Foundations of Module Theory

Sean K. Sather-Wagstaff (they/them/theirs)

Clemson University

17–19 December 2020
Online CIMPA Pre-School El Salvador 2020
Algebraic Methods in Topology

Day 2 Preface

Take-Home Points (from Day 1)

1. Rings and modules have many applications.
2. Modules give a unified way to study vector spaces, abelian groups, and other constructions.

Take-Home Points (from Day 2)

1. Understand a ring by understanding its modules.
2. This is like the Sylow Theorems, but for rings.
3. This is representation theory for rings.
4. Elementary rings have only elementary modules and conversely.

Day 2 Outline

1. Niceness
2. Free Modules
3. Submodules
4. Conclusion

2.1. Niceness

Assumption

Let $R \neq 0$ be a ring, i.e., a commutative ring with identity.
Let $\mathbb{K}$ be a field.

In general, modules can be complicated, not of a nice form.

Question

When are all R-modules nice?

Answer

When R is particularly nice.

Day 2 Outline

2.2. Free Modules - Definitions and Examples

Definition (Let M be an R-module)

A subset $B \subseteq M$ is a basis for M if every element of M can be written uniquely as a finite linear combination of elements of B.

span: For each $x \in M$, can write $x = \sum_{b \in B}^{\text{finite}} r_b b$ with $r_b \in R$.

LI: If $\sum_{b \in B}^{\text{finite}} r_b b = \sum_{b \in B}^{\text{finite}} s_b b$, then $r_b = s_b$ for all b.

M is free if it has a basis.

Examples

1. $R^n = \left\{ \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \mid r_1, \ldots, r_n \in R \right\}$ is free with basis $e_1, \ldots, e_n$.
2. $M \cong R^n$ if and only if M has a basis with n elements.
3. If $n \in \mathbb{Z}_{\geq 2}$, then $\mathbb{Z}/\langle n \rangle$ is not free over $\mathbb{Z}$.
4. If $I \subsetneq R$ is an ideal with $I \neq 0$, then R/I is not free over R.

2.2. Finitely Generated Modules

Definition (Let M be an R-module)

A subset $B \subseteq M$ is a generating set for M if for each $x \in M$, we can write $x = \sum_{b \in B}^{\text{finite}} r_b b$ with $r_b \in R$.
M is finitely generated if it has a finite generating sequence.

Examples

1. Basis $\Longrightarrow$ generating set, but the converse fails in general.
2. R^n is finitely generated.
3. If $I \subsetneq R$ is an ideal with $I \neq 0$, then R/I is finitely generated.

Proposition (Let M be an R-module.)

1. *M has a generating set, namely M itself.*
2. *There is a module epimorphism $F \to M$ such that F is free.*

Proposition (For an R-module M, the following are equivalent.)

1. *M is finitely generated*
2. *There is an R-module epimorphism $\tau\colon R^n \to M$ for some n.*
3. *There is a submodule $K \subseteq R^n$ for some n such that $M \cong R^n/K$.*

Proof.

(1) $\implies$ (2). If $x_1, \ldots, x_n \in M$ is a generating sequence, then the map $R^n \to M$ given by $\sum_i r_i e_i \mapsto \sum_i r_i x_i$ is a well-defined R-module empimorphism.

(2) $\implies$ (1). $\tau(e_1), \ldots, \tau(e_n)$ is a generating sequence for M.

(2) $\implies$ (3). First Isomorphism Theorem.

(3) $\implies$ (2). $R^n \to R^n/K \xrightarrow{\cong} M$. □

Theorem

The following conditions are equivalent:

1. *Every R-module is free.*
2. *Every finitely generated R-module is free.*
3. *R is a field.*

Proof.

(1) $\implies$ (2). Trivial.
(2) $\implies$ (3). By contrapositive: if R is not a field, then there is an ideal $I \subsetneq R$ such that $I \neq 0$. A previous example implies that R/I is a finitely generated R-module that is not free.
(3) $\implies$ (2). Standard from linear algebra. Show that a finite generating sequence (i.e., a finite spanning set) can be pruned to one that is a basis.
(3) $\implies$ (1). Non-trivial. Uses Zorn's Lemma. □

2.2. Take-Home Point

Theorem

The following conditions are equivalent:

1. *Every R-module is free.*
2. *Every finitely generated R-module is free.*
3. *R is a field.*

Take-Home Points

1. Understand a ring by understanding its modules.
2. Elementary rings have only elementary modules and conversely.

Day 2 Outline

Definition

A principal ideal domain (PID) is an integral domain such that every ideal is principal, i.e., every ideal is of the form $\langle x \rangle$.

Examples

1. $\mathbb{Z}$ and $\mathbb{K}[X]$ are principal ideal domains.
2. $\mathbb{Z}[X]$ and $\mathbb{K}[X, Y]$ are not principal ideal domains.

Fact (The following conditions are equivalent.)

1. R is a principal ideal domain.
2. Every submodule of R is free.
3. Every submodule of a free R-module is free.
4. For all $n \geq 1$, for every submodule $K \subseteq R^n$, there is an integer $k \leq n$ such that $K \cong R^k$.

2.3. Take-Home Point

Fact (The following conditions are equivalent.)

1. R is a principal ideal domain.
2. Every submodule of a free R-module is free.
3. R is an integral domain and every finitely generated R module is a direct sum of cyclic modules.

Take-Home Points

1. Understand a ring by understanding its modules.
2. Elementary rings have only elementary modules and conversely.

Example

1. In $\mathbb{K}[X, Y] \supseteq \langle X, Y \rangle$ is finitely generated, not a direct sum of cyclic modules; it is a submodule of R that is not free.
2. Similarly for $\mathbb{Z}[X] \supseteq \langle 2, X \rangle$.

2.3. Submodules of Finitely Generated Modules

Fact (The following conditions are equivalent.)

1. R is a principal ideal domain.
2. For all $n \geq 1$, for every submodule $K \subseteq R^n$, there is an integer $k \leq n$ such that $K \cong R^k$.

Corollary (Let R be a principal ideal domain)

For all $n \geq 1$, every submodule of R^n is finitely generated.

Question

What other rings satisfy the conclusion of the corollary?

If R satisfies the conclusion of the corollary, then in particular, every submodule of R is finitely generated ($n = 1$), i.e., every ideal of R is finitely generated.

2.3. Submodules of Finitely Generated Modules, cont.

Theorem (The following conditions are equivalent.)

1. *Every ideal of R is finitely generated.*
2. *For all $n \geq 1$, every submodule of R^n is finitely generated.*
3. *Every submodule of every finitely generated R-module is finitely generated.*

(Proof)

(3) $\implies$ (1). Follows as R is finitely generated as an R-module.
(1) $\implies$ (2). Careful induction on n.

2.3. Submodules of Finitely Generated Modules, cont.

Theorem (The following conditions are equivalent.)

1. *Every ideal of R is finitely generated.*
2. *For all $n \geq 1$, every submodule of R^n is finitely generated.*
3. *Every submodule of every finitely generated R-module is finitely generated.*

Proof.

(2) $\implies$ (3). Let M be a finitely generated R-module, and let $K \subseteq M$ be a submodule. Need to show K is finitely generated.

M finitely generated $\implies$ there is an epimorphism $\tau\colon R^n \to M$.

$\tau^{-1}(K) \subseteq R^n$ submodule, so (2) $\implies$ $\tau^{-1}(K)$ finitely generated.

τ surjective $\implies$ $K = \tau(\tau^{-1}(K))$.

So $\tau^{-1}(K)$ finitely generated $\implies$ K finitely generated. □

2.3. Noetherian Rings

Theorem (The following conditions are equivalent.)

1. *Every ideal of R is finitely generated.*
2. *For all $n \geq 1$, every submodule of R^n is finitely generated.*
3. *Every submodule of every finitely generated R-module is finitely generated.*

Definition

R is noetherian if every ideal of R is finitely generated.

Examples

1. field $\implies$ noetherian
2. principal ideal domain $\implies$ noetherian
3. R noetherian and $I \subseteq R$ an ideal $\implies$ R/I is noetherian.

2.3. Noetherian Rings, cont.

Emmy Noether, 1882–1935. One of the most influential mathematicians of the 20th century. She persisted despite incredible obstacles.

“Noether’s theorem, has been called ‘one of the most important mathematical theorems ever proved in guiding the development of modern physics’.”

Her work ”changed the face of [abstract] algebra.”

She “developed the theory of ideals in commutative rings.”

She “is credited with several lines of research published by other mathematicians, even in fields far removed from her main work, such as algebraic topology.”[1]

[1] https://en.wikipedia.org/wiki/Emmy_Noether

2.3. Noetherian Rings, cont.

Theorem (Hilbert Basis Theorem)

If R is noetherian, then $R[X]$ is noetherian.

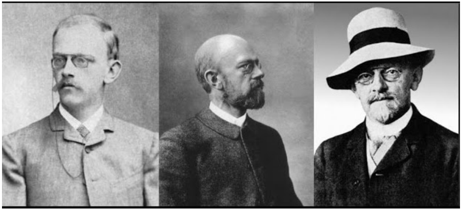
Corollary (Assume that R is noetherian.)

1. *Then $S = R[X_1, \ldots, X_d]$ is noetherian.*
2. *S/I is noetherian for every ideal $I \subseteq S$*

Corollary (Let $\mathbb{K}$ be a field, and let $V \subseteq \mathbb{K}^d$.)

Then $I(V) \subseteq \mathbb{K}[X_1, \ldots, X_d]$ is finitely generated.

David Hilbert, 1862–1943. One of the last universal mathematicians. His 23 problems steered much of the mathematical research of the 20th century.



Day 2 Outline

2.5. Conclusion

Take-Home Points

1. Understand a ring by understanding its modules.
2. This is like the Sylow Theorems, but for rings.
3. This is representation theory for rings.
4. Elementary rings have only elementary modules and conversely.

Next Lecture

1. In general, modules are not nice like they are over a principal ideal domain.
2. There are other niceness conditions one can see via modules.

Exercise (1)

1. Let F, M be R-modules such that F is free with basis B. For every $b \in B$, let $m_b \in M$ be given. Prove that there is a unique R-module homomorphism $\phi\colon F \to M$ such that $\phi(b) = m_b$ for all $b \in B$.
2. Let M be an R-module, and let $m_1, \ldots, m_n \in M$. Prove that there is a unique R-module homomorphism $\phi\colon R^n \to M$ such that $\phi(e_i) = m_i$ for all $i = 1, \ldots, n$.

Exercise (2)

Let $\mathbb{K}$ be a field, and let
$R = \mathbb{K}[X_1, X_2, X_3, \ldots] = \mathbb{K}[X_1] \cup \mathbb{K}[X_1, X_2] \cup \mathbb{K}[X_1, X_2, X_3] \cup \cdots$.
Prove that the ideal $I = \langle X_1, X_2, X_3, \ldots \rangle = \{\sum_{i=1}^{\text{finite}} f_i X_i \mid f_i \in R\}$ is not finitely generated. Conclude that R is not noetherian.