Ext-vanishing and ascent of module structures

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Motivation

Throughout this talk, $(R, \mathfrak{m}, k)$ is a local ring.

Theorem. (Buchweitz-Flenner, '06) *If F and K are R-modules such that F is flat and K is complete, then* $\operatorname{Ext}_R^{\geqslant 1}(F, K) = 0$.

Question. If $\operatorname{Ext}_R^{\geqslant 1}(F, K) = 0$ for each flat R-module F, must K be complete? What if we only assume that $\operatorname{Ext}_R^{\geqslant 1}(\widehat{R}, K) = 0$?

Example. Let R be a local noetherian domain that is not a field. The field of fractions K of R is an injective R-module, and so $\operatorname{Ext}_R^{\geqslant 1}(F, K) = 0$ for each R-module F. But K is not complete.

Theorem. (Buchweitz-Flenner, '06) *If* $\dim(R) \geq 1$*, then $R[\![X]\!]$ is not a projective R-module.*

Question. What conditions on R and $\mathfrak{a} \subset R$ guarantee that $\widehat{R}^{\mathfrak{a}}$ is not projective over R? Similarly for R^h or for a pointed étale neighborhood R'?

More Questions.

Let $\varphi\colon R \to S$ be a flat local ring homomorphism such that S has maximal ideal $\mathfrak{m}S$ and residue field k.

Shorthand. When the above conditions are satisfied, we write "$\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{m}S, k)$ is a flat local homomorphism".

Examples. The natural maps $R \to \widehat{R}^{\mathfrak{a}}$ and $R \to R^h$.
Any pointed étale neighborhood $R \to R'$.

Question. If M is a finitely generated R-module such that $\operatorname{Ext}_R^{\geqslant 1}(S, M) = 0$, must the R-module structure on M ascend along φ? that is, must M have an S-module structure that is compatible with its R-module structure via φ?

Question. If S is projective as an R-module via φ, must φ be bijective?

Main Theorem

Theorem. (AJF-SSW-RAW, '07) *Let* $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{m}S,k)$ *be a flat local homomorphism and* M *a finitely generated* R*-module. The following conditions are equivalent:*

(i) *The* R*-module structure on* M *ascends along* φ*.*

(ii) *The evaluation map* $\mathrm{Hom}_R(S,M) \to M$ *is bijective.*

(iii) $\mathrm{Ext}^i_R(S,M)$ *is finitely generated over* R *for each* $i \geqslant 1$*.*

(iv) $\mathrm{Ext}^{\geqslant 1}_R(S,M) = 0$*.*

Proof of (iii) $\Longrightarrow$ (ii) *and* (iii) $\Longrightarrow$ (iv)*.* Let K be the Koszul complex of R. Apply $\mathrm{Hom}_R(K,-)$ to the evaluation morphism $\mathbf{R}\mathrm{Hom}_R(S,M) \to M$ to yield an isomorphism in $\mathrm{D}(R)$

$$\mathrm{Hom}_R(K,\mathbf{R}\mathrm{Hom}_R(S,M)) \xrightarrow{\simeq} \mathrm{Hom}_R(K,M).$$

Because M and $\mathbf{R}\mathrm{Hom}_R(S,M)$ are homologically finite over R, this implies that the original morphism $\mathbf{R}\mathrm{Hom}_R(S,M) \to M$ is an isomorphism. □

Consequences of Main Theorem

Corollary. (AJF-SSW-RAW, '07) *Let* $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{m}S, k)$ *be a flat local homomorphism. The following conditions are equivalent:*

(i) φ *is bijective.*

(ii) S *is finitely generated as an* R*-module via* φ*.*

(iii) S *is projective as an* R*-module via* φ*.*

(iv) $\mathrm{Ext}^i_R(S, R)$ *is finitely generated over* R *for each* $i \geqslant 1$*.*

(v) φ *is part of a retract pair.*

Corollary. (AJF-SSW-RAW, '07)

(a) $\widehat{R}^{\mathfrak{a}}$ *is* R*-projective if and only if* R *is* $\mathfrak{a}$*-adically complete.*

(b) R^h *is* R*-projective if and only if* R *is henselian.*

(c) R *does not admit a pointed étale neighborhood* $R \to R'$ *such that* R' *is* R*-projective.*

Structural Implications of the Ascent Result

Theorem. (AJF-SSW-RAW, '07) *Let* $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{m}S, k)$ *be a flat local homomorphism and* M *a finitely generated* R*-module. The following conditions are equivalent:*

(i) *The* R*-module structure on* M *ascends along* φ*.*

(ii) *The map* $R/\mathfrak{p} \to S/\mathfrak{p}S$ *is bijective for each* $\mathfrak{p} \in \operatorname{Supp}_R(M)$*.*

(iii) *The map* $R/\mathfrak{p} \to S/\mathfrak{p}S$ *is bijective for each* $\mathfrak{p} \in \operatorname{Min}_R(M)$*.*

(iv) *The map* $R/\operatorname{Ann}_R(M) \to S/\operatorname{Ann}_R(M)S$ *is bijective.*

Corollary. (AJF-SSW-RAW, '07) *Let* $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{m}S, k)$ *be a flat local homomorphism and* M, N *finitely generated* R*-modules such that* $\operatorname{Supp}_R(N) \subseteq \operatorname{Supp}_R(M)$*. If the* R*-module structure on* M *ascends along* φ*, then the* R*-module structure on* N *ascends along* φ*.*

Comparing Module Structures over R and S

Proposition. (AJF-SSW-RAW, '07) *Let* $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{m}S, k)$ *be a flat local homomorphism and let* M, N *be* R*-modules such that* M *is finitely generated over* R*.*

(a) $\operatorname{Ext}^n_R(N, M) \cong \operatorname{Ext}^n_S(N, M)$ *for each* n*.*

(b) $\operatorname{Tor}^R_n(N, M) \cong \operatorname{Tor}^S_n(N, M)$ *for each* n*.*

(c) $\operatorname{H}^n_{\mathfrak{a}}(M) \cong \operatorname{H}^n_{\mathfrak{a}S}(M)$ *for each ideal* $\mathfrak{a} \subseteq R$ *and each* n*.*

(d) $\operatorname{Ann}_R(M)S = \operatorname{Ann}_S(M)$*.*

(e) $\operatorname{length}_R(M) = \operatorname{length}_S(M)$*.*

(f) $\dim_R(M) = \dim_S(M)$*.*

(g) $\operatorname{depth}_R(\mathfrak{a}, M) = \operatorname{depth}_S(\mathfrak{a}S, M)$ *for each ideal* $\mathfrak{a} \subseteq R$*.*

(h) $\beta^R_n(M) = \beta^S_n(M)$ *and* $\mu^n_R(M) = \mu^n_S(M)$ *for each* n*.*

(i) $\operatorname{pd}_R(M) = \operatorname{pd}_S(M)$ *and* $\operatorname{id}_R(M) = \operatorname{id}_S(M)$*.*

(j) $\text{G-dim}_R(M) = \text{G-dim}_S(M)$ *and* $\text{G-id}_R(M) = \text{G-id}_S(M)$

(k) $\text{CI-dim}_R(M) = \text{CI-dim}_S(M)$

(l) $\text{CM-dim}_R(M) = \text{CM-dim}_S(M)$

Necessity of Hypotheses on φ

Example. Let k be a perfect field of prime characteristic p. Set $S = R = k[\![X]\!]$ and consider the Frobenius endomorphism $\varphi\colon R \to S$. The map φ is flat and induces an isomorphism of residue fields, and S is finitely generated and projective R-module via φ. However, the map φ is not an isomorphism.

Example. Let $\varphi_0\colon k \to l$ be a nontrivial finite field extension. Set $R = k[\![X]\!]$ and $S = l[\![X]\!]$ and consider the induced local ring homomorphism $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$. The map φ is flat and $\mathfrak{m}S = \mathfrak{n}$, and S is finitely generated and projective R-module via φ. However, the map φ is not an isomorphism.

Final Questions

Question. Let $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{m}S, k)$ be a flat local homomorphism. If M is a finitely generated R-module such that $\operatorname{Ext}_R^1(S, M) = 0$, must the R-module structure on M ascend along φ?

Question. Let $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{m}S, k)$ be a flat local homomorphism. If M is an $\mathfrak{m}$-adically separated R-module such that $\operatorname{Ext}_R^{\geqslant 1}(S, M) = 0$, must the R-module structure on M ascend along φ?