

# Adic Semidualizing Modules

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## Goals

- 1 Describe a notion that covers both “semidualizing modules” and “quasi-dualizing modules”.
- 2 Show how to construct dualizing complexes over arbitrary complete local rings without using Cohen structure theorem.

## Assumption

$R$  is a commutative noetherian ring with identity.

# Two Examples

Fact (Assume  $R$  is local and CM with canonical module  $D$ .)

- 1  $D$  is finitely generated over  $R$ ,
- 2  $\text{Ext}_R^i(D, D) = 0$  for all  $i \neq 0$ , and
- 3  $R \xrightarrow{\chi} \text{Hom}_R(D, D)$  is an isomorphism.

Fact (The module  $R$  has similar properties.)

- 1  $R$  is finitely generated over  $R$ ,
- 2  $\text{Ext}_R^i(R, R) = 0$  for all  $i \neq 0$ , and
- 3  $R \xrightarrow{\chi} \text{Hom}_R(R, R)$  is an isomorphism.

Remark

Each of these examples is used for an important duality.

# Semidualizing Modules

## Definition (Foxby '72)

An  $R$ -module  $C$  is **semidualizing** if

- 1  $C$  is finitely generated over  $R$ ,
- 2  $\text{Ext}_R^i(C, C) = 0$  for all  $i \neq 0$ , and
- 3  $R \xrightarrow{X} \text{Hom}_R(C, C)$  is an isomorphism.

## Applications

- 1 Understanding compositions of local ring homomorphisms of finite G-dimension.
- 2 Understanding growth of Bass numbers of local rings.

# Quasi-dualizing Modules

Fact (Assume  $(R, \mathfrak{m}, k)$  is local with  $E = E_R(k)$ .)

- 1 If  $A$  is an artinian  $R$ -module, then it is  $\mathfrak{m}$ -torsion, so  $A$  has:
  - a natural  $\widehat{R}$ -module structure such that  $A \xrightarrow{\cong} \widehat{R} \otimes_R A$ , and
  - a natural homothety map  $\widehat{R} \xrightarrow{\chi} \text{Hom}_R(A, A)$ .
- 2  $E$  is artinian over  $R$ ,
- 3  $\text{Ext}_R^i(E, E) = 0$  for all  $i \neq 0$ , and
- 4  $\widehat{R} \xrightarrow{\chi} \text{Hom}_R(E, E)$  is an isomorphism.

Definition (Kubik '14)

An  $R$ -module  $A$  over a local ring is **quasi-dualizing** if

- 1  $A$  is artinian over  $R$ ,
- 2  $\text{Ext}_R^i(A, A) = 0$  for all  $i \neq 0$ , and
- 3  $\widehat{R} \xrightarrow{\chi} \text{Hom}_R(A, A)$  is an isomorphism.

# Similarities and Differences

## Definition

- An  $R$ -module  $C$  is **semidualizing** if
  - 1  $C$  is finitely generated (i.e., noetherian) over  $R$ ,
  - 2  $\text{Ext}_R^i(C, C) = 0$  for all  $i \neq 0$ , and
  - 3  $R \xrightarrow{\chi} \text{Hom}_R(C, C)$  is an isomorphism.
- An  $R$ -module  $A$  over a local ring is **quasi-dualizing** if
  - 1  $A$  is artinian over  $R$ ,
  - 2  $\text{Ext}_R^i(A, A) = 0$  for all  $i \neq 0$ , and
  - 3  $\widehat{R} \xrightarrow{\chi} \text{Hom}_R(A, A)$  is an isomorphism.

## Remark

- Similarities: a finiteness condition, Ext-vanishing, and a good endomorphism algebra.
- Differences: the finiteness condition, and the endomorphism algebra.

# Common Ground (Noetherian vs. Artinian)

## Assumption

$\alpha \subsetneq R$  with finite generating sequence  $\underline{x}$ . Set  $K = K^R(\underline{x})$ .

## Fact (Melkerson '05)

*Given an  $R$ -module  $M$ , the following conditions are equivalent:*

- 1  $\text{Ext}_R^i(R/\alpha, M)$  is finitely generated for all  $i$ ,
- 2  $\text{Tor}_i^R(R/\alpha, M)$  is finitely generated for all  $i$ , and
- 3  $H_i(K \otimes_R M)$  is finitely generated for all  $i$ .

## Definition (SW-Wicklein)

An  $R$ -module  $M$  is  **$\alpha$ -adically finite** if

- 1  $M$  is  $\alpha$ -torsion, and
- 2  $\text{Ext}_R^i(R/\alpha, M)$  is finitely generated for all  $i$ .

# Common Ground (Noetherian vs. Artinian), cont.

## Definition (SW-Wicklein)

An  $R$ -module  $M$  is  $\alpha$ -adically finite if

- 1  $M$  is  $\alpha$ -torsion, and
- 2  $\text{Ext}_R^i(R/\alpha, M)$  is finitely generated for all  $i$ .

## Examples (Common Ground - Let $M$ be an $R$ -module.)

- 1  $M$  is 0-adically finite if and only if it is finitely generated (i.e., noetherian) over  $R$ .
- 2 if  $(R, \mathfrak{m})$  is local, then  $M$  is  $\mathfrak{m}$ -adically finite if and only if it is artinian over  $R$ .
- 3  $M$  is  $\alpha$ -adically finite for some ideal  $\alpha$  of finite colength if and only if it is artinian over  $R$ .

## Remark

Nice properties include a version of Nakayama's Lemma.



# Common Ground (Homothety Maps)

Fact (Let  $M$  be an  $\alpha$ -torsion  $R$ -module.)

- 1 Then  $M$  has a natural  $\widehat{R}^\alpha$ -module structure  $M \xrightarrow{\cong} \widehat{R}^\alpha \otimes_R M$ .
- 2 This yields a natural homothety map  $\widehat{R}^\alpha \xrightarrow{\chi} \text{Hom}_R(M, M)$ .

Examples (Let  $M$  be an  $R$ -module.)

- 1  $\widehat{R}^0 \cong R \xrightarrow{\chi} \text{Hom}_R(M, M)$  is the same homothety map discussed for semidualizing modules.
- 2 If  $(R, \mathfrak{m}, k)$  is local and  $M$  is  $\mathfrak{m}$ -torsion, then  $\widehat{R}^{\mathfrak{m}} = \widehat{R} \xrightarrow{\chi} \text{Hom}_R(M, M)$  is the same homothety map discussed for quasidualizing modules.

# Goal 1: A notion that covers both semidualizing modules and quasi-dualizing modules.

## Definition (SW-Wicklein)

An  $R$ -module  $M$  is  $\alpha$ -adically semidualizing if

- 1  $M$  is  $\alpha$ -adically finite,
- 2  $\text{Ext}_R^i(M, M) = 0$  for  $i \neq 0$ , and
- 3  $\widehat{R}^\alpha \xrightarrow{\chi} \text{Hom}_R(M, M)$  is an isomorphism.

## Examples

- 1 an  $R$ -module is 0-adically semidualizing if and only if it is semidualizing.
- 2 if  $(R, \mathfrak{m})$  is local, an  $R$ -module is  $\mathfrak{m}$ -adically semidualizing if and only if it is quasi-dualizing.

## Goal 2: Construct dualizing complexes over complete local rings without using Cohen structure theorem.

### Theorem (SW-Wicklein)

Let  $M$  be an  $R$ -module with (truncated) flat resolution  $F$ .

- 1  $M$  is  $\alpha$ -adically finite over  $R$  if and only if  $H_i(\widehat{F}^\alpha)$  is finitely generated over  $\widehat{R}^\alpha$  for all  $i$ , i.e.,  $\mathbf{L}\widehat{\Lambda}^\alpha(M) \simeq \widehat{F}^\alpha \in \mathcal{D}_b^f(\widehat{R}^\alpha)$ .
- 2  $M$  is  $\alpha$ -adically semidualizing over  $R$  if and only if  $\mathbf{L}\widehat{\Lambda}^\alpha(M)$  is a semidualizing complex over  $\widehat{R}^\alpha$ .
- 3  $M$  is  $\alpha$ -adically semidualizing and  $\text{id}_R(M) < \infty$  if and only if  $\mathbf{L}\widehat{\Lambda}^\alpha(M)$  is a dualizing complex over  $\widehat{R}^\alpha$ .
- 4 If  $(R, \mathfrak{m}, k)$  is local and  $E = E_R(k)$ , then  $\mathbf{L}\widehat{\Lambda}^{\mathfrak{m}}(E)$  is a dualizing complex over  $\widehat{R}$ .

### Remark

All of this can be done for complexes.

# Relations with Semidualizing Complexes

$$\begin{array}{ccc}
 \mathfrak{S}^a(R) & \begin{array}{c} \xrightarrow{\mathbf{R}\widehat{\Gamma}_a = \widehat{R}^a \otimes_R^{\mathbf{L}} -} \\ \xleftarrow{Q} \end{array} & \mathfrak{S}^{a\widehat{R}^a}(\widehat{R}^a) \\
 \updownarrow \begin{array}{c} ? \\ K \otimes_R^{\mathbf{L}} - \end{array} & \searrow \mathbf{L}\widehat{\Lambda}^a & \updownarrow \begin{array}{c} \mathbf{R}\Gamma_{a\widehat{R}^a} \\ \mathbf{L}\Lambda^{a\widehat{R}^a} \end{array} \\
 \mathfrak{S}(K) & \begin{array}{c} \xrightarrow{Q \circ \mathbf{R}\Gamma_{a\widehat{R}^a} = \mathbf{R}\Gamma_a \circ Q} \\ \xleftarrow{?} \end{array} & \mathfrak{S}(\widehat{R}^a) \\
 \updownarrow \begin{array}{c} Q \\ \widehat{K}^a \otimes_K^{\mathbf{L}} - \end{array} & \searrow \begin{array}{c} ? \\ \widehat{K}^a \otimes_{\widehat{R}^a}^{\mathbf{L}} - \end{array} & \\
 \mathfrak{S}(\widehat{K}^a) & & 
 \end{array}$$

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[4, 1, 3, 5, 2, 6]