{\phi} N \rightarrow 0$$

- 2 If ϕ is 1-1, then we have an exact sequence

$$0 \rightarrow M \xrightarrow{\phi} N \xrightarrow{\tau} N/\text{Im}(\phi) \rightarrow 0$$

- 3 We have exact sequences (see next proposition)

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Ker}(\phi) & \xrightarrow{i} & M & \xrightarrow{\phi} & N & \xrightarrow{\tau} & N/\text{Im}(\phi) & \rightarrow & 0 \\ & & & & \searrow & & \nearrow & & & & \\ & & & & \phi' & & j & & & & \\ & & & & & & \text{Im}(\phi) & & & & \\ & & & & \nearrow & & \searrow & & & & \\ & & & & 0 & & & & & & 0 \end{array}$$

3.3. Exact Sequences, cont.

Proposition

Given exact sequences

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & & D & \xrightarrow{\delta} & E \\ & & \searrow \beta & & \nearrow \gamma & & \\ & & C & & & & \\ & \nearrow 0 & & & \searrow 0 & & \end{array}$$

the following spliced sequence is also exact.

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\gamma\beta} & D & \xrightarrow{\delta} & E \\ & & \searrow \beta & & \nearrow \gamma & & \\ & & C & & & & \\ & \nearrow 0 & & & \searrow 0 & & \end{array}$$

Proof.

Diagram chase - if time.



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3.4. Hilbert's Syzygy Theorem

Definition (Let M be an R -module.)

M is **approximately free** if there is an epimorphism $\tau: F \rightarrow M$ such that F and $\text{Ker}(\tau)$ are both free.

Proposition (Let M be an R -module.)

- 1 M is approximately free iff there is an exact sequence $0 \rightarrow G \rightarrow F \rightarrow M \rightarrow 0$ such that F and G are free.
- 2 R noetherian and M finitely generated and approximately free $\implies F, G$ can be chosen finitely generated.
- 3 R is a principal ideal domain if and only if every (finitely generated) R -module is approximately free.

Example

- 1 The $\mathbb{K}[X, Y]$ -module $\mathbb{K}[X, Y]/\langle X, Y \rangle$ is not approx. free.
- 2 Similarly for $\mathbb{Z}[X]/\langle 2, X \rangle$ over $\mathbb{Z}[X]$.

3.4. Hilbert's Syzygy Theorem, cont.

Definition (Let M be an R -module.)

M is **approximately approximately free** if there is an epimorphism $\tau: F \rightarrow M$ with F free and $\text{Ker}(\tau)$ approximately free.

Proposition (Let M be an R -module.)

M is *approximately approximately free* iff there is an exact sequence $0 \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ with F_0, F_1, F_2 free.

Proof. Slice and splice.

$$\begin{array}{ccccccccc} 0 & \rightarrow & F_2 & \xrightarrow{\alpha} & F_1 & \xrightarrow{\gamma\beta} & F_0 & \xrightarrow{\tau} & M & \longrightarrow & 0 \\ & & & & & \searrow \beta & \nearrow \gamma & & & & \\ & & & & & & K & & & & \\ & & & & & \nearrow & \searrow & & & & \\ & & & & 0 & & & & 0 & & \square \end{array}$$

Question

When is every R -module approximately approximately free?

3.4. Hilbert's Syzygy Theorem, cont.

Theorem (Hilbert, Quillen, Suslin)

- 1 Every finitely generated $\mathbb{K}[X, Y]$ -module is approximately approximately free.
- 2 If A is a principal ideal domain, then every finitely generated $A[X]$ -module is approximately approximately free.

In each case, can choose $0 \rightarrow R^{b_2} \rightarrow R^{b_1} \rightarrow R^{b_0} \rightarrow M \rightarrow 0$.

Example

- 1 Over $R = \mathbb{K}[X, Y]$ the module $R/\langle X, Y \rangle$ is approximately approximately free.

$$0 \rightarrow R \xrightarrow{\begin{pmatrix} -Y \\ X \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} X & Y \end{pmatrix}} R \xrightarrow{\tau} R/\langle X, Y \rangle \rightarrow 0$$

- 2 Similarly for $\mathbb{Z}[X]/\langle 2, X \rangle$ over $\mathbb{Z}[X]$.

3.4. Hilbert's Syzygy Theorem, cont.

Example

- 1 Over $R = \mathbb{K}[X, Y, Z]$ the module $R/\langle X, Y, Z \rangle$ is not approximately approximately free.
- 2 Similarly for $\mathbb{Z}[X, Y]/\langle 2, X, Y \rangle$ over $\mathbb{Z}[X, Y]$.

Question

What can be said for these rings?

Theorem (Hilbert, Quillen, Suslin)

Let A be a principal ideal domain. For every finitely generated module M over $\mathbb{K}[X_1, \dots, X_n]$ or $A[X_1, \dots, X_{n-1}]$ there is an exact sequence $0 \rightarrow R^{b_n} \rightarrow \dots \rightarrow R^{b_1} \rightarrow R^{b_0} \rightarrow M \rightarrow 0$.

3.4. Hilbert's Syzygy Theorem, cont.

Theorem (Hilbert, Quillen, Suslin)

Let A be a principal ideal domain. For every finitely generated module M over $\mathbb{K}[X_1, \dots, X_n]$ or $A[X_1, \dots, X_{n-1}]$ there is an exact sequence $0 \rightarrow R^{b_n} \rightarrow \dots \rightarrow R^{b_1} \rightarrow R^{b_0} \rightarrow M \rightarrow 0$.

This conclusion fails for most other rings.

Example

Let $R = A[X]/\langle X^2 \rangle \ni \bar{X} = x$.

$$\dots R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{\tau} R/\langle x \rangle \rightarrow 0.$$

Similarly over $A[X, Y]/\langle XY \rangle$ and $\mathbb{Z}/\langle 4 \rangle$ etc.

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3.5. Conclusion

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- ① Understand a ring by understanding its modules.
 - ② This is like the Sylow Theorems, but for rings.
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-
- ① In general, modules are not nice like they are over a principal ideal domain.
 - ② There are other niceness conditions one can see via modules.

3.5. Exercises

Exercise (1)

Let $R = \mathbb{K}[X, Y]$. Prove that the following sequence is exact.

$$0 \rightarrow R \xrightarrow{\begin{pmatrix} -Y \\ X \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} X & Y \end{pmatrix}} R \xrightarrow{\tau} R/\langle X, Y \rangle \rightarrow 0$$

Exercise (2)

Let $R = A[X]/\langle X^2 \rangle \ni \bar{X} = x$. Prove that the sequence is exact:

$$\dots R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{\tau} R/\langle x \rangle \rightarrow 0.$$

Exercise (3 - Hint: First Isomorphism Theorem $\times 2$)

- 1 Consider an exact sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$. Prove that $C \cong B/\text{Im}(\alpha)$.
- 2 Consider an exact sequence $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C$. Prove that $A \cong \text{Ker}(\beta)$.

3.5. DCC. P. Vámos. The dual of the notion of “finitely generated”. J. London Math. Soc. 43 (1968), 643–646.

Definition

An R -module M is **finitely embedded** if $E(M) \cong \bigoplus_{i=1}^n E(R/\mathfrak{m}_i)$ for (not necessarily distinct) maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n \subseteq R$.

Theorem (Vámos)

An R -module M is artinian (i.e., satisfies DCC on submodules) if and only if M/K is finitely embedded for all submodules K .

Theorem (Vámos. The following conditions are equivalent.)

- 1 R is artinian (i.e., satisfies DCC on ideals).
- 2 every finitely generated R -module is finitely embedded.
- 3 **R/I is finitely embedded for all ideals I .**
- 4 R is finitely embedded.